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A NOTE ON THE DRAZIN INVERSE OF A MODIFIED MATRIX*

Dragana S. Cvetkovic
Faculty of Sciences and Mathematics, Department of Mathematics, University of Nis, Visegradska 33, 18000 Nis, Serbia
Jelena Ljubisavljevic
Faculty of Medicine, University of Niš
E-mail: dragana@pmf.ni.ac.rs; gagamaka@ptt.rs

Abstract In this article, the expression for the Drazin inverse of a modified matrix is considered and some interesting results are established. This contributes to certain recent results obtained by Y.Wei [9].

Key words Drazin inverse; modified matrix; perturbation bound

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1 Introduction

Let $C^{n\times m}$ denote the set of all complex $n\times m$ matrices. For $A\in C^{n\times m}$, the set of inner inverses are given by

$$A\{1\}=\{X:AXA=A\}. \tag{1}$$

Let us recall that the Drazin inverse of $A\in C^{n\times n}$ [3] is the matrix $A^D\in C^{n\times n}$ that satisfies

$$A^{k+1}X=A^k,\quad XAX=X,\quad AX=XA,$$

for some nonnegative integer k. The least such k is the index of A, denoted by $\mathrm{ind}(A)$. Some interesting properties of Drazin inverse, among other articles, were investigated in [4, 8, 10].

In this article, we consider a matrix $A\in C^{(m+p)\times(n+q)}$ partitioned as

$$M=\begin{bmatrix} A & B\\ C & D\end{bmatrix}, \tag{2}$$

where $A\in C^{m\times n}$ and $D\in C^{p\times q}$.

The motivation for this research is the article of Y.Wei [9], in which he derived various expressions for the Drazin inverse of a modified matrix.

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It is well-known that the generalized Schur complement of D in M is defined as

$$S(M) = A - BD^{-}C, \tag{3}$$

where $D^{-} \in D\{1\}$.

If we replace $D^{-} \in D\{1\}$ by the Drazin inverse of D in (3), we obtain the Drazin-Schur complement of D in M, denoted by

$$S_D(M) = A - BD^D C.$$

The Drazin-Schur complement of A in M is denoted by

$$Z_D(M) = D - CA^D B.$$

For interesting results concerning Schur complements, see [1, 2, 6, 7].

In this article, we derive some expressions for the Drazin inverse of Drazin-Schur complement for the matrix M given by (2). As a corollary, we obtain the results of Wei [9].

2 Results

For an arbitrary matrix A, we denote $E_A = I - AA^D$. Let

$$K = A^D B, \quad H = CA^D, \quad G = HK.$$

We use S and Z instead of $S_D(M)$ and $Z_D(M)$, respectively.

When the partitioned matrix M and the submatrix D are both nonsingular, then the Schur complement of D in M is also nonsingular. When M, A, and D are all nonsingular, then,

$$(A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}B(D - CA^D B)^{-1}CA^{-1}$$

which was observed by Duncan [5]. We have the analogous result concerning the Drazin inverse and the Drazin-Schur complement.

Theorem 2.1 Suppose that $E_A B = 0$, $CE_A = 0$, $BE_D Z^D C = 0$, $BD^D E_Z C = 0$, $BZ^D E_D C = 0$, and $BE_Z D^D C = 0$. Then,

$$S^D = A^D + A^D BZ^D CA^D.$$

Proof Let $X = A^D + A^D BZ^D CA^D$. Then,

$$\begin{aligned} SX &= (A - BD^D C)(A^D + A^D BZ^D CA^D) \\ &= AA^D + AA^D BZ^D CA^D - BD^D CA^D - BD^D CA^D BZ^D CA^D \\ &= AA^D + BZ^D CA^D - BD^D CA^D - BD^D (D - Z)Z^D CA^D \\ &= AA^D + BE_D Z^D CA^D - BD^D E_Z CA^D \\ &= AA^D. \end{aligned}$$

Similarly, $XS = A^D A$, that is, $XS = SX$. Furthermore,

$$\begin{aligned} XSX &= A^D A(A^D + A^D BZ^D CA^D) \\ &= A^D + A^D BZ^D CA^D \\ &= X. \end{aligned}$$

By induction, it follows that

$$(A - BD^DC)^{m+1}X = (A - BD^DC)^m + (A^{m+1}A^D - A^m).$$

Hence, $(A - BD^DC)^{m+1}X = (A - BD^DC)^m$ holds for $m \geq \text{ind}(A)$.

In case $D = I$, it follows that $E_D = 0$. Hence, we obtain Theorem 2.1 in [9] as a corollary of our Theorem 2.1.

Corollary 2.1 If $E_AB = 0,\ CE_A = 0,\ BE_ZC = 0$, then,

$$(A - BC)^D = A^D + A^DBZ^DCA^D,$$

where $Z = I - CA^DB$.

If Z is invertible, then, from Theorem 2.1, we obtain the following result.

Corollary 2.2 Suppose that Z is nonsingular and $E_AB = 0$, $CE_A = 0$, $BE_DZ^{-1}C = 0$, and $BZ^{-1}E_DC = 0$. Then,

$$S^D = A^D + A^DBZ^{-1}CA^D.$$

In case $B = I$, we have the following corollary.

Corollary 2.3 Suppose that $CE_A = 0$, $E_AD^D = 0$, and $\|A^D\| \cdot \|D^DC\| \leq 1$. Then,

$$(A - D^DC)^D = (I - A^DD^DC)^{-1}A^D = A^D(I - D^DCA^D)^{-1}$$

and

$$(A - D^DC)^D - A^D = (A - D^DC)^DD^DCA^D = A^DD^DC(A - D^DC)^D,$$

with

$$\frac{\|(A - D^DC)^D - A^D\|}{\|A^D\|} \leq \frac{k_D(A)\|D^DC\|/\|A\|}{1 - k_D(A)\|D^DC\|/\|A\|},$$

where $k_D(A) = \|A\|\|A^D\|$ is the condition number with respect to the Drazin inverse.

Proof For the proof of this corollary, see Theorem 3.2 and Corollary 3.2 in [10].

Theorem 2.2 Let $Z = 0$, $E_AB = 0$, $CE_A = 0$, $BE_DG^DC = 0$, $BD^DE_GC = 0$, $BG^DE_DC = 0$, and $BE_GD^DC = 0$. Then,

$$\begin{aligned} S^D &= (I - KG^DH)A^D(I - KG^DH) \\ &= (I - KH(KH)^D)A^D(I - KH(KH)^D). \end{aligned}$$

Proof Denote $X = (I - KG^DH)A^D(I - KG^DH)$. We obtain

$$\begin{aligned} SX &= (A - BD^DC)(I - KG^DH)A^D(I - KG^DH) \\ &= (A - BD^DC - BG^DCA^D + BD^DDG^DCA^D)(A^D - (A^D)^2BG^DCA^D) \\ &= (A - BD^DC)(A^D - (A^D)^2BG^DCA^D) \\ &= AA^D - BD^DCA^D - A(A^D)^2BG^DCA^D + BD^DC(A^D)^2BG^DCA^D \\ &= AA^D - BD^DCA^D - A^DBG^DCA^D + BD^DGG^DCA^D \\ &= AA^D - A^DBG^DCA^D \\ &= AA^D - KG^DH \end{aligned}$$

and $XS = AA^D - KG^DH$, that is, $XS = SX$. Also,

$$\begin{aligned} XSX &= (AA^D - KG^DH)(I - KG^DH)A^D(I - KG^DH) \\ &= (AA^D - KG^DH - AA^DKG^DH + KG^DHKG^DH)(A^D - A^DKG^DH) \\ &= (AA^D - KG^DH)(A^D - A^DKG^DH) \\ &= (I - KG^DH)AA^DA^D(I - KG^DH) \\ &= X. \end{aligned}$$

We prove that $(A - BD^DC)^{m+1}X = (A - BD^DC)^m$ by induction.

If $D = I$, then we obtain Theorem 2.2 of [9]:

Corollary 2.4 Suppose that $Z = 0,\ E_AB = 0,\ CE_A = 0$, and $BE_GC = 0$. Then,

$$\begin{aligned} (A - BC)^D &= (I - KG^DH)A^D(I - KG^DH) \\ &= (I - KH(KH)^D)A^D(I - KH(KH)^D). \end{aligned}$$

Theorem 2.3 Let $\mathrm{ind}(Z) = 1$ and $E_AB = 0,\ CE_A = 0,\ BE_D = 0,\ E_DC = 0,\ ZZ^{\#}G = GZZ^{\#},\ BD = DB,\ CD = DC,\ BE_GC = 0$. Then,

$$S^D = (I - KE_ZG^DH)A^D(I - KE_ZG^DH) + KZ^{\#}H. \tag{4}$$

Proof Denote by X the right side of (4). We have

$$\begin{aligned} SX &= (A - BD^DC - BE_ZG^DH + BD^DCA^DBE_ZG^DH) \\ &\quad \times A^D(I - KE_ZG^DH) + AKZ^{\#}H - BD^DCA^DBZ^{\#}H \\ &= (A - BD^DC - BE_ZG^DH + BD^D(D - Z)E_ZG^DH) \\ &\quad \times A^D(I - KE_ZG^DH) + BZ^{\#}H - BD^D(D - Z)Z^{\#}H \\ &= (A - BD^DC)A^D(I - KE_ZG^DH) + BD^DZZ^{\#}H \\ &= AA^D - KE_ZG^DH - BD^DCA^D + BD^DGG^DE_ZH + BD^DZZ^{\#}H \\ &= AA^D - KE_ZG^DH - BD^DE_GCA^D + BD^DE_GZZ^{\#}CA^D \\ &= AA^D - KE_ZG^DH \end{aligned}$$

and

$$\begin{aligned} XS &= (I - KE_ZG^DH)A^D(A - BD^DC - KE_ZG^DC \\ &\quad + KE_ZG^DCA^DBD^DC) + KZ^{\#}C - KZ^{\#}CA^DBD^DC \\ &= (I - KE_ZG^DH)A^D(A - BD^DC - KE_ZG^DC \\ &\quad + KE_ZG^D(D - Z)D^DC) + KZ^{\#}C - KZ^{\#}(D - Z)D^DC \\ &= (I - KE_ZG^DH)A^D(A - BD^DC) + KZ^{\#}ZD^DC \\ &= A^DA - A^DBD^DC - KE_ZG^DH + KE_ZG^DGD^DC + KZ^{\#}ZD^DC \\ &= A^DA - KG^DE_ZH - A^DBE_GD^DC + KZ^{\#}ZE_GD^DC \\ &= A^DA - KG^DE_ZH. \end{aligned}$$

Furthermore,

$$\begin{aligned} XSX &= (A^DA - KG^DE_ZH)(I - KE_ZG^DH)A^D \\ &\quad \times (I - KE_ZG^DH) + (A^DA - KG^DE_ZH)KZ^{\#}H \\ &= (A^DA - KG^DE_ZH)A^D(I - KE_ZG^DH) + KZ^{\#}H \\ &= X. \end{aligned}$$

By induction, it follows that

$$(A - BD^DC)^{m+1}X = (A - BD^DC)^m + (A^{m+1}A^D - A^m).$$

Hence, $(A - BD^DC)^{m+1}X = (A - BD^DC)^m$, for $m \geq \operatorname{ind}(A)$.

Obviously, for $D = I$ we have the following result.

Corollary 2.5 Let $E_AB = 0,\ CE_A = 0,\ BE_GC = 0,\ G^DE_Z = E_ZG^D$, and $\operatorname{ind}(Z) = 1$. Then,

$$(A - BC)^D = (I - KE_ZG^DH)A^D(I - KE_ZG^DH) + KZ^{\#}H.$$

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