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Short communication

Integral representations of the Riemann zeta function for odd-integer arguments

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Abstract

We deduce four new integral representations for $\zeta(2n+1)$, $n \in N$, where $\zeta(s)$ is the Riemann zeta function.

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1. Introduction

The Riemann zeta function $\zeta(s)$ is defined for Re $s > 1$ as [1, p. 807, Eq. 23.2.1]

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}. \tag{1}$$

Recall that there exists a formula which expresses $\zeta(2n)$, $n \in N$, as a rational multiple of π^{2n}. However, there is no analogous closed evaluation for $\zeta(2n+1)$, which is usually given by the integral representation

$$\zeta(2n+1) = (-1)^{n+1} \frac{(2\pi)^{2n+1}}{2(2n+1)!} \int_0^1 B_{2n+1}(x) \cot(\pi x) \, \mathrm{dx} \tag{2}$$

[1, p. 807, Eq. 23.2.17] involving the Bernoulli polynomials $B_n(x)$, or by various series representations. By making use of elementary arguments, we will derive (2) and several related integral representations for $\zeta(2n+1)$. Three of the four integrals involved cannot be found in [3] or [2].

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2. Statement of the results

Throughout the text $B_n(x)$ and $E_n(x)$ are the Bernoulli and Euler polynomials [1, p. 804, Eq. 23.1.1]

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad |t| < 2\pi, \tag{3a}$$

$$\frac{2e^{tx}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} \quad |t| < \pi. \tag{3b}$$

We also use [1, p. 807, Eqs. 23.2.19, 23.2.20 and 23.2.21]

$$\eta(s) = (1 - 2^{1-s})\,\zeta(s) = \sum_{k=1}^{\infty} (-1)^{k-1}\frac{1}{k^s}, \tag{4a}$$

$$\lambda(s) = (1 - 2^{-s})\,\zeta(s) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^s}, \tag{4b}$$

$$\beta(s) = \sum_{k=0}^{\infty} (-1)^k\frac{1}{(2k+1)^s}. \tag{4c}$$

Our results are as follows.

Theorem 1. *Assume that n is a positive integer and that* $\delta = 1$ *and* $\frac{1}{2}$. *Let* $\zeta(s), \eta(s), \lambda(s)$ *and* $\beta(s)$ *be defined as in Eqs.* (1) *and* (4), *respectively. Let* $B_n(x)$ *and* $E_n(x)$ *be the Bernoulli and Euler polynomials given by Eqs.* (3*a*) *and* (3*b*), *respectively. We then have*:

$$\zeta(2n+1) = (-1)^{n+1}\frac{(2\pi)^{2n+1}}{2\delta(2n+1)!}\int_0^{\delta} B_{2n+1}(t)\cot(\pi t)\,dt,$$

$$\eta(2n+1) = (-1)^{n+1}\frac{(2\pi)^{2n+1}}{2\delta(2n+1)!}\int_0^{\delta} B_{2n+1}(t)\tan(\pi t)\,dt,$$

$$\lambda(2n+1) = (-1)^{n}\frac{\pi^{2n+1}}{4\delta(2n)!}\int_0^{\delta} E_{2n}(t)\csc(\pi t)\,dt,$$

$$\beta(2n) = (-1)^{n}\frac{\pi^{2n}}{4\delta(2n-1)!}\int_0^{\delta} E_{2n-1}(t)\sec(\pi t)\,dt.$$

Note 1. Observe that the existence of the above integrals is assured, since the integrands on $[0, \delta]$ have only removable singularities. To demonstrate this, we need some basic properties of $B_n(x)$ and

$E_n(x)$. For instance, knowing that the odd-indexed Bernoulli numbers B_n are zero [1, p. 805, Eq. 23.1.19], we have

$$\lim_{t\to 1/2} B_{2n+1}(t)\tan(\pi t) = \lim_{t\to 1/2}\frac{B_{2n+1}(t)}{\cos(\pi t)} = \lim_{t\to 1/2}\frac{(2n+1)B_{2n}(t)}{-\pi\sin(\pi t)}$$
$$= (1-2^{1-2n})(2n+1)B_{2n}/\pi$$

since $B_n(1/2) = (2^{1-n}-1)B_n$ [1, p. 805, Eq. 23.1.21].

Note 2. The integrals for $\zeta(2n+1)$ and B_{2n} when $\delta = 1$ are well known [1, p. 807, Eqs. 23.2.17 and 23.2.23], but are included for completeness. These two integrals for $\delta = \frac{1}{2}$ are given in [4, Eqs. 10 and 10′]. However, we have been unable to find the remaining integral representations in the literature.

3. Proof of the results

First, we will show that

$$\sin(2kx)\cot x = 1 + \sum_{j=1}^{k-1}\cos(2jx) + \sum_{j=1}^{k}\cos(2jx), \tag{5a}$$

$$\sin(2kx)\tan x = (-1)^{k-1} + \sum_{j=1}^{k-1}(-1)^{k-1-j}\cos(2jx) - \sum_{j=1}^{k}(-1)^{k-j}\cos(2jx), \tag{5b}$$

$$\frac{\sin(2k+1)x}{\sin x} = 1 + 2\sum_{j=1}^{k}\cos(2jx), \tag{5c}$$

$$\frac{\cos(2k+1)x}{\cos x} = (-1)^k + 2\sum_{j=1}^{k}(-1)^{k-j}\cos(2jx). \tag{5d}$$

Indeed, starting from

$$2\sin x\cos(2mx) = \sin(2m+1)x - \sin(2m-1)x$$

we have

$$2\sin x\sum_{j=1}^{k}\cos(2jx) = \sin(2k+1)x - \sin x$$

and formula (5c) follows without difficulty. On setting $x = t + \pi/2$ we obtain formula (5d) from (5c).

Further, knowing that

$$\sin(2kx)\cot x = \frac{1}{2}\left[\frac{\sin(2k+1)x}{\sin x} + \frac{\sin(2k-1)x}{\sin x}\right],$$

$$\sin(2kx)\tan x = \frac{1}{2}\left[\frac{\cos(2k-1)x}{\cos x} - \frac{\cos(2k+1)x}{\cos x}\right],$$

we have (5a) and (5b) from (5c) and (5d), respectively.

Next, we need the following Fourier expansions for the Bernoulli and Euler polynomials [1, p. 805, Eqs. 23.1.17 and 23.1.18]

$$B_{2n+1}(x) = \frac{(-1)^{n+1}2(2n+1)!}{(2\pi)^{2n+1}} \sum_{k=1}^{\infty} \frac{\sin(2k\pi x)}{k^{2n+1}}, \tag{6a}$$

$$E_{2n}(x) = \frac{(-1)^n 4(2n)!}{\pi^{2n+1}} \sum_{k=0}^{\infty} \frac{\sin(2k+1)\,\pi x}{(2k+1)^{2n+1}}, \tag{6b}$$

$$E_{2n-1}(x) = \frac{(-1)^n 4(2n-1)!}{\pi^{2n}} \sum_{k=0}^{\infty} \frac{\cos(2k+1)\,\pi x}{k^{2n}}, \tag{6c}$$

where $n = 1, 2, 3, \ldots$ and $0 < x \leqslant 1$.

Finally, the formulae proposed in the Theorem are obtained from the expansions in Eq. (6) in conjunction with the identities in Eq. (5) and the definitions in Eqs. (1) and (4). For instance,

$$(-1)^{n+1}\frac{(2\pi)^{2n+1}}{2(2n+1)!}\int_0^{\delta} B_{2n+1}(t)\tan(\pi t)\,\mathrm{d}t = \sum_{k=1}^{\infty}\frac{1}{k^{2n+1}}\int_0^{\delta}\sin(2k\pi t)\tan(\pi t)\,\mathrm{d}t$$

$$= \delta\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k^{2n+1}} = \delta\eta(2n+1).$$

Observe that inverting the order of summation and integration is justified by absolute convergence. This completes our proof. □

4. Concluding remarks

We deduce the following representations for $\zeta(2n+1)$, $n \in N$:

$$\zeta(2n+1) = \frac{(-1)^{n+1}(2\pi)^{2n+1}}{2\delta(1-2^{-2n})(2n+1)!}\int_0^{\delta} B_{2n+1}(t)\tan(\pi t)\,\mathrm{d}t,$$

$$\zeta(2n+1) = \frac{(-1)^n \pi^{2n+1}}{4\delta(1-2^{-(2n+1)})(2n)!}\int_0^{\delta} E_{2n}(t)\csc(\pi t)\,\mathrm{d}t,$$

where $\delta = 1$ and $\frac{1}{2}$.

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