Global attractors for 3-dimensional stochastic Navier-Stokes equations

Nigel J. Cutland and H. Jerome Keisler

April 17, 2003

Abstract

Sell's approach [32] to the construction of attractors for the Navier–Stokes equations in 3-dimensions is extended to the 3-D stochastic equations with a general multiplicative noise.

1 Introduction

The existence of global attractors for deterministic time-homogeneous incompressible Navier–Stokes equations

$$\begin{cases} du = [\nu\Delta u \,-\, \langle u,\nabla\rangle u + f(u) - \nabla p]dt \\ \\ \text{div } u = 0 \end{cases} \tag{1}$$

on a bounded domain $D \subseteq \mathbb{R}^2$ is well known (see Temam [33] for an exposition): there is a compact global attractor in $\mathbf{H}$, the subspace of divergence free vector fields in $L^2(D)$, which is the phase space for the equations (1). (The solutions to (1) have $u(t) \in \mathbf{H}$ for all t.)

Equally well known is the fundamental problem with the very idea of attractors for these equations in 3-dimensions; the question of uniqueness of weak solutions to the equations is still open, with the consequence that the conventional formulation of the notion of an attractor may not even make sense. It would require the existence of a semiflow of solutions S_t say, so that $S_t v$ is the (unique) solution for the given initial condition v. If attention is restricted to strong solutions, where uniqueness *is* known, the fundamental open problem is that of existence for all time – which again means that the usual notion of attractor cannot be used. In order to overcome the difficulties mentioned above when considering attractors for the 3-dimensional *deterministic* Navier–Stokes equations, a number of approaches have been suggested, beginning with Foias & Temam [23], and more recently Sell [32], Cutland & Capiński [8] and Ball [2].

Sell's radical approach [32] to the 3-D problem of attractors was to replace the phase space $\mathbf{H}$ by a space $\mathbf{W}$ of entire solutions to the Navier–Stokes equations. That is, each point in $\mathbf{W}$ is the complete trajectory in $\mathbf{H}$ of a solution. There is a simple semigroup action S_t on $\mathbf{W}$ – namely time translation. Thus, if $u = u(\cdot) \in \mathbf{W}$ then $S_t u = v \in \mathbf{W}$ is given by

$$v(s) = u(t + s).$$

Clearly this is well defined, and has the crucial semi-flow property

$$S_{t_1} \circ S_{t_2} = S_{t_1+t_2}$$

along with $S_0 u = u$.

Using this idea, Sell was able to establish the existence of a global attractor for the 3-D (deterministic) Navier–Stokes equations.

The goal of this paper is to extend Sell's approach to the general time-homogeneous *stochastic* Navier–Stokes equations (2) with multiplicative noise in dimensions 2 and 3, taking the form

$$\begin{cases} du = [\nu\Delta u \,-\, \langle u, \nabla\rangle u + f(u) - \nabla p]dt + g(u)dw_t \\ \\ \text{div } u = 0 \end{cases} \tag{2}$$

where $u = u(t, \omega)$ is now a *random* velocity field at time $t \geq 0$ of a fluid in a bounded domain $D \subseteq \mathbb{R}^3$ (or $D \subseteq \mathbb{R}^2$). For simplicity the driving noise w_t is taken to be a 1-dimensional Wiener process.

Even in 2-dimensions there are considerable difficulties when seeking stochastic attractors. These can be overcome in a number of ways – for example by considering *measure attractors* (see [31, 6]), or by working with the notion of stochastic attractor developed by Crauel & Flandoli [10]. For this it is necessary first to show the existence of a flow of solutions with a stochastic equivalent of the semi-group property – known as a *perfect cocycle*. This was achieved in [9] for a very special form of the noise $g(u)$ when the system has periodic boundary conditions.

For the 3-dimensional stochastic case, Sell's idea was used by Flandoli & Schmalfuss in the paper [12] for the Navier–Stokes equations with a special form of multiplicative noise, using a mild solution concept. The equation considered allowed essentially a pathwise solution, and then a random attractor was obtained by combining Sell's approach with the idea of pulling back in time to $-\infty$, as developed by Crauel & Flandoli [12]. In a later paper [13] Flandoli & Schmalfuss consider in the same framework 3-D-Navier–Stokes equations with an irregular forcing term, but no feedback.

In the current paper we consider 3-D stochastic Navier–Stokes equations with a general multiplicative noise $g(u)$ as above. The idea is to use Sell's approach at the level of *processes* rather than paths. In this way the idea of an attractor is formulated in the conventional sense, examining the long term behavior of solutions as $t \to \infty$. To do this, it is necessary to have a single underlying probability space, rich enough to carry a supply of solutions to the 3-D stochastic Navier–Stokes equations that is sufficient for the concepts to make sense. For this we need a filtered Loeb space.

2 Preliminaries and assumptions

We will always state our definitions for the three dimensional case, but it should be understood that these definitions can be modified in the natural way to get the analogous notions for two dimensions. We use $\mathbb{N}$ for the set of *positive* integers.

2.1 Stochastic Navier–Stokes equations

We consider the stochastic Navier–Stokes equations (2) in a bounded domain $D \subseteq \mathbb{R}^3$ with boundary of class C^2, and adopt the conventional Hilbert space approach as follows.

Let $\mathbf{H}$ be the closure of the set

$$\{u \in C_0^\infty(D, \mathbb{R}^3) : \text{div } u = 0\}$$

in the L^2 norm $|u| = (u, u)^{1/2}$, where $u = \langle u_1, u_2, u_3 \rangle$ and

$$(u, v) = \sum_{j=1}^{3} \int_D u_j(x) v_j(x) dx.$$

The letters u, v, w will be used for elements of $\mathbf{H}$. The subspace $\mathbf{V}$ is the closure of the set $\{u \in C_0^\infty(D, \mathbb{R}^3) : \text{div } u = 0\}$ in the stronger norm $|u| + \|u\|$ where $\|u\| = ((u, u))^{1/2}$ and

$$((u, v)) = \sum_{j=1}^{3} \left(\frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j} \right).$$

$\mathbf{H}$ and $\mathbf{V}$ are Hilbert spaces with scalar products $(\cdot, \cdot)$ and $((\cdot, \cdot))$ respectively, and $|\cdot| \leq c\|\cdot\|$ for some constant c.

By A we denote the self adjoint extension of the projection of $-\Delta$ in $\mathbf{H}$. Classical theory shows that there is an orthonormal basis $\{e_k : k \in \mathbb{N}\}$ of eigenfunctions of A with corresponding eigenvalues $\lambda_k > 0$ such that $\lambda_k \nearrow \infty$. For $u \in \mathbf{H}$ we write $u_{(k)} = (u, e_k)$, and write Pr_m for the projection of $\mathbf{H}$ on the subspace $\mathbf{H}_m$ spanned by $\{e_1, \ldots, e_m\}$. Since each $e_k \in \mathbf{V}$, then $\mathbf{H}_m \subseteq \mathbf{V}$. If $u = \sum u_{(k)} e_k \in \mathbf{V}$ then $\|u\|^2 = \sum \lambda_k u_{(k)}^2$, so that the constant c above is $\lambda_1^{\frac{1}{2}}$.

A trilinear form b is defined by

$$b(u,v,w) = \sum_{i,j=1}^{3} \int_D u_j(x) \frac{\partial v_i}{\partial x_j}(x) w_i(x) dx = (\langle u, \nabla \rangle v, w)$$

whenever the integrals make sense. Note the following well-known properties of the trilinear form b, where c is a real constant.

$$b(u,v,w) = -b(u,w,v),$$

$$b(u,v,v) = 0,$$

$$|b(u,v,w)| \leq c\|u\|\,\|v\|\,\|w\|.$$

The last is a continuity property of b with respect to the norm $\|\cdot\|$. There are a number of other important continuity properties for other norms which we will not need.

2.2 Functional formulation of the Navier–Stokes equations

In the above framework, the stochastic Navier–Stokes equations may be formulated as the following stochastic differential equation in $\mathbf{V}'$ (the dual of $\mathbf{V}$):

$$du = [-\nu Au - B(u) + f(u)]dt + g(u)dw_t, \tag{3}$$

where $B(u) = b(u,u,\cdot)$. Note that the pressure has disappeared, because $\nabla p = 0$ in $\mathbf{V}'$ (using div $v = 0$ in $\mathbf{V}$ and an integration by parts). Although equation (3) is regarded as an equation in $\mathbf{V}'$, it turns out that solutions can be found that live in $\mathbf{H}$ (and in fact in $\mathbf{V}$ for almost all times).

We take w_t to be a 1-dimensional adapted Wiener process on a filtered probability space

$$\mathbf{\Omega} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P).$$

The term *adapted* is always taken to mean adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Further assumptions on $\mathbf{\Omega}$ that are needed to formulate the idea of an attractor for a class of stochastic processes are given in the next section.

The equation (3) is really an *integral* equation, with the first integral being the Bochner integral and the second an extension of the Itô integral to Hilbert spaces in the weak sense. Thus, when we write

$$u(t_1) = u(t_0) + \int_{t_0}^{t_1} [-\nu Au(t) - B(u(t)) + f(u(t))]dt + \int_{t_0}^{t_1} g(u(t))dw_t$$

we mean that for all $v \in \mathbf{V}$ we have

$$(u(t_1), v) =$$

$$(u(t_0),v)+\int_{t_0}^{t_1}[-\nu(Au(t),v)-(B(u(t)),v)+(f(u(t)),v)]dt+\int_{t_0}^{t_1}(g(u(t)),v)dw_t$$

as a stochastic equation in $\mathbb{R}$.

We make the following assumptions on the coefficients f and g, where c_0, d_1, d_2 are positive real constants.

(H1) $f:\mathbf{H}\to\mathbf{H}$ and $|f(u)|\le c_0+d_1|u|$.

(H2) $g:\mathbf{H}\to\mathbf{H}$ and $|g(u)|\le c_0+d_2|u|$.

(H3) f and g are continuous.

(H4) $2d_1+3d_2^2<2\nu\lambda_1$.

The general theory of stochastic Navier–Stokes equations expounded in [7] shows that the equation (3) can be solved with only the assumptions (H1)–(H3). The additional growth restriction (H4) on f, g is needed here to obtain the attractor.

2.3 Truncation functions

For technical reasons (in connection with testing for S-integrability) we will need an explicit family of "truncation functions". We give the details here but suggest that the reader refer back to this section only when needed later.

The following real C^2 function $\psi:[0,\infty)\to[0,1]$ is designed to be concave on $[0,1]$ and constant (with value 1) on $[1,\infty)$.

Definition 2.1 (a)

$$\psi(x)=\begin{cases}(x-1)^3+1 & \text{if } \ 0\le x\le 1\\ 1 & \text{if } \ x\ge 1\end{cases}$$

(b) For each $n\in\mathbb{N}$,

$$\psi_n(x)=\psi(x/n^2).$$

(c) For each $n\in\mathbb{N}$ a function $\varphi_n(u)$ is defined for u in any Hilbert space (finite or infinite dimensional) by

$$\varphi_n(u)=u^2\psi_n(u^2),$$

where we write u^2 to mean $|u|^2$ to ease the notation.

Direct calculation gives:

(d)

$$\psi_n'(x)=\begin{cases}3n^{-2}(1-x/n^2)^2 & \text{if } \ 0\le x\le n^2\\ 0 & \text{if } \ n^2\le x\end{cases}$$

(e)

$$\psi_n''(x)=\begin{cases}-6n^{-4}(1-x/n^2) & \text{if } \ 0\le x\le n^2\\ 0 & \text{if } \ n^2\le x\end{cases}$$

(f)

$$\varphi_n'(u)=2u\Big(\psi_n(u^2)+u^2\psi_n'(u^2)\Big).$$

(g)

$$\varphi_n''(u) = 2I\Big[\psi_n(u^2) + u^2\psi_n'(u^2)\Big] + 4uu^T\Big[2\psi_n'(u^2) + u^2\psi_n''(u^2)\Big]$$

(again writing u^2 for $|u|^2$).

The significance of the functions φ_n involves the concept of S-integrability from Loeb measure theory, to be defined later. We will show that given an internal random vector $U(\omega)$, $|U|^2$ is S-integrable if and only if $^\circ\mathbb{E}(\varphi_n(U)) \to 0$ as $n \to \infty$. The application of this fact will involve the following particular properties of ψ_n and φ_n.

Lemma 2.2 (Properties of ψ_n and φ_n) *The functions ψ_n and φ_n have the following properties:*
(a) $x\psi_n'(x) \le \psi_n(x)$ *all n and $x \ge 0$.*
(b) $2|u|\,\psi_n(u^2) \le |\varphi_n'(u)| \le 4|u|\psi_n(u^2)$.
(c) $2\varphi_n(u) \le |u|\,|\varphi_n'(u)| \le 4\varphi_n(u)$.

Proof Elementary calculation. □

Lemma 2.3 *For any random vector $u(\omega)$ in a Hilbert space:*
(a) $\mathbb{E}(|u|\psi_n(u^2)) \le n^{-1/2}(3 + \mathbb{E}(u^2))$.
(b) $\mathbb{E}(|\varphi_n'(u)|) \le 4n^{-1/2}(3 + \mathbb{E}(u^2))$.
(c) $\mathbb{E}(\psi_n(u^2)) \le n^{-1}(3 + \mathbb{E}(u^2))$.

Proof (a) We have

$$\begin{aligned}
\mathbb{E}(|u|\psi_n(u^2)) &= \int_{|u|^2\le n} |u|\psi_n(u^2) + \int_{|u|^2>n} |u|\psi_n(u^2) \\
&\le n^{1/2}\psi_n(n) + \int_{|u|>\sqrt{n}} |u| \qquad \text{(since } \psi_n \le 1\text{)} \\
&\le n^{1/2}\psi_n(n) + \int_{|u|>\sqrt{n}} |u||u|/\sqrt{n} \\
&\le n^{1/2}\psi_n(n) + n^{-1/2}\mathbb{E}(u^2) \\
&\le n^{-1/2}(3 + \mathbb{E}(u^2))
\end{aligned}$$

using the fact that $n\psi_n(n) = n^{-2} - 3n^{-1} + 3 \le 3$.
(b) This follows from (a) since $|\varphi_n'(u)| \le 4|u|\psi_n(u^2)$.
(c) We have

$$\begin{aligned}
\mathbb{E}(\psi_n(u^2)) &= \int_{|u|^2\le n} \psi_n(u^2) + \int_{|u|^2>n} \psi_n(u^2) \\
&\le \psi_n(n) + \mathbb{P}(u^2 > n) \qquad \text{(since } \psi_n \le 1\text{)} \\
&\le \psi_n(n) + n^{-1}\mathbb{E}(u^2) \\
&\le n^{-1}(3 + \mathbb{E}(u^2)).
\end{aligned}$$

□

3 Semiflows and attractors

3.1 Semiflows

We now assume that the space $\mathbf{\Omega}$ is equipped with a family of measure preserving maps $\theta_t : \Omega \to \Omega$ for $t \geq 0$ with the following properties:

(θ1) $\theta_0 =$ identity and $\theta_t \circ \theta_s = \theta_{t+s}$;

(θ2) $\theta_t \mathcal{F}_s = \mathcal{F}_{t+s}$ for all $s, t \geq 0$;

(θ3) $w(t+s, \theta_t\omega) - w(t, \theta_t\omega) = w(s, \omega)$ for all $s \geq 0$.

Note that the property (θ3) tells us that for a fixed t the increments of the process $w(t+s, \theta_t\omega)$ are the same as those of the process $w(s, \omega)$. Thus θ_t can be thought of as a shift of the noise to the right by t.

The family (θ_t) allows the following definition of a semiflow S_r of stochastic processes.

Definition 3.1 (Semiflow of Processes) (a) Suppose that $u = u(t, \omega)$ is a stochastic process defined for $t > 0$. Then for any $r \geq 0$ the process $v = S_r u$ is defined by

$$v(t, \omega) = u(r+t, \theta_r\omega).$$

(b) By a *semiflow* $(S_t)_{t\geq 0}$ on a filtered space $\mathbf{\Omega}$ we mean that there is a measure preserving family $(\theta_t)_{t\geq 0}$ obeying $(\theta 1)-(\theta 3)$ from which S_t is defined as above.

Proposition 3.2

(a) *S_r is a semigroup on the class of all stochastic processes $u : [0, \infty) \times \Omega \to$ $\mathbf{H}$.*

(b) *If u is adapted (to the filtration $(\mathcal{F}_t)_{t\geq 0}$) then so is $v = S_r u$.*

(c) *If u is adapted and $v = S_r u$ then for $t \geq 0$:*

(i) For appropriate continuous g,

$$\int_0^t g(v(s, \omega))dw_s(\omega) = \int_r^{r+t} g(u(s, \theta_r\omega))dw_s(\theta_r\omega)$$

(meaning that $I(\omega) = J(\theta_r\omega)$ as random variables, where $I(\omega)$ is the left-hand integral and $J(\omega) = \int_r^{r+t} g(u(s, \omega))dw_s(\omega)$).

(ii) For appropriate continuous f,

$$\int_0^t f(v(s, \omega))ds = \int_r^{r+t} f(u(s, \theta_r\omega))ds.$$

(By appropriate *we mean that the integrals are defined.)*

(d) *If u is adapted and $v = S_r u$ and $t_1 \geq t_0 \geq 0$ with*

$$u(t_1 + r) = u(t_0 + r) + \int_{t_0+r}^{t_1+r} f(u(s))ds + \int_{t_0+r}^{t_1+r} g(u(s))dw_s,$$

then

$$v(t_1) = u(t_0) + \int_{t_0}^{t_1} f(v(s))ds + \int_{t_0}^{t_1} g(v(s))dw_s.$$

Proof Elementary. □

3.2 Attractors for the 3-D stochastic Navier-Stokes equations

Suppose now that a filtered probability space $\mathbf{\Omega}$ and a Wiener process w is given, together with a family of measure preserving maps as above.

A natural space for paths of solutions to the stochastic Navier–Stokes equations is the space M defined as follows.

Definition 3.3 (a) For a measurable (deterministic) function $\xi : [0, \infty) \to \mathbf{H}$ define a norm

$$|\xi| = \left(\int_0^\infty \xi(t)^2 \exp(-t) dt \right)^{\frac{1}{2}} = \left(\int_0^\infty \xi(t)^2 \mu(dt) \right)^{\frac{1}{2}}$$

where $\mu(dt) = \exp(-t)dt$, and write

$$M = \{\xi : |\xi| < \infty\}$$

for this space of paths, which is a separable Hilbert space.

(b) For a process $u(t, \omega)$ with paths in M define

$$|u| = \left(\mathbb{E}(|u(\cdot, \omega)|^2)\right)^{\frac{1}{2}} = \left(\mathbb{E} \int_0^\infty |u(t, \omega)|^2 \exp(-t) dt \right)^{\frac{1}{2}},$$

which is simply the norm of $L^2(\Omega, M)$. Let ρ be the corresponding metric $\rho(u, v) = |u - v|$.

We will be interested in the laws of solutions viewed as probability distributions on the space of paths M, so we need the following definitions.

Definition 3.4 Let $u(t, \omega)$ be a process with paths in M.

(a) $\mathrm{law}(u)$ is the probability law on M induced by u; i.e.

$$\mathrm{law}(u)(E) = \mathbb{P}(u(\cdot, \omega) \in E)$$

for Borel $E \subseteq M$.

(b) $\mathrm{law}_w(u) = \mathrm{law}(u, w)$, the probability law induced on $M \times C_0$ by the pair of processes $(u(t, \omega), w(t, \omega))$, where $C_0 = C_0[0, \infty)$.

For the space of probability laws $\mathcal{M}_1(S)$ on a separable metric space S a fundamental metric is the Prohorov metric, which we denote by d_0; this makes $\mathcal{M}_1(S)$ separable. Here we are thinking of $S = M$ and $S = M \times C_0$.

There is a natural projection mapping $\pi : \mathcal{M}_1(M \times C_0) \to \mathcal{M}_1(M)$ defined by

$$\pi(\lambda)(E) = \lambda(E \times C_0).$$

In the current situation the laws on the space M that we are interested are laws of L^2 random variables, so it is appropriate to define a stronger metric to reflect this.

Definition 3.5 (a) $\mathcal{M}_{1,2}(M) = \{\mu \in \mathcal{M}_1(M) : \mathbb{E}_\mu(|u|^2) < \infty\}$.

(b) the metric d on $\mathcal{M}_{1,2}(M)$ is defined by

$$d(\mu_1, \mu_2) = d_0(\mu_1, \mu_2) + \left|\mathbb{E}_{\mu_1}(|u|^2) - \mathbb{E}_{\mu_2}(|u|^2)\right|.$$

(c) $\mathcal{M}_{1,2}(M \times C_0) = \{\lambda \in \mathcal{M}_1(M \times C_0) : \pi(\lambda) \in \mathcal{M}_{1,2}(M)\}$.

(d) the metric d on $\mathcal{M}_{1,2}(M \times C_0)$ is defined by

$$d(\lambda_1, \lambda_2) = d_0(\lambda_1, \lambda_2) + \left|\mathbb{E}_{\mu_1}(|u|^2) - \mathbb{E}_{\mu_2}(|u|^2)\right|,$$

where $\mu_i = \pi(\lambda_i)$ $(i = 1, 2)$.

The following lemma is easily checked.

Lemma 3.6 (a) *The function* law *maps* $L^2(\Omega, M)$ *into* $\mathcal{M}_{1,2}(M)$ *and is continuous with respect to the metrics* ρ *and* d.

(b) *The function* law_w *maps* $L^2(\Omega, M)$ *into* $\mathcal{M}_{1,2}(M \times C_0)$ *and is continuous with respect to the metrics* ρ *and* d.

(c) *The mapping* $\pi : \mathcal{M}_1(M \times C_0) \to \mathcal{M}_1(M)$ *defined by*

$$\pi(\lambda)(E) = \lambda(E \times C_0)$$

is continuous with respect to the metric d.

Suppose now that $X \subseteq L^2(\Omega, M)$ is a class of solutions to the stochastic Navier–Stokes equations on $\mathbf{\Omega}$ for the given Wiener process w. Then each $u \in X$ is a stochastic process such that $|u|$ as defined above is finite. By a *bounded subset of* X we will mean a subset of X which is bounded in the norm $|\cdot|$.

Let us assume further that $S_t X \subseteq X$ for all $t \geq 0$, that is, S_t is a semigroup on X. A semigroup $\widehat{S}_t : \mathcal{M}_{1,2}(M \times C_0) \to \mathcal{M}_{1,2}(M \times C_0)$ is induced in a natural way: in detail

$$\widehat{S}_t(\lambda) = \lambda \circ h_t^{-1},$$

where $h_t : M \times C_0 \to M \times C_0$ is given by

$$h_t(\xi, w)(s) = (\xi(t+s), w(t+s) - w(t)).$$

For future reference note that

$$\widehat{S}_t \circ \mathrm{law}_w = \mathrm{law}_w \circ S_t. \tag{4}$$

This allows the following definition of an *attractor* for a semiflow S_t on X.

Definition 3.7 (a) A set of laws $\mathcal{A} \subseteq \mathrm{law}_w(X)$ is a *law-attractor* for the semiflow S_t on X if:

(i) **(Invariance)** $\widehat{S}_t\mathcal{A} = \mathcal{A}$ for all $t \geq 0$.

(ii) **(Attraction)** For any open set $\mathcal{O} \supset \mathcal{A}$ and d-bounded set $\mathcal{Z} \subseteq \mathrm{law}_w(X)$,

$$\widehat{S}_t\mathcal{Z} \subseteq \mathcal{O}$$

eventually (i.e. for some $t_0 = t_0(\mathcal{O}, \mathcal{Z})$, this holds for all $t \geq t_0$).

(iii) **(Compactness)** $\mathcal{A}$ is compact in the metric d.

(b) An *attractor* for the semiflow S_t on X is a set of processes $A \subseteq X$ such that:

(i) $\mathrm{law}_w(A)$ is a law-attractor (in particular $\mathrm{law}_w(A)$ is compact in the metric d, and so A is bounded).

(ii) **(Invariance)** $S_tA = A$ for all $t \geq 0$.

(iii) **(Attraction)** For any bounded set $Z \subseteq X$ and compact set $K \subseteq L^2(\Omega, M)$,

$$\underline{\lim}_{t\to\infty}\rho(S_tZ, K) \geq \rho(A, K).$$

(iv) A is closed in the space $L^2(\Omega, M)$.

Remarks on Definition 3.7.

1. Since existence results for the stochastic Navier–Stokes equations require a rather large probability space, it is to be expected that any space carrying a whole class of solutions X as above will be too big to allow an attractor $A \subseteq X$ that is compact in the usual sense. In a later section we will give evidence to support this remark. We will see, however, that on a suitable space there is an attractor A in the above sense. In the sequel [18] to this paper we improve this result, and show that the attractor to be constructed in this paper is *neocompact* in the sense of [20]. Neocompact sets share many properties with compact sets. Among other things, they are closed and have compact laws.

2. The attraction property 3.7(b)(iii) is equivalent to the following:

$$S_tZ \subseteq \mathcal{O} \text{ eventually} \tag{5}$$

for any bounded Z and any open $\mathcal{O} \supset A$ of the form $\mathcal{O} = L^2(\Omega, M) \setminus K^{\leq\varepsilon}$, with K compact. Property 3.7(b)(i) means that in addition (5) holds for any open set $\mathcal{O}$ of the form $\mathcal{O} = \mathrm{law}_w^{-1}(\mathcal{O}')$ where $\mathcal{O}'$ is an open set of laws with $\mathrm{law}_w(A) \subseteq \mathcal{O}' \subseteq \mathcal{M}_{1,2}(M \times C_0)$.

3. The usual attraction property for attractors, namely that property (5) holds for any bounded Z and *any* open $\mathcal{O} \supset A$, is probably too much to expect. However, in the sequel [18] we will show that the attractor to be constructed in this paper has property (5) for a smaller class of open sets – those that are *neoopen* in the sense of [20]. Sets $\mathcal{O}$ of the form $L^2(\Omega, M) \setminus K^{\leq\varepsilon}$ or $\mathrm{law}_w^{-1}(\mathcal{O}')$ as above are neoopen.

We can now state the main theorem of this paper.

Theorem 3.8 *There is a filtered probability space* $\mathbf{\Omega}$*, a class* X *of adapted weak solutions to the stochastic Navier-Stokes equation in* $\mathbf{\Omega}$*, and a semiflow* S_t *on* X*, such that*

(a) *there exists* $u \in X$ *for all* L^2 $\mathcal{F}_0$*-measurable initial conditions;*

(b) *there is an attractor for the semiflow* S_t *on* X*.*

Before embarking on the proof of this result we note some further properties of an attractor (if it exists) that can be deduced immediately.

Theorem 3.9 *Given a semiflow* S_t *on a set* $X \subseteq L^2(\Omega, M)$*, there is at most one attractor* A *for* S_t *on* X*.* A *has the following properties:*

(a) $\mathrm{law}(A)$ *is compact in the metric* d*.*

(b) *For any open set of laws* $\mathcal{O} \supset \mathrm{law}(A)$*, and bounded* $Z \subseteq X$*,* $S_t Z \subseteq \mathrm{law}^{-1}(\mathcal{O})$ *eventually.*

(c) $A = \bigcap_{n\in\mathbb{N}} S_n(Z) = \bigcap_{t\geq 0} S_t(Z)$ *for any bounded set* Z *with* $A \subseteq Z \subseteq X$*.*

In particular

(d) $A = \bigcap_{n\in\mathbb{N}} S_n(\mathrm{law}^{-1}(\mathrm{law}(A)) \cap X)$*.*

Remark The significance of (d) is that although A may not be compact, it may be defined from a compact set of laws.

Proof For uniqueness, suppose that A and A' are attractors with $u \in A' \setminus A$. Since A is closed, $\rho(A, u) = \varepsilon > 0$. Since $A' \subseteq X$ is bounded we have $\underline{\lim}_{t\to\infty} \rho(S_t A', u) \geq \varepsilon$, and hence $\rho(S_t A', v) > \varepsilon/2$ for some t, contradicting $v \in A' = S_t A'$.

(a) and (b) follow immediately from the continuity of π.

For (c), it is clear that A is contained in the right-hand side, since $A = S_t A \subseteq S_t Z$ for all $t \geq 0$. For the other inclusion, suppose that $u \in \bigcap_{n\in\mathbb{N}} S_n(Z) \setminus A$. Since A is closed, $\varepsilon = \rho(A, u) > 0$. Since Z is bounded the attraction property gives $\rho(S_n Z, v) > \varepsilon/2$ for some n, contradicting $v \in S_n Z$.

For (d), apply (c) to $Z = \mathrm{law}^{-1}(\mathrm{law}(A)) \cap X$. Clearly $A \subseteq Z \subseteq X$ and Z has the same bound as A. □

4 Solutions to the stochastic Navier–Stokes equations

We define below a special class X of weak solutions to the stochastic Navier–Stokes equations (3). Each element u of X is thus an adapted stochastic process (with $u(t,\omega) \in \mathbf{H}$ for all t,ω). The properties required for membership of X are among those that can be deduced heuristically from (3) using elementary stochastic calculus. (Of course we will later show rigorously that X is non empty!)

Definition 4.1 (a) Given positive real constants $k_1, k_2, k_3, \alpha, \beta$, denote by X the class of adapted stochastic processes $u : (0,\infty) \times \Omega \to \mathbf{H}$ with the following properties.

(X1) For a.a. ω the path $u(\cdot,\omega)$ belongs to the following spaces:

$$L^\infty_{\text{loc}}(0,\infty;\mathbf{H}) \cap L^2_{\text{loc}}[0,\infty;\mathbf{H}) \cap L^2_{\text{loc}}(0,\infty;\mathbf{V}) \cap C(0,\infty;\mathbf{H}_{\text{weak}}).$$

(X2) For all $t_1 \geq t_0 > 0$

$$u(t_1) = u(t_0) + \int_{t_0}^{t_1} [-\nu Au(t) - B(u(t)) + f(u(t))]dt + \int_{t_0}^{t_1} g(u(t))dw_t.$$

(X3) For a.a. $t_0 > 0$ and all $t_1 \geq t_0$,

$$\mathbb{E}(|u(t_1)|^2) \leq \mathbb{E}(|u(t_0)|^2)\exp(-k_1(t_1 - t_0)) + k_2. \tag{6}$$

(X4) For a.a. $t_0 > 0$ and all $t_1 \geq t_0$,

$$\mathbb{E}\left(\sup_{t_0 \leq s \leq t_1} |u(s)|^2 + \int_{t_0}^{t_1} \|u(s)\|^2 ds\right) \leq \alpha\mathbb{E}(|u(t_0)|^2) + \beta(t_1 - t_0). \tag{7}$$

(X5) For a.a. $t_0 > 0$ and all $t_1 \geq t_0$ and $n \in \mathbb{N}$,

$$\mathbb{E}(\varphi_n(u(t_1))) \leq \mathbb{E}(\varphi_n(u(t_0))\exp(-k_3(t_1 - t_0))) + n^{-\frac{1}{2}}(\alpha\mathbb{E}(|u(t_0)|^2) + \beta). \tag{8}$$

(X6) $\mathbb{E}\int_0^1 |u(t)|^2 dt < \infty$.

(b) Denote by X_k the set of $u \in X$ with

($\mathrm{X6}_k$) $\mathbb{E}\int_0^1 |u(t)|^2 dt \leq k$.

Remarks

1. The class X depends on the constants $k_1, k_2, k_3, \alpha, \beta$. We will show later that there is an explicit choice of constants for which X is non-empty.

2. The sets X_k obviously increase with k.

3. The above conditions tell us nothing about $u(t,\omega)$ at $t = 0$ and there may be a singularity there. In this sense the class X is a class of *generalized weak solutions* to the stochastic Navier–Stokes equations (cf. [32], p.12).

4. The meaning of "loc" in the path properties (X1) is as follows: $L^p_{\text{loc}}(0,\infty)$ means $L^p[1/n, n]$ for all n, whereas $L^p_{\text{loc}}[0,\infty)$ means $L^p[0,n]$ for all n.

5. The conditions (X5) follow naturally from the Foias equation for the stochastic Navier-Stokes equations (see [5]), which may be deduced heuristically from the equation (3). The choice of the functions φ_n makes (X5) a uniform integrability condition for $|u(t)|^2$ on any $[t_0, \infty)$.

The following lemma is our motivation for singling out the sets X_k; it relates these sets to bounded sets.

Lemma 4.2 (a) $X \subseteq M$.

(b) *If $u \in X$ and $|u|^2 e \leq k$ then $u \in X_k$.*

(c) *If $u \in X_k$ then $|u|^2 e \leq k(1+e) + k_2$.*

(d) *If $Z \subseteq X$, then Z is bounded if and only if $Z \subseteq X_k$ for some $k \in \mathbb{N}$.*

Proof (a) Let $u \in X$. It follows from (X6) that $\mathbb{E}(|u(t))|^2) < \infty$ for a.a. $t \in (0,1)$. Thus, from (X3) we see that $\mathbb{E}(|u(t))|^2)$ is bounded on $[\frac{1}{n}, \infty)$ for all $n \in \mathbb{N}$. It follows from $u \in L^2_{\text{loc}}[0,\infty)$ that $\mathbb{E}(|u|^2) < \infty$, so $u \in M$.

(b) For $u \in M$ we have

$$\mathbb{E}\int_0^1 |u(t)|^2 dt \leq e\mathbb{E}\int_0^1 |u(t)|^2 \exp(-t) dt \leq e|u|^2.$$

So if $|u|^2 e \leq k$ then $\mathbb{E}\int_0^1 |u(t)|^2 dt \leq k$.

(c) Let $u \in X_k$. Then the set $\{t \in [0,1] : \mathbb{E}(|u(t)|^2) \leq k\}$ has positive Lebesgue measure. By (X3) there is a $t_0 \in (0,1]$ such that (6) holds for all $t_1 \geq t_0$ and $\mathbb{E}(|u(t)|^2) \leq k$. Then for all $t_1 \geq t_0$, $\mathbb{E}(|u(t_1)|^2) \leq k + k_2$. Thus

$$|u|^2 = \int_0^1 \mathbb{E}|u(t)|^2 exp(-t) dt + \int_1^\infty \mathbb{E}|u(t)|^2 exp(-t) dt$$

$$\leq k + \int_1^\infty (k_2 + k) exp(-t) dt = k + (k_2 + k)/e,$$

and (c) follows.

(d) follows easily from (b) and (c). □

We can now reformulate the main theorem for the particular class of solutions X that has just been introduced.

Theorem 4.3 *There is a filtered probability space* $\mathbf{\Omega}$ *with a semiflow* S_t *on* X*, and constants* k_1*,* k_2*,* k_3*,* α*,* β*, such that there exist* $u \in X$ *for all* L^2 $\mathcal{F}_0$*-measurable initial conditions, and there is an attractor for the semiflow* S_t *on* X*.*

In fact, we will show that there exists weak solutions in the following smaller class Y whose members are defined at 0.

Definition 4.4 Denote by Y the class of stochastic processes $u : [0, \infty) \times \Omega \to \mathbf{H}$ with $u \in X$ (that is, the restriction of u to $(0, \infty)$ lies in X) with the following additional properties:

(Y1) For a.a. ω, the path $u(\cdot, \omega)$ is in

$$L^\infty_{\text{loc}}[0, \infty; \mathbf{H}) \cap L^2_{\text{loc}}[0, \infty; \mathbf{H}) \cap L^2_{\text{loc}}[0, \infty; \mathbf{V}) \cap C[0, \infty; \mathbf{H}_{\text{weak}}).$$

(Y2) For all $t_1 \geq t_0 \geq 0$,

$$u(t_1) = u(t_0) + \int_{t_0}^{t_1} [-\nu A u(t) - B(u(t)) + f(u(t))]dt + g(u(t))dw_t.$$

(Y3) $\mathbb{E}(|u(t)|^2)$ is bounded on $[0, \infty)$.

Note that (Y1) implies (X1), (Y2) implies (X2), and (Y3) implies (X6).

5 The space Ω and the semiflow

The particular space $\mathbf{\Omega}$ that we use is a filtered Loeb space similar to that used in [7] for the construction of solutions to the stochastic Navier–Stokes equations. Loeb spaces constitute a special class of probability spaces that are very rich – in a sense that can be made precise (see for example [28]). The richness is needed to be able to solve the general stochastic Navier–Stokes equations in dimension 3, and it will also come into play when showing that the single space $\mathbf{\Omega}$ has solutions to (3) with the same (prescribed) Wiener process w_t for any random initial condition.

From this point on we assume the basics of nonstandard analysis and in particular the Loeb construction. Some details are provided in the appendix; for a full exposition see any of [1, 7, 14, 16, 17, 29, 30].

We set $\Omega = {}^*(C_0(\mathbb{R}))$, the internal space of *continuous functions $\omega : {}^*\mathbb{R} \to {}^*\mathbb{R}$ with $\omega(0) = 0$, and let Q be the internal *Wiener measure on Ω. Thus the canonical process

$$W(\tau, \omega) = \omega(\tau)$$

is a two-sided *Wiener process under Q. This gives the internal filtered probability space

$$\bar{\mathbf{\Omega}} = (\Omega, \mathcal{G}, (\mathcal{G}_\tau)_{\tau \in {}^*\mathbb{R}}, Q),$$

where $\mathcal{G}_\tau = {}^*\sigma(\{\mathcal{W}(\tau') : \tau' \leq \tau\})$ and $\mathcal{G} = \bigvee_{\tau \in {}^*\mathbb{R}} \mathcal{G}_\tau$.

A family of internal measure preserving maps $\Theta_\tau : \Omega \to \Omega$ is defined for $\tau \in {}^*\mathbb{R}$ by

$$(\Theta_\tau(\omega))(\sigma) = \omega(\sigma - \tau) - \omega(-\tau).$$

That is, Θ_τ is a shift of the path ω to the right by τ and then adjusted to be 0 at 0.

Now let $P = Q_{\mathrm{L}}$ be the Loeb measure obtained from Q with the corresponding Loeb σ-algebra $\mathcal{F} = \mathrm{L}(\mathcal{G})$, giving the Loeb probability space $(\Omega, \mathrm{L}(\mathcal{G}), \mathrm{Q_L}) = (\Omega, \mathcal{F}, \mathrm{P})$, and denote the P-null sets by $\mathcal{N}$.

Definition 5.1

(a) The filtered probability space $\mathbf{\Omega}$ is

$$\mathbf{\Omega} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P),$$

where the right continuous filtration $(\mathcal{F}_t)_{t \geq 0}$ is defined by

$$\mathcal{F}_t = \bigcap_{t < {}^\circ\tau} \sigma(\mathcal{G}_\tau) \vee \mathcal{N}.$$

(b) The Wiener process $w(t, \omega)$ on $\mathbf{\Omega}$ is defined by

$$w(t, \omega) = {}^\circ W(t, \omega). \tag{9}$$

(c) The family of measure preserving transformations $(\theta_t)_{t \geq 0}$ is given by

$$\theta_t = \Theta_t.$$

That is, the restriction of the family (Θ_τ) to non-negative standard times.

It is well known that (9) defines an almost surely continuous Wiener process on $\mathbf{\Omega}$. It is clear that the family θ_t satisfies conditions $(\theta 1), (\theta 2), (\theta 3)$.

With the space $\mathbf{\Omega}$ and the family $(\theta_t)_{t \geq 0}$ now fixed, the semiflow of processes S_t defined by Definition 3.1 is also fixed for the rest of the paper.

6 Construction of solutions

From now on all discussion is in the context of the fixed space $\mathbf{\Omega}$ of the previous section, so that in particular the classes of processes X and Y are classes of processes on $\mathbf{\Omega}$. It is clear from Proposition 3.2 that both classes of solutions X and Y are closed under S_t for all t. In fact $S_tX \subseteq Y$ for $t > 0$.

This section is devoted to a proof that, for an explicit choice of the constants, the class Y (and hence X) is non-empty, and in particular contains solutions with any prescribed $\mathcal{F}_0$-measurable L^2 initial condition. (In this connection, note that the richness of the space $\mathbf{\Omega}$ means that for any probability law on $\mathbf{H}$ there is an $\mathcal{F}_0$-measurable $u_0(\omega)$ on $\mathbf{\Omega}$ with the same law.) In the course of the proof the constants in the definition of the class X (Definition 4.1) are given explicitly.

Theorem 6.1 *There are constants $k_1, k_2, k_3, \alpha, \beta$ such that for any $\mathcal{F}_0$-measurable $u_0(\omega) \in L^2(\Omega, \mathbf{H})$ there is a process $u(t,\omega) \in Y$ with $u(0,\omega) = u_0(\omega)$.*

Proof

As in [7] a solution $u \in Y$ is constructed as the standard part of an internal Galerkin approximate solution on the internal space $\bar{\mathbf{\Omega}}$.

Suppose that $u_0 \in L^2(\Omega, \mathbf{H})$ is $\mathcal{F}_0-$measurable. Fix an infinite natural number N and take an SL^2 lifting $U_0(\omega) \in \mathbf{H_N}$ of u_0 that is $\mathcal{G}_\delta$-measurable for some positive $\delta \approx 0$. Consider the following internal SDE on ${}^*[\delta, \infty)$ in $\mathbf{H_N}$, where B, F, G denote the projections of ${}^*B, {}^*f, {}^*g$ respectively onto $\mathbf{H_N}$. (Here and elsewhere we use interchangeably the notation $U_\tau(\omega) = U(\tau, \omega)$, etc.)

$$\begin{cases} dU_\tau = [-\nu\, {}^*A\, U_\tau - B(U_\tau) + F(U_\tau)]d\tau + G(U_\tau)dW_\tau \\ U_\delta(\omega) = U_0(\omega). \end{cases}$$

Elementary SDE theory and the usual energy considerations give an internal solution U (not necessarily unique) to this Galerkin approximation. Fix any one such solution U. The task now is to derive some properties of U that will show that its standard part $u = {}^\circ U$ belongs to Y as required.

Using Itô's lemma and the fact that $(B(U), U) = 0$ we have

$$d|U_\tau|^2 = (-2\nu\|U_\tau\|^2 + 2(F(U_\tau), U_\tau) + |G(U_\tau)|^2)d\tau + 2(U_\tau, G(U_\tau))dW_\tau. \quad (10)$$

Put $Z_\tau = \mathbb{E}(|U_\tau|^2)$. The growth conditions on f and g, together with an application of Young's inequality give positive constants k_1, l such that

$$\frac{d}{d\tau} Z_\tau + k_1 Z_\tau \leq l. \quad (11)$$

(See the appendix to this section for details.) Hence (using Gronwall's lemma), for all $\tau_0 \leq \tau_1$

$$Z_{\tau_1} \leq Z_{\tau_0} \exp(-k_1(\tau_1 - \tau_0)) + \frac{l}{k_1}(1 - \exp(-k_1(\tau_1 - \tau_0))). \quad (12)$$

So

$$\mathbb{E}(|U_{\tau_1}|^2) \leq \mathbb{E}(|U_{\tau_0}|^2)\exp(-k_1(\tau_1-\tau_0))+k_2, \tag{13}$$

where $k_2 = l/k_1$.

We will see below that this is almost sufficient to obtain the inequality (X3) in the definition of the class X. Turning now to the condition (X4), we go back to (10).

The growth conditions on f and g, together with an application of Young's inequality, give

$$2(F(U),U)+|G(U)|^2 \leq c(1+|U|^2),$$

where $c = c_0^2 + c_0(1+d_1)$.

¿From (10), this gives, for $\delta \leq \tau_0 \leq \tau_1$,

$$\sup_{\tau_0\leq\sigma\leq\tau_1}|U_\sigma|^2 + 2\nu\int_{\tau_0}^{\tau_1}\|U_\sigma\|^2)d\sigma$$

$$\leq |U_{\tau_0}|^2 + c\int_{\tau_0}^{\tau_1}(1+|U_\sigma|^2)d\sigma + \sup_{\tau_0\leq\sigma\leq\tau_1}|M_\sigma|, \tag{14}$$

where M_τ is the martingale

$$M_\tau = 2\int_{\tau_0}^{\tau_1}(U_\sigma, G(U_\sigma))dW_\sigma,$$

and so

$$[M]_\tau = 4\int_{\tau_0}^{\tau_1}(U_\sigma, G(U_\sigma))^2 d\sigma.$$

Now using the bound on g we obtain

$$[M]_\tau \leq 4c' \sup_{\tau_0\leq\sigma\leq\tau}|U_\tau|^2\int_{\tau_0}^{\tau}(1+|U_\sigma|^2)d\sigma, \tag{15}$$

where $c' = c_0 d_2 + \max(c_0, d_2^2)$.

Using Young's inequality again gives

$$[M]_\tau^{1/2} \leq \frac{c'}{\gamma}\sup_{\tau_0\leq\sigma\leq\tau}|U_\tau|^2 + \gamma\int_{\tau_0}^{\tau}(1+|U_\sigma|^2)d\sigma \tag{16}$$

for γ to be specified below. The following Burkholder-Davis-Gundy inequality is now applicable (κ is a universal constant independent of the dimension N):

$$\mathbb{E}\left(\sup_{\tau_0\leq\sigma\leq\tau_1}|M_\sigma|\right) \leq \kappa\mathbb{E}\left([M]_{\tau_1}^{1/2}\right). \tag{17}$$

Use (14) and (16), with $\gamma = 2(\kappa c')^{-1}$, together with the Burkholder-Davis-Gundy inequality (17) to obtain

$$\mathbb{E}\left(\sup_{\tau_0\le\sigma\le\tau_1}|U_\sigma|^2+\int_{\tau_0}^{\tau_1}\|U_\sigma\|^2d\sigma\right)\le c''\mathbb{E}\left(|U_{\tau_0}|^2+\int_{\tau_0}^{\tau_1}(1+|U_\sigma|^2)d\sigma\right), \tag{18}$$

where $c'' = \max(1, c + 2/c')/\min(\frac{1}{2}, 2\nu)$. This, together with (12) gives

$$\mathbb{E}\left(\sup_{\tau_0\le\sigma\le\tau_1}|U_\sigma|^2+\int_{\tau_0}^{\tau_1}\|U_\sigma\|^2d\sigma\right)\le \alpha\mathbb{E}(|U_{\tau_0}|^2)+\beta(\tau_1-\tau_0) \tag{19}$$

for $\delta \le \tau_0 \le \tau_1$, provided that $\alpha \ge c''(1 + k_1^{-1})$, $\beta \ge c''(1 + lk_1^{-1})$.

The construction of the standard process $u(t)$ from U_τ now proceeds as in the main existence result Theorem 6.4.1 of [7] using the inequality (19). This shows that for a.a. ω the function $|U_\tau(\omega)|$ is finite for all finite $\tau \ge \delta$ and U_τ is weakly S-continuous. Thus we can define a standard process $u(t,\omega)$ for a.a. ω by

$$u(t,\omega) = {}^\circ U(\tau,\omega)$$

(the weak standard part) for any finite $\tau \approx t$ with $\tau \ge \delta$.

The inequality (19) also shows that for a.a. ω, for almost all finite τ we have $\|U_\tau(\omega)\|$ finite and so $u = {}^\circ U$ in the strong topology, hence $F(U) \approx f(u)$ and similarly with g. From the theory developed in [7] it follows that $u(t,\omega)$ is a solution; that is, condition (Y2) is satisfied. It is also clear that condition (Y1) holds. From (13) we also have (Y3) since $|{}^\circ U| \le {}^\circ|U|$ always and $\mathbb{E}(|U_\delta|^2) \approx \mathbb{E}(|u_0|^2) < \infty$.

It follows that conditions (X1), (X2), and (X6) hold. It is now necessary to check conditions (X3)–(X5).

We have the internal energy decay inequality (13), but in order to translate this into condition (X3) we need to know that $|U_\tau(\omega)|^2$ is S-integrable for all finite $\tau \ge \delta$. To show this we need to prove an internal analogue of the formula (X5).

Applying Itô's lemma to the process $\varphi_n(U_\tau(\omega))$ for any $n \in {}^*\mathbb{N}$ (of course we mean ${}^*\varphi_n$ as usual in such contexts) gives:

$$\begin{aligned} d\varphi_n(U_\tau) \;=\; & \left[-\nu(\!(U_\tau,\varphi_n'(U_\tau))\!)+(F(U_\tau),\varphi_n'(U_\tau))+\frac{1}{2}G(U_\tau)^T\varphi_n''(U_\tau)G(U_\tau)\right]d\tau \\ & +(G(U_\tau),\varphi_n'(U_\tau))dW_\tau \end{aligned}$$

since $(B(U),\varphi_n'(U)) = 0$.

Now from the explicit form of φ_n' and φ_n'' (see Definition 2.1(f),(g)) and the growth conditions on f, g we have:

$$(\!(U,\varphi_n'(U))\!) \ge \lambda_1|U||\varphi_n'(U)|,$$

$$(F(U),\varphi_n'(U)) \le (c_0 + d_1|U|)|\varphi_n'(U)|,$$

and

$$G^T(U)\varphi_n''(U)G(U) \le 3d_2^2|U||\varphi_n'(U)| + 12c_0(c_0 + 2d_2|U|)\psi_n(U^2).$$

Putting $Y_\tau = \mathbb{E}(\varphi_n(U_\tau))$ and $\tilde{Y}_\tau = \mathbb{E}(|U_\tau||\varphi_n'(U_\tau)|)$, it follows that for all $\delta \le \tau_0 \le \tau$,

$$\frac{dY_\tau}{d\tau} + \nu\lambda_1\tilde{Y}_\tau \le$$

$$(d_1 + \frac{3}{2}d_2^2)\tilde{Y}_\tau + c_0\mathbb{E}(|\varphi_n'(U_\tau)|) + 6c_0\mathbb{E}((c_0 + 2d_2|U_\tau|)\psi_n(U_\tau^2)). \tag{20}$$

Hence, using the estimates in Lemma 2.3 together with 2.2(c)

$$\begin{aligned} \frac{dY_\tau}{d\tau} + k_3Y_\tau &\le n^{-\frac{1}{2}}(\beta_0 + \alpha_0 Z_\tau) \\ &\le n^{-\frac{1}{2}}(\beta_1 + \alpha_0 Z_{\tau_0}\exp(-k_1(\tau - \tau_0))) \\ &\le n^{-\frac{1}{2}}(\beta_1 + \alpha_0 Z_{\tau_0}), \end{aligned}$$

where $k_3 = 2\nu\lambda_1 - 2d_1 - 3d_2^2$ and $Z_\tau = \mathbb{E}(|U_\tau|^2)$ as before, and we have used the inequality (13). The constants are given explicitly as follows:

$$\begin{aligned} \alpha_0 &= 2c_0(2 + 3c_0 + 6d_2), \\ \beta_0 &= 3\alpha_0, \\ \beta_1 &= \beta_0 + \alpha_0 k_2. \end{aligned}$$

Finally, using Gronwall's inequality, we get an internal analogue of (X5):

$$\mathbb{E}\varphi_n(U_{\tau_1}) \le \mathbb{E}\varphi_n(U_{\tau_0})\exp[-k_3(\tau_1 - \tau_0) + n^{-\frac{1}{2}}(\alpha\mathbb{E}(|U_{\tau_0}|^2) + \beta)] \tag{21}$$

for all $\tau_1 \ge \tau_0 \ge \delta$, provided that $\beta \ge \beta_1/k_3$ and $\alpha \ge \alpha_0/k_3$.

We can now see that $|U_\tau(\omega)|^2$ is S-integrable for all finite $\tau \ge \delta$. We know that $|U_\delta(\omega)|^2$ is S-integrable (it was chosen thus). The property of the "truncation" function φ_n given by Corollary 13.8 (in the Appendix) means that ${}^\circ\mathbb{E}\varphi_n(U_\delta) \to 0$ as $n \to \infty$. Now use (21) to see that ${}^\circ\mathbb{E}\varphi_n(U_\tau) \to 0$ as $n \to \infty$ for all $\tau \ge \delta$, and so $|U_\tau(\omega)|^2$ is S-integrable for all $\tau \ge \delta$.

We can now proceed to verify conditions (X3)–(X5). First note that applying the internal Fubini theorem to (19) shows that for all finite τ_0 in a Loeb full set T_0, say, we have

$$\mathbb{E}(\|U_{\tau_0}\|^2) < \infty.$$

Thus for any such τ_0 we have $\|U_{\tau_0}\| < \infty$ for a.a. ω and hence ${}^\circ|U_{\tau_0}| = |{}^\circ U_{\tau_0}|$. Consequently, now that we have S-integrability of $|U_\tau(\omega)|^2$ for all $\tau \ge \delta$, Loeb theory gives the following: for all $\tau_0 \in T_0$,

$$\begin{aligned} \mathbb{E}(|U_{\tau_0}|^2) &\approx \mathbb{E}({}^\circ|U_{\tau_0}|^2) \\ &= \mathbb{E}(|{}^\circ U_{\tau_0}|^2) \\ &= \mathbb{E}(|u(t_0)|^2), \end{aligned}$$

where $t_0 = {}^\circ\tau_0$. Loeb theory also gives that for *all* τ and $t_1 = {}^\circ\tau$,

$$\begin{aligned} {}^\circ\mathbb{E}(|U_\tau|^2) &\geq \mathbb{E}({}^\circ|U_\tau|^2) \\ &\geq \mathbb{E}(|{}^\circ U(\tau)|^2) \\ &= \mathbb{E}(|u(t_1)|^2). \end{aligned}$$

Putting this together and using the internal energy decay inequality (13), it follows that for all $t_0 = {}^\circ\tau_0 \in {}^\circ T_0$ and $t_1 = {}^\circ\tau \geq t_0$,

$$\begin{aligned} \mathbb{E}(|u(t_1)|^2) &\leq {}^\circ\mathbb{E}(|U_\tau|^2) \\ &\leq {}^\circ(\mathbb{E}(|U_{\tau_0}|^2)\exp(-k_1(\tau-\tau_0)) + k_2) \\ &= \mathbb{E}(|u(t_0)|^2)\exp(-k_1(t_1-t_0)) + k_2 \end{aligned}$$

which is (X3), since ${}^\circ T_0$ is a full set.

A similar argument shows that (X4) follows from (19). For (X5) note that since $\varphi_n(U) \leq |U|^2$ then $\varphi_n(U_\tau)$ is also S-integrable. Thus, for $t_0 = {}^\circ\tau_0 \in {}^\circ T_0$ we have, using the continuity of ψ_n and the fact that ${}^\circ|U_{\tau_0}(\omega)| = |{}^\circ U_{\tau_0}(\omega)|$ for a.a. ω,

$$\begin{aligned} \mathbb{E}\varphi_n(U_{\tau_0}) &\approx \mathbb{E}{}^\circ\varphi_n(U_{\tau_0}) \\ &= \mathbb{E}\varphi_n({}^\circ U_{\tau_0}) \\ &= \mathbb{E}\varphi_n(u(t_0)). \end{aligned}$$

For all other $t = {}^\circ\tau$ we have

$$\mathbb{E}\varphi_n(u(t)) \leq {}^\circ\mathbb{E}\varphi_n(U_\tau),$$

using $|{}^\circ U(\tau,\omega)| \leq {}^\circ|U_\tau|$ and the continuity of ψ_n. These facts, together with (21), give (X5), and we are done. □

Appendix. Calculation of constants

Starting from (10), use $\|U\|^2 \geq \lambda_1|U|^2$ and the growth conditions on f, g to give

$$\begin{aligned} \frac{d}{d\tau}Z_\tau + 2\nu\lambda_1\mathbb{E}(|U|^2) &\leq \mathbb{E}(2|U|(c_0 + d_1|U|) + (c_0 + d_2|U|)^2) \\ &= \mathbb{E}(c_0^2 + 2c_0(1+d_2)|U| + (2d_1 + d_2^2)|U|^2) \\ &\leq c_0^2 + K/\varepsilon + \mathbb{E}((\varepsilon K + 2d_1 + d_2^2)|U|^2) \end{aligned}$$

for any given $\varepsilon > 0$, where $K = c_0(1+d_2)$. Here we have used Young's inequality $2|U| \leq \varepsilon^{-1} + \varepsilon|U|^2$.

Now let $\delta = 2\nu\lambda_1 - 2d_1 - d_2^2 > 0$ and set $\varepsilon = \delta/2K$. Then, putting $k_1 = \delta/2$ and $l = c_0^2 + K/\varepsilon = c_0^2 + 2K^2/\delta$, we have

$$\frac{d}{d\tau}Z_\tau + k_2\mathbb{E}(|U|^2) \leq l,$$

which is (11). For the constants α, β we simply set

$$\alpha = \max\{c''(1 + k_1^{-1}), \alpha_0/k_3\}$$

and

$$\beta = \max\{c''(1 + lk_1^{-1}), \beta_1/k_3\}$$

to ensure that (19) and (21) hold.

Stipulation 6.2 *From now on we fix a set X of solutions as given by Definition 4.1, corresponding to a given choice of the constants $k_1, k_2, k_3, \alpha, \beta$ such that $X \neq \emptyset$.*

7 Internal approximate solutions

In order to construct an attractor we need to obtain an internal representation of *all* solutions $u \in X$. There may be many more than those that are obtained by means of Theorem 6.1, and to represent these we need the notion of an internal *approximate* solution. The following definition explains this. Here we work with the internal filtered probability space $\bar{\mathbf{\Omega}} = (\Omega, \mathcal{G}, (\mathcal{G}_\tau)_{\tau \geq 0}, Q)$. For any element $V \in \mathbf{H}_N$, we let $V_{(n)} = (V, {}^*e_n)$.

Definition 7.1 (a) For each $k \in \mathbb{N}$ and $n \in {}^*\mathbb{N}$ denote by $\mathcal{X}_{k,n}$ the internal class of *-adapted (with respect to $(\mathcal{G}_\tau)_{\tau \geq 0}$) processes

$$U : {}^*[0, \infty) \times \Omega \to \mathbf{H}_N$$

with the following properties (numbered to match with the corresponding properties in the Definition 4.1 of X_k).

($\mathcal{X}1$) $U_\tau(\omega)$ has paths *a.s. in ${}^*M$ and $U \in {}^*L^2(\Omega, {}^*M)$, i.e.

$$\mathbb{E}\left(\int_0^{^*\infty} |U_\tau(\omega)|^2 \exp(-\tau) d\tau\right) < {}^*\infty.$$

($\mathcal{X}2_n$) With Q-probability $\geq 1 - \frac{1}{n}$ on Ω, for all $\tau_1 \in {}^*[\frac{1}{n}, n]$ and all $m \leq n$,

$$\left| U(\tau_1)_{(m)} - U(1/n)_{(m)} - \int_{\frac{1}{n}}^{\tau_1} [-(\nu A U_\tau)_{(m)} - B(U_\tau)_{(m)} + F(U_\tau)_{(m)}] d\tau - \int_{\frac{1}{n}}^{\tau_1} G(U_\tau)_{(m)} dW_\tau \right| \leq 2^{-n}. \quad (22)$$

($\mathcal{X}3_n$) For all $\tau_0 \in {}^*[0,\infty)$ except for a set of *Lebesgue measure $\frac{1}{n}$, for all $\tau_1 \geq \tau_0$,

$$\mathbb{E}(|U_{\tau_1}|^2) \leq \mathbb{E}(|U_{\tau_0}|^2)\exp(-k_1(\tau_1-\tau_0)) + k_2 + \tfrac{1}{n}. \tag{23}$$

($\mathcal{X}4_n$) For all $\tau_0 \in {}^*[0,\infty)$ except for a set of *Lebesgue measure $\frac{1}{n}$, for all $\tau_1 \geq \tau_0$,

$$\mathbb{E}\left(\sup_{\tau_0\leq\sigma\leq\tau_1}|U_\sigma| + \int_{\tau_0}^{\tau_1}\|U_\sigma\|^2 d\sigma\right) \leq \alpha\mathbb{E}(|U_{\tau_0}|^2) + \beta(\tau_1-\tau_0) + \tfrac{1}{n}. \tag{24}$$

($\mathcal{X}5_n$) For all $\tau_0 \in {}^*[0,\infty)$ except for a set of *Lebesgue measure $\frac{1}{n}$, for all $\tau_1 \geq \tau_0$ and all $m \leq n$,

$$\mathbb{E}(\varphi_m(U_{\tau_1})) \leq \mathbb{E}(\varphi_m(U_{\tau_0}))\exp(-k_3(\tau_1-\tau_0)) + m^{-\frac{1}{2}}(\alpha\mathbb{E}(|U_{\tau_0}|)^2+\beta) + \tfrac{1}{n}. \tag{25}$$

($\mathcal{X}6_{k,n}$) $$\mathbb{E}\int_0^1 |U_\tau|^2 d\tau \leq k + \frac{1}{n}.$$

(b) For each $k \in \mathbb{N}$, define

$$\mathcal{X}_k = \bigcap_{n\in\mathbb{N}} \mathcal{X}_{k,n}.$$

(c) Define $\mathcal{X} = \bigcup_{k\in\mathbb{N}} \mathcal{X}_k$.

Remark 7.2 1. The sets $\mathcal{X}_k$ and $\mathcal{X}_{k,n}$ obviously increase with k.

2. The choice of the bound 2^{-n} on the right of ($\mathcal{X}2_n$) ensures that ($\mathcal{X}2_{n+1}$) $\Rightarrow$ ($\mathcal{X}2_n$) for all n. Hence the sets $\mathcal{X}_{k,n}$ decrease with n.

3. Each of the sets $\mathcal{X}_{k,n}$ is internal.

Note that although $\mathcal{X}_{k,n}$ is defined for all $n \in {}^*\mathbb{N}$, the sets X_k only involve $\mathcal{X}_{k,n}$ for finite n. However, it important to note that for each fixed k the whole family $(\mathcal{X}_{k,n})_{n\in{}^*\mathbb{N}}$ is defined and is internal, and we will make use of certain $\mathcal{X}_{k,J}$ for infinite $J \in {}^*\mathbb{N}$.

To help explain the next definition note that if $U \in \mathcal{X}$ then for a.a. ω, $|U(\tau,\omega)| < \infty$, and so $U(\tau,\omega)$ is weakly nearstandard, for all finite $\tau \not\approx 0$ (see Lemma 13.3(a) in the Appendix); the weak standard part has coordinates ${}^\circ(U(\tau,\omega)_{(m)})$ for $m \in \mathbb{N}$.

Definition 7.3 Given $U \in \mathcal{X}$, a *weak standard part of* U is a process

$$u : (0,\infty)\times\Omega \to \mathbf{H}$$

such that for a.a. ω, whenever $t \in (0,\infty)$ and ${}^\circ\tau = t$,

$$u(t,\omega)_{(m)} = {}^\circ(U(\tau,\omega)_{(m)})$$

for each $m \in \mathbb{N}$. If U has a weak standard part, it is a.s. unique and is denoted by ${}^\circ U$.

For a set $Z \subseteq \mathcal{X}$, ${}^\circ Z = \{{}^\circ U : U \in Z \text{ and } {}^\circ U \text{ exists}\}$.

Proposition 7.4 *Let $U \in \mathcal{X}$.*

(a) *U has a weak standard part if and only if for each $m \in \mathbb{N}$, for P-almost all ω, $U(\cdot,\omega)_{(m)}$ is S-continuous on $(0,\infty)$, that is, whenever $\sigma \approx \tau$ and ${}^\circ\tau \in (0,\infty)$, $U(\sigma,\omega)_{(m)} \approx U(\tau,\omega)_{(m)}$ and $U(\tau,\omega)_{(m)}$ is finite.*

(b) *If the weak standard part $u = {}^\circ U$ exists, then for P-almost all ω, $u(\cdot,\omega)$ is weakly continuous on $(0,\infty)$, that is, for each $m \in \mathbb{N}$, $u(\cdot,\omega)_{(m)}$ is continuous on $(0,\infty)$.*

Proof This follows from Theorem 2.5 in [26], which is the analogous result for processes $V : [0,1] \times \Omega \to \mathbb{R}$. For (b) note that weak continuity for a norm bounded function $v(t)$ with values in $\mathbf{H}$ is equivalent to continuity of the coordinate functions $v(t)_{(m)}$.

□

Definition 7.5 We denote by $SL^2[a,b]$ the set of internal processes

$$U(\tau,\omega) : {}^*[0,\infty) \to \mathbf{H}_N$$

such that the restriction of $|U(\tau,\omega)|^2$ to ${}^*[a,b]\times\Omega$ is S-integrable with respect to the product measure on ${}^*[a,b] \times \Omega$.

The importance of $\mathcal{X}$ lies in the following result.

Theorem 7.6 (a) *For each $k \in \mathbb{N}$, if $U \in \mathcal{X}_k \cap SL^2[0,1]$ then the weak standard part $u = {}^\circ U$ exists and is in X_k.*

(b)

$$^\circ(\mathcal{X}_k \cap SL^2[0,1]) = X_k,$$

and hence

$$^\circ(\mathcal{X} \cap SL^2[0,1]) = X.$$

We will prove this in two halves. First we have:

Proof of Theorem 7.6(a) Let $U \in \mathcal{X}_k$. By overspill, $U \in \mathcal{X}_{k,J}$ for some infinite J. From $(\mathcal{X}6_{k,J})$ there is a Loeb-full subset of $\tau_0 \le 1$ such that $\mathbb{E}(|U_{\tau_0}|^2) < \infty$, so by (23), $\mathbb{E}(|U_{\tau_0}|^2) < \infty$ for all $\tau_0 \not\approx 0$. By $(\mathcal{X}4_J)$, there is a Loeb-full set of finite τ_0 such that (24) holds for all finite τ_1. Taking arbitrarily small such τ_0 gives

$$\mathbb{E}\left(\sup_{\frac{1}{n}\le\sigma\le\tau_1}|U_\sigma|^2 + \int_{\frac{1}{n}}^{\tau_1} \|U_\sigma\|^2 d\sigma\right) < \infty \tag{26}$$

for all $n \in \mathbb{N}$ and all finite $\tau_1 \ge \frac{1}{n}$. So $|U_\tau(\omega)|$ is finite for all finite $\tau \not\approx 0$, for a.a. ω.

Let $m \in \mathbb{N}$. Fixing n for the moment and putting

$$V(\tau)_{(m)} = \int_{\frac{1}{n}}^{\tau} [-(\nu A U_\sigma)_{(m)} - B(U_\sigma)_{(m)} + F(U_\sigma)_{(m)}]d\sigma - \int_{\frac{1}{n}}^{\tau} G(U_\sigma)_{(m)} dW_\sigma,$$

the theory of [7] shows that for a.a. ω the process $V(\tau)_{(m)}$ is S-continuous on $(0,\infty)$. Then $(\mathcal{X}2_J)$ gives that for a.a. ω the process $U(\tau)_{(m)}$ is S-continuous on $[\frac{1}{n},\infty)$. Since this holds for all n we have established the existence and almost sure continuity of the weak standard part $u = {}^\circ U$.

We must now verify the properties of u that are needed to place it in X_k. First we have from $(\mathcal{X}2_J)$ that for $m \in \mathbb{N}$,

$$u(t,\omega)_{(m)} = u(\tfrac{1}{n},\omega)_{(m)} + {}^\circ V(t,\omega)_{(m)}.$$

The book [7] shows that

$${}^\circ V(t)_{(m)} = \left(\int_{\frac{1}{n}}^{t} [-\nu A u(t) - B(u(t)) + f(u(t))]dt + \int_{\frac{1}{n}}^{t} g(u(t))dw_t\right)_{(m)}$$

almost surely, and this is sufficient to establish (X2).

The inequalities (X3)–(X5) follow from $(\mathcal{X}3_J) - (\mathcal{X}5_J)$ provided that, as in the proof of Theorem 6.1, it can be shown that $|U_\tau(\omega)|^2$ is S-integrable for all finite $\tau \not\approx 0$. The condition that $U \in SL^2[0,1]$ ensures that $|U_\tau(\omega)|^2$ is S-integrable for a.a. $\tau \in {}^*[0,1]$. The formula (21) which was established in the proof of Theorem 6.1 is the same as condition $(\mathcal{X}5_J)$. As in the proof of Theorem 6.1, this condition together with Corollary 13.8 imply that $|U_\tau(\omega)|^2$ is S-integrable for all $\tau \not\approx 0$. Then (X3)–(X5) follow routinely from $(\mathcal{X}3_J) - (\mathcal{X}5_J)$, using $(\mathcal{X}4_J)$ to give that for almost all $\tau_0 \not\approx 0$ we have $\mathbb{E}(\|U(\tau_0)\|^2) < \infty$. Hence, putting $t_0 = {}^\circ\tau_0$,

$$\mathbb{E}|u(t_0)|^2 = \mathbb{E}^\circ|U(\tau_0)|^2 = {}^\circ\mathbb{E}|U(\tau_0)|^2,$$

and for all $t = {}^\circ\tau$ we have

$$\mathbb{E}|u(t)|^2 \le \mathbb{E}^\circ|U_\tau|^2 \le {}^\circ\mathbb{E}|U(\tau)|^2.$$

We have

$$\mathbb{E}\int_0^1 |u(t)|^2 dt \le \mathbb{E}\int_0^1 {}^\circ|U_\tau|^2 d_L\tau = \mathbb{E}\int_{\frac{1}{J}}^1 {}^\circ|U_\tau|^2 d_L\tau \le {}^\circ\mathbb{E}\int_0^1 |U_\tau|^2 d\tau \le k,$$

using basic Loeb theory and $(\mathcal{X}6_{k,J})$, so that $(X6_k)$ holds.

For the path properties (X1), we have $u(\cdot,\omega) \in L^\infty_{\mathrm{loc}}(0,\infty;\mathbf{H}) \cap L^2_{\mathrm{loc}}[0,\infty;\mathbf{H})$ from $(\mathcal{X}3_J)$ combined with $\mathbb{E}\int_0^1 |u(t)|^2 dt \le k$. The weak continuity of $u(\cdot,\omega)$ has already been established. Finally, the condition $u(\cdot,\omega) \in L^2_{\mathrm{loc}}(0,\infty;\mathbf{V})$ is immediate from (26). □

Implicit in the proof of Theorem 7.6 (a) is the following.

Theorem 7.7

$$\mathcal{X} \cap SL^2[0,1] = \mathcal{X} \cap \mathrm{NS},$$

where $\mathrm{NS} = \mathrm{ns}^2(\Omega, M)$ *as defined in the Appendix. Hence for* $U \in \mathcal{X} \cap SL^2[0,1]$ *the weak standard part* ${}^\circ U$ *as defined in Definition 7.3 is the same as the standard part* ${}^\circ U$ *of* $U \in \mathrm{NS}$ *in* $L^2(\Omega, M)$ *as defined in the Appendix.*

Proof Take $U = U_\tau(\omega) \in \mathcal{X} \cap SL^2[0,1]$ and let $u = {}^\circ U$ be its weak standard part. Regarded as random functions we have $U \in {}^*L^2(\Omega, {}^*M)$ and $u \in L^2(\Omega, M)$. To see that $U \in \mathrm{NS}$ we first show that ${}^\circ U(\omega) = u(\omega)$ in M for a.a. ω.

For a.a. ω, the proof of Theorem 7.6 and basic Loeb theory gives for $V = U(\omega)$ and $v = u(\omega)$:

(a) for a.a. finite τ, $V(\tau) \approx v({}^\circ\tau)$ and (by Anderson's Lusin Theorem) ${}^*v(\tau) \approx v({}^\circ\tau)$ (both strongly in $\mathbf{H}$), so ${}^\circ|V(\tau) - {}^*v(\tau)| = 0$.

(b) $|V(\tau)| \in SL^2[0,1]$ and is bounded for $\tau \geq 1$, so $|V(\tau)| \in SL^2[0,n]$ for each $n \in \mathbb{N}$.

(c) $|{}^*v(\tau)| \in SL^2[0,n]$ for each $n \in \mathbb{N}$.

Hence, for each $n \in \mathbb{N}$,

$$\int_0^n |V(\tau) - {}^*v(\tau)|^2 \exp(-\tau) d\tau \approx \int_0^n {}^\circ|V(\tau) - {}^*v(\tau)|^2 \exp(-\tau) d_L\tau = 0.$$

To get $V \approx v$ in M it is now sufficient to show that

$${}^\circ\!\left(\int_n^{{}^*\infty} |V(\tau) - {}^*v(\tau)|^2 \exp(-\tau) d\tau\right) \;\rightarrow\; 0 \text{ as } n \rightarrow \infty.$$

We have $\int_n^\infty |{}^*v(\tau)|^2 \exp(-\tau) d\tau = \int_n^\infty |v(t)|^2 \exp(-t) dt \;\rightarrow\; 0$ as $n \rightarrow \infty$. It is enough then to show that for a.a. ω, $I_n = {}^\circ(\int_n^{{}^*\infty} |V(\tau)|^2 \exp(-\tau) d\tau) \;\rightarrow 0$. Now from ($\mathcal{X}3$) and ($\mathcal{X}6$) we deduce that there is $k \in \mathbb{N}$ such that

$$\mathbb{E} I_n \leq k \exp(-n),$$

and so $P(I_n \geq \exp(-\frac{n}{2})) \leq k \exp(-\frac{n}{2})$ by Chebychev. Borel-Cantelli gives that $P(I_n \geq \exp(-\frac{n}{2})$ i.o.$\} = 0$ so that $I_n \rightarrow 0$ a.s. as required.

To obtain $U \in \mathrm{NS}$ it remains to show that $|U(\omega)|^2$ is S-integrable; that is

$$\mathbb{E}|U(\omega)|^2 \approx \mathbb{E}({}^\circ|U(\omega)|^2) = \mathbb{E}|u(\omega)|^2.$$

Now

$$\begin{aligned}\mathbb{E}(|U(\omega)|^2) &= \mathbb{E}\int_0^{{}^*\infty} |U_\tau(\omega)|^2 \exp(-\tau) d\tau \\ &= \mathbb{E}\int_0^n |U_\tau(\omega)|^2 \exp(-\tau) d\tau + \mathbb{E}\int_n^{{}^*\infty} |U_\tau(\omega)|^2 \exp(-\tau) d\tau\end{aligned}$$

for each $n \in \mathbb{N}$, and the second term becomes infinitesimal as $n \rightarrow \infty$. The same is true for $\mathbb{E}(|u(\omega)|^2)$, so it suffices to show that for each $n \in \mathbb{N}$,

$$\mathbb{E}\int_0^n |U_\tau(\omega)|^2 \exp(-\tau) d\tau \approx \mathbb{E}\int_0^n |u(t,\omega)|^2 \exp(-t) dt.$$

Since $|U_\tau(\omega)|^2$ is assumed to be S-integrable on $[0,1]\times\Omega$, we have

$$\mathbb{E}\int_0^1 |U_\tau(\omega)|^2\exp(-\tau)d\tau \approx \mathbb{E}\int_0^1 |u(t,\omega)|^2\exp(-t)dt.$$

Now $\mathbb{E}|U_\tau(\omega)|^2$ is bounded for $\tau\geq 1$ and, from the proof of the previous theorem, $|U_\tau(\omega)|^2$ is S-integrable for a.a. $\tau\not\approx 0$. Putting this together we have

$$\begin{aligned}\mathbb{E}\int_1^n |U_\tau(\omega)|^2\exp(-\tau)d\tau &= \int_1^n \mathbb{E}|U_\tau(\omega)|^2\exp(-\tau)d\tau\\ &\approx \int_1^n {}^\circ\mathbb{E}|U_\tau(\omega)|^2\exp(-\tau)d_L\tau\\ &= \int_1^n \mathbb{E}\,{}^\circ|U_\tau(\omega)|^2\exp(-\tau)d_L\tau\\ &= \mathbb{E}\int_1^n {}^\circ|U_\tau(\omega)|^2\exp(-\tau)d_L\tau\\ &= \mathbb{E}\int_0^n |(u(t,\omega)|^2\exp(-t)dt\end{aligned}$$

from the fact that ${}^\circ U(\omega)=u(\omega)$ in M for a.a. ω.

For the other direction, we show that $\mathrm{NS}\subseteq SL^2[0,1]$. This is routine: if $U\in\mathrm{NS}$ with ${}^\circ U=u$ then from $\mathbb{E}(|U|^2)\approx\mathbb{E}(|u|^2)$ and $U_\tau(\omega)\approx u({}^\circ t,\omega)$ in $\mathbf{H}$ a.s. in $[0,1]\times\Omega$, we have that $|U_\tau(\omega)|^2\exp(-\tau)$ is S-integrable over $[0,1]\times\Omega$. □

Now we turn to part (b) of Theorem 7.6. In view of part (a) and Theorem 7.7, it suffices to prove the following.

Theorem 7.8 *If $u\in X_k$ then there is $U\in\mathcal{X}_k\cap\mathrm{NS}$ with weak standard part $u={}^\circ U$.*

Proof Take $u\in X_k$. By Theorem 7.7, it suffices to find a $U\in\mathcal{X}_k\cap SL^2[0,1]$ with $u={}^\circ U$. Proposition 13.10 in the Appendix gives a positive $\delta\approx 0$, and for each $i\in\mathbb{N}$ an internal $\mathcal{G}_\tau$-adapted process $\bar U(\tau,\omega)_{(i)}$ such that for a.a. ω and for all $i\in\mathbb{N}$, $\bar U(\tau,\omega)_{(i)}=0$ for all $\tau<\delta$, and

$$\bar U(\tau,\omega)_{(i)}\approx u({}^\circ\tau,\omega)_{(i)}$$

for all finite $\tau\geq\delta$. By $\aleph_1$-saturation the sequence of co-ordinate processes extends to an internal family $(\bar U_{(i)})_{i\leq N}$ of $\mathcal{G}_\tau$-adapted processes, giving a $\mathcal{G}_\tau$-adapted process $\bar U(\tau,\omega)\in\mathbf{H}_N$. This process must now be suitably truncated in order meet the conditions for membership of $\mathcal{X}_k\cap\mathrm{NS}$.

For each $m\leq N$ define a process $\bar U^{(m)}(\tau,\omega)\in\mathbf{H}_N$ by

$$\bar U^{(m)}_{(i)}=\begin{cases}(\bar U(\tau,\omega)_{(i)}\wedge m)\vee -m & \text{if}\quad i\leq m\\ 0 & \text{otherwise}\end{cases}$$

This process is again $\mathcal{G}_\tau$-adapted. For each $m \in \mathbb{N}$ and all $n \leq m$ we have

$$\begin{aligned} \mathbb{E}\left(\int_0^n |\bar{U}_\tau^{(m)}|^2 d\tau\right) &\approx \mathbb{E}\left(\int_0^n |{}^\circ\bar{U}_\tau^{(m)}|^2 d_L\tau\right) \\ &< \mathbb{E}\left(\int_0^n |u(t)|^2 dt\right) + \frac{1}{m} \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}\left(\int_{\frac{1}{n}}^n \|U_\tau^{(m)}\|^2 d\tau\right) &\approx \mathbb{E}\left(\int_{\frac{1}{n}}^n \|{}^\circ\bar{U}_\tau^{(m)}\|^2 d_L\tau\right) \\ &< \mathbb{E}\left(\int_{\frac{1}{n}}^n \|u(t)\|^2 dt\right) + \frac{1}{m}. \end{aligned}$$

By overspill there is an infinite $J \leq N$ such that for all $n \in \mathbb{N}$,

$$\mathbb{E}\left(\int_0^n |\bar{U}_\tau^{(J)}|^2 d\tau\right) < \mathbb{E}\left(\int_0^n |u(t)|^2 dt\right) + \frac{1}{J}$$

and

$$\mathbb{E}\left(\int_{\frac{1}{n}}^n \|\bar{U}_\tau^{(J)}\|^2 d\tau\right) < \mathbb{E}\left(\int_{\frac{1}{n}}^n \|u(t)\|^2 dt\right) + \frac{1}{J}.$$

Let $V = \bar{U}^{(J)}$. Then for $i \in \mathbb{N}$, for a.a. ω we have

$$V(\tau,\omega)_{(i)} \approx u({}^\circ\tau,\omega)_{(i)}$$

for all finite $\tau \geq \delta$, and so $|u({}^\circ\tau,\omega)|^2 \leq {}^\circ|V(\tau,\omega)|^2$ for all finite $\tau \geq \delta$. Hence for $n \in \mathbb{N}$,

$$\begin{aligned} {}^\circ\mathbb{E}\int_0^n |V(\tau,\omega)|^2 d\tau &\leq \mathbb{E}\left(\int_0^n |u(t,\omega)|^2 dt\right) \\ &= \mathbb{E}\left(\int_0^n |u({}^\circ\tau,\omega)|^2 d_L\tau\right) \\ &\leq \mathbb{E}\left(\int_0^n {}^\circ|V(\tau,\omega)|^2 d_L\tau\right) \\ &\leq {}^\circ\mathbb{E}\int_0^n |V(\tau,\omega)|^2 d\tau. \end{aligned}$$

So $V(\tau,\omega)$ is an adapted lifting of $u(t,\omega)$ which is in $SL^2[0,n]$ for each n. Therefore $V \in SL^2[0,1]$. Moreover, for each $n \in \mathbb{N}$ and a.a. $(\tau,\omega) \in {}^*[0,n] \times \Omega$ we have $V(\tau,\omega) \approx u(\tau,\omega)$ in $\mathbf{H}$ (strongly). Note also that for each $n \in \mathbb{N}$ and a.a. $\tau \in {}^*[0,n]$ we have

$$\mathbb{E}(|V(\tau)|^2) \approx \mathbb{E}(|u({}^\circ\tau)|^2). \tag{27}$$

Similar reasoning shows that for all $n \in \mathbb{N}$, $\|V(\tau,\omega)\| \in SL^2[\frac{1}{n}, n]$ and $\|V(\tau,\omega)\| \approx \|u(\tau,\omega)\|$ for a.a. $(\tau,\omega) \in {}^*[0,n] \times \Omega$. Thus, for a.a. $(\tau,\omega) \in {}^*[0,n] \times \Omega$ we also have $V(\tau,\omega) \approx u(\tau,\omega)$ in $\mathbf{V}$ (strongly).

It is now necessary to truncate further (i.e. take $V' = \bar{U}^{(J')}$ for some infinite $J' \leq J$) in order to ensure that the approximate inequalities $(\mathcal{X}3_n)$–$(\mathcal{X}5_n)$ are all satisfied. Note that *a fortiori* all the properties of V noted above are valid for any such V' also.

Let T_0 be the Loeb-full set of finite τ_0 such that

(i) (27) holds for $\tau = \tau_0$.

(ii) $t_0 = {}^\circ\tau_0$ is a value for which (6), (7) and (8) hold for all $t_1 \geq t_0$.

Choose an increasing chain of *internal* sets $A_m \subseteq {}^*[0,m] \cap T_0$ with ${}^*\mathrm{Leb}(A_m) \geq m - \frac{1}{m}$. Then for all $m \in \mathbb{N}$, $\tau_0 \in A_m$, and $\tau_0 \leq \tau_1 \leq m$ we have $\mathbb{E}(|V(\tau_0)|^2) \approx \mathbb{E}(|u(t_0)|^2)$ and ${}^\circ\mathbb{E}(|\bar{U}^{(m)}(\tau_1)|^2) \leq \mathbb{E}(|u(t_1)|^2)$ (where $t_0 = {}^\circ\tau_0$ and $t_1 = {}^\circ\tau_1$). Hence, using (6), for $m \in \mathbb{N}$, $\tau_0 \in A_m$, and all $\tau_0 \leq \tau_1 \leq m$,

$$\mathbb{E}(|\bar{U}^{(m)}(\tau_1)|^2) \leq \mathbb{E}(|V(\tau_0)|^2)\exp(-k_1(\tau_1-\tau_0)) + k_2 + \tfrac{1}{m}. \tag{28}$$

Similar reasoning gives

$$\mathbb{E}\left(\sup_{\tau_0\leq\sigma\leq\tau_1}|\bar{U}^{(m)}(\sigma)|^2 + \int_{\tau_0}^{\tau_1}\|\bar{U}^{(m)}(\sigma)\|^2 d\sigma\right) \leq \alpha\mathbb{E}(|V(\tau_0)|^2)+\beta(\tau_1-\tau_0)+\tfrac{1}{m}, \tag{29}$$

and for all $n \leq m$,

$$\mathbb{E}(\varphi_n(\bar{U}^{(m)}_{\tau_1})) \leq \mathbb{E}(\varphi_n(V_{\tau_0}))\exp(-k_3(\tau_1-\tau_0))+n^{-\frac{1}{2}}(\alpha\mathbb{E}(|V_{\tau_0}|^2)+\beta)+\tfrac{1}{m}. \tag{30}$$

By $\aleph_1$-saturation we extend $A_m, m \in \mathbb{N}$ to an internal increasing chain of sets $A_m, m \in {}^*\mathbb{N}$ such that $A_m \subseteq {}^*[0,m]$ and A_m has *Lebesgue measure $\geq m - 1/m$. Overspill gives an infinite $J' \leq J$ such that (28)–(30) all hold with $m = J'$ for $\tau_0 \in A_{J'}$ and $\tau_0 \leq \tau_1 \leq J'$.

Writing $V' = \bar{U}^{(J')}$, for all $m \in \mathbb{N}$ and $\tau_0 \in A_m$ we have $\mathbb{E}(|V(\tau_0)|^2) \approx \mathbb{E}(|u(t_0)|^2) \approx \mathbb{E}(|V'(\tau_0)|^2)$, and so for $m \in \mathbb{N}$, $\tau_0 \in A_m$, and $\tau_0 \leq \tau_1 \leq m$,

$$\mathbb{E}(|V'(\tau_1)|^2) \leq \mathbb{E}(|V'(\tau_0)|^2)\exp(-k_1(\tau_1-\tau_0)) + k_2 + \tfrac{1}{m}. \tag{31}$$

Similar reasoning gives

$$\mathbb{E}\left(\sup_{\tau_0\leq\sigma\leq\tau_1}|V'(\sigma)|^2 + \int_{\tau_0}^{\tau_1}\|V'(\sigma)\|^2 d\sigma\right) \leq \alpha\mathbb{E}(|V'(\tau_0)|^2)+\beta(\tau_1-\tau_0)+\tfrac{1}{m}, \tag{32}$$

and for all $n \leq m$,

$$\mathbb{E}(\varphi_n(V'(\tau_1)) \leq \mathbb{E}(\varphi_n(V'_{\tau_0}))\exp(-k_3(\tau_1-\tau_0))+n^{-\frac{1}{2}}(\alpha\mathbb{E}(|V'_{\tau_0}|^2)+\beta)+\tfrac{1}{m}. \tag{33}$$

By overspill, there is some infinite $J'' \leq J'$ such that (31)–(33) all hold with $m = J''$ and for all $\tau_0 \in A_{J''}$ and $\tau_0 \leq \tau_1 \leq J''$. Put

$$U = \begin{cases} V'(\tau,\omega) & \text{for} \quad \tau \leq J'' \\ 0 & \text{for} \quad \tau > J'' \end{cases}$$

Since $V' \in SL^2[0,1]$, we have $U \in SL^2[0,1]$. It follows from (27) with $\tau = \tau_0$ and (28) that U satisfies the condition $(\mathcal{X}1)$. Using (31)–(33), it is routine to check that $(\mathcal{X}3_n) - (\mathcal{X}5_n)$ hold for this U for each $n \in \mathbb{N}$, the exceptional set in each case being ${}^*[0,J''] \setminus A_{J''}$.

As noted earlier, V' (like V) is a lifting of $u(t,\omega)$ which is in $SL^2[0,1]$, so

$$^\circ\mathbb{E}\int_0^1 |U_\tau|^2 d\tau = \mathbb{E}\int_0^1 |u_t|^2 dt,$$

giving $(\mathcal{X}6_{k,n})$ for each n.

To see that $U \in \mathcal{X}_k$ as required we now have only to check that $(\mathcal{X}2_n)$ holds.

For this, fix n and note that U is an adapted lifting of u in $SL^2[0,1]$ with the property (24) for $\tau \geq \delta$. Thus the theory of [7] shows that for a.a. ω, for all $\tau_1 \in [\frac{1}{n}, n]$ and $m \leq n$,

$$U(\tau_1)_{(m)} - U(1/n)_{(m)} \approx u(^\circ\tau_1)_{(m)} - u(1/n)_{(m)}$$

$$= \int_{\frac{1}{n}}^{^\circ\tau_1} [-\nu Au(t)_{(m)} - B(u(t))_{(m)} + f(u(t))_{(m)}]dt - \int_{\frac{1}{n}}^{^\circ\tau_1} g(u(t))_{(m)} dw_t$$

$$\approx \int_{\frac{1}{n}}^{\tau_1} [-(\nu AU_\tau)_{(m)} - B(U(\tau))_{(m)} + F(U(\tau))_{(m)}]d\tau - \int_{\frac{1}{n}}^{\tau_1} G(U(\tau))_{(m)} dW_\tau.$$

□

Corollary 7.9 $\mathcal{X} \cap \mathrm{NS}$ *is nonempty.*

Proof By Theorem 6.1, there is a $k \in \mathbb{N}$ such that X_k is nonempty. Then by Theorem 7.8, $\mathcal{X}_k \cap SL^2[0,1]$ is nonempty, and by Theorem 7.7, $\mathcal{X}_k \cap \mathrm{NS}$ is nonempty. □

8 The internal semiflow

Let $\mathcal{H}$ be the internal set consisting of all internal stochastic processes $U : {}^*[0,\infty) \times \Omega \to H_N$. There is a natural internal semiflow $T_\tau : \mathcal{H} \to \mathcal{H}$ defined for all $0 \leq \tau \in {}^*\mathbb{R}$, as follows.

Definition 8.1 (Semiflow of internal processes) Suppose that $U \in \mathcal{H}$. Then for any $\tau \geq 0$ the process $V = T_\tau U \in \mathcal{H}$ is defined by

$$V(\sigma,\omega) = U(\sigma + \tau, \Theta_\tau\omega).$$

This semiflow has internal properties corresponding to those for S_t as given in Proposition 3.2

Proposition 8.2

(a) T_τ *is an internal semigroup on the class* $\mathcal{H}$.

(b) *If* U *is adapted to the filtration* $(\mathcal{G}_\tau)_{\tau\geq 0}$, *then so is* $V = T_\tau U$.

(c) *If* U *is adapted and* $V = T_\tau U$, *then for* $\tau_1 \geq 0$:

(i) For appropriate internal *continuous F,

$$\int_0^{\tau_1} F(V(\sigma,\omega))d\sigma = \int_\tau^{\tau+\tau_1} F(U(\sigma,\Theta_\tau\omega))d\sigma.$$

(By appropriate *we mean that the integrals are defined.)*

(ii) For appropriate internal *continuous G,

$$\int_0^{\tau_1} G(V(\sigma,\omega))dW(\sigma,\omega) = \int_\tau^{\tau+\tau_1} G(V(\sigma,\Theta_\tau\omega))dW(\sigma,\Theta_\tau\omega)$$

(meaning that $I(\omega) = J(\Theta_\tau\omega)$ *as random variables, where* $I(\omega)$ *is the left-hand integral and* $J(\omega) = \int_\tau^{\tau+\tau_1} G(u(\sigma,\omega))dW(\sigma,\omega)$*).*

(d) *If* U *is adapted and* $V = T_\tau U$ *and* $\tau_1 \geq \tau_0$, *then*

$$\left|U(\tau_1+\tau) - U(\tau_0+\tau) - \int_{\tau_0+\tau}^{\tau_1+\tau} F(U(\sigma))d\sigma - \int_{\tau_0+\tau}^{\tau_1+\tau} G(U(\sigma))dW(\sigma)\right|$$

and

$$\left|V(\tau_1) - V(\tau_0) - \int_{\tau_0}^{\tau_1} F(V(\sigma))d\sigma - \int_{\tau_0}^{\tau_1} G(V(\sigma))dW(\sigma)\right|$$

have the same internal probability distribution.

Proposition 8.3 *For finite* $\tau > 0$:

(a) $T_\tau\mathcal{X} \subseteq \mathcal{X}$.

(b)

$$T_\tau(\mathcal{X} \cap SL^2[0,1]) \subseteq \mathcal{X} \cap SL^2[0,1],$$

and for $U \in SL^2[0,1]$ *we have*

$$^\circ(T_\tau U) = S_{^\circ\tau}{}^\circ U. \tag{34}$$

Proof (a) Clauses $(\mathcal{X}3_n)$–$(\mathcal{X}5_n)$ are clearly invariant under the operation of T_τ. For $(\mathcal{X}2)$, it is easy to check using Proposition 8.2 that $T_\tau U$ has property $(\mathcal{X}2_n)$ provided that U has the property $(\mathcal{X}2_{n'})$ for some $n' \geq n+\tau$.

If $\mathbb{E}\int_0^1 |U_\sigma|^2 d\sigma < \infty$ then it is clear from $(\mathcal{X}3_n)$ that $\mathbb{E}\int_0^1 |(T_\tau U)_\sigma|^2 d\sigma < \infty$ also.

(b) Suppose that $U \in \mathcal{X} \cap SL^2[0,1]$. From the proof of Theorem 7.6 (a) we see that $|U_\sigma(\omega)|^2$ is S-integrable for all $\sigma \not\approx 0$. Further, from $(\mathcal{X}3)$ we know that $\mathbb{E}(|U_\sigma|^2)$ is bounded on ${}^*[\sigma, \infty)$ for any $s \not\approx 0$. Thus $|U_\sigma(\omega)| \in SL^2[1,n]$ for any $n \geq 1$, and hence on ${}^*[\tau, \tau+1]$ for any $\tau > 0$. This means that $T_\tau U \in SL^2[0,1]$ as required. □

We next show that for a suitable $\rho \in \mathbb{N}$ the set $\mathcal{X}_\rho$ is *S-absorbing*.

Lemma 8.4 *There is a $\rho \in \mathbb{N}$ such that $\mathcal{X}_\rho$ is S-absorbing. That is, for each $k \in \mathbb{N}$ there is an $r(k) \in \mathbb{N}$ such that*

$$T_\tau \mathcal{X}_k \subseteq \mathcal{X}_\rho$$

for all finite $\tau \geq r(k)$.

Proof This follows from the fact that $U \in \mathcal{X}_k$ has the property $(\mathcal{X}3_n)$ for all n and $\mathbb{E}\int_0^1 |U_\tau|^2 d\tau \leq k + \frac{1}{n}$ for all n. From these properties we can deduce that for any $\tau \not\approx 0$

$$\mathbb{E}\int_\tau^{\tau+1} |U_\sigma|^2 d\sigma \leq k\exp(-k_1\tau) + k_2 + \frac{1}{n}$$

for any n. To see this, let $Z_\sigma = \mathbb{E}|U_\sigma|^2$ and note that from $(\mathcal{X}6)$ and $(\mathcal{X}3)$, Z_σ is bounded and hence S-integrable on $[\tau, \tau+1]$. Moreover, for a.a. σ in this interval, $\sigma - \tau$ has the property of τ_0 in $(\mathcal{X}3_n)$ for all n. Thus

$$\begin{aligned}
\int_0^1 \mathbb{E}|(T_\tau U)(\sigma)|^2 d\sigma = \int_\tau^{\tau+1} Z_\sigma d\sigma \quad &\approx \quad \int_\tau^{\tau+1} {}^\circ Z_\sigma d_L\sigma \\
&\leq \quad \int_\tau^{\tau+1} {}^\circ(Z_{\sigma-\tau}\exp(-k_1\tau) + k_2) d_L\sigma \\
&= \quad \exp(-k_1{}^\circ\tau) \int_0^1 {}^\circ Z_\sigma d_L\sigma + k_2 \\
&\leq \quad \exp(-k_1{}^\circ\tau)^\circ\left(\int_0^1 Z_\sigma d\sigma\right) + k_2 \\
&< \quad k\exp(-k_1\tau) + k_2 + \frac{1}{n}
\end{aligned}$$

as required. To verify the lemma we may take any natural number $\rho > k_2$ and then $r(k) = r$ such that

$$k\exp(-k_1 r) + k_2 = \rho.$$

□

Now write $\mathcal{B} = \mathcal{X}_\rho$ and $\mathcal{B}_n = \mathcal{X}_{\rho,n}$. Thus $\mathcal{B}$ is the intersection of a decreasing chain of internal sets, $\mathcal{B} = \bigcap_{n\in\mathbb{N}} \mathcal{B}_n$.

Corollary 8.5 *For each τ,*

$$T_\tau \mathcal{B} = \bigcap_{n\in\mathbb{N}} T_\tau \mathcal{B}_n.$$

Proof This follows from $\mathcal{B} = \bigcap_{n\in\mathbb{N}} \mathcal{B}_n$ by $\aleph_1$-saturation. □

Corollary 8.6 $\mathcal{B} \cap \mathrm{NS}$ *is nonempty.*

Proof By Corollary 7.9, $\mathcal{X} \cap \mathrm{NS} \neq \emptyset$, so there is a $k \in \mathbb{N}$ such that $\mathcal{X}_k \cap \mathrm{NS} \neq \emptyset$. Take finite τ such that $T_\tau \mathcal{X}_k \subseteq \mathcal{B}$ and then

$$\mathcal{B} \cap \mathrm{NS} \supseteq T_\tau(\mathcal{X}_k \cap \mathrm{NS}) \neq \emptyset$$

using Proposition 8.3(b) and Theorem 7.7. □

Definition 8.7 Define $r_0 = r(\rho)$; then r_0 is finite and $T_\tau \mathcal{B} \subseteq \mathcal{B}$ for finite $\tau \geq r_0$.

The following will be useful in several situations below.

Proposition 8.8 *For any $n, m \in {}^*\mathbb{N}$, if $r_0 \leq \sigma \leq n - m$ then*

$$T_\sigma \mathcal{B}_n \subseteq \mathcal{B}_m.$$

Proof Let $U \in \mathcal{B}_n$. It is routine to check from the definitions that for any σ the properties ($\mathcal{X}1$) and ($\mathcal{X}3_n$) – ($\mathcal{X}5_n$) are preserved under T_σ. If $\sigma \geq r_0$ then the calculation in the proof of Lemma 8.4 shows that $T_\sigma U$ has property ($\mathcal{X}6_{\rho,n}$) also. Finally, as noted in the proof of Proposition 8.3, $T_\sigma U$ has the property ($\mathcal{X}2_m$) provided that $\sigma + m \leq n$. Then, since $m \leq n$, $T_\sigma U$ belongs to $\mathcal{X}_{\rho,m} = \mathcal{B}_m$. □

9 Construction of global attractors

The main theorem of this section shows the existence of a global attractor A for the semiflow S_t on X. The idea for constructing A is similar to that used in the earlier papers [6, 8, 9] of the first author and Capiński: $A = {}^\circ\mathcal{C}$ for a set $\mathcal{C} \subseteq \mathcal{X}$ that is an S-attractor for the internal semiflow T_τ.

From the S-absorbing set $\mathcal{B}$ we define the following set $\mathcal{C}$, which we will call the S-*attractor* for the internal semiflow T_τ on $\mathcal{X}$.

Definition 9.1 Define sets $\mathcal{C}, \mathcal{C}_n$ and $\widehat{\mathcal{C}}_n$ (for $n \in \mathbb{N}$) as follows.

(a) $$\mathcal{C} = \bigcap_{\substack{0 \le \tau \\ \tau \text{ finite}}} T_\tau \mathcal{B}.$$

(b) $$\mathcal{C}_n = \bigcap_{0 \le \tau \le n} T_\tau \mathcal{B}, \text{ so that } \mathcal{C} = \textstyle\bigcap_{n \in \mathbb{N}} \mathcal{C}_n.$$

(c) $$\widehat{\mathcal{C}}_n = \bigcap_{0 \le \tau \le n} T_\tau \mathcal{B}_n.$$

We first show that $\mathcal{C}$ is nonempty.

Proposition 9.2 (a) *The sets $\widehat{\mathcal{C}}_n$ are internal and decreasing.*

(b) $\widehat{\mathcal{C}}_n \supseteq \mathcal{C}_n \supseteq T_{r_0+n}\mathcal{B}$, *hence* $\widehat{\mathcal{C}}_n \neq \emptyset$.

(c) $\mathcal{C} = \bigcap_{n \in \mathbb{N}} \widehat{\mathcal{C}}_n$, *and so* $\mathcal{C} \neq \emptyset$.

Proof (a) is obvious.

(b) Since $\mathcal{B}_n \supseteq \mathcal{B}$, we have $\mathcal{C}_n \supseteq \widehat{\mathcal{C}}_n$. The second inclusion follows from the choice of r_0: for $V \in \mathcal{B}$ we have $T_{r_0+n}V = T_\tau T_{r_0+n-\tau} V \in \mathcal{T}_\tau \mathcal{B}$ for all $\tau \le n$. Since $\mathcal{B}$ is nonempty, it follows that $\widehat{\mathcal{C}}_n$ is nonempty.

(c)
$$\bigcap_{n \in \mathbb{N}} \widehat{\mathcal{C}}_n = \bigcap_{n \in \mathbb{N}} \bigcap_{0 \le \tau \le n} T_\tau \mathcal{B}_n = \bigcap_{\substack{0 \le \tau \\ \tau \text{ finite}}} \bigcap_{n \in \mathbb{N}} T_\tau \mathcal{B}_n = \bigcap_{\substack{0 \le \tau \\ \tau \text{ finite}}} T_\tau \mathcal{B} = \mathcal{C}$$

(the third equality following from Corollary 8.5). The fact that $\mathcal{C} \neq \emptyset$ now follows from (a) and $\aleph_1$-saturation. □

In preparation for the next theorem, which gives some of the key properties of $\mathcal{C}$, we have:

Proposition 9.3

$$T_\tau \mathcal{C} = \bigcap_{n \in \mathbb{N}} T_\tau \mathcal{C}_n = \bigcap_{n \in \mathbb{N}} T_\tau \widehat{\mathcal{C}}_n.$$

Proof The equality of the outer sets is an elementary application of $\aleph_1$-saturation, since $\mathcal{C} = \bigcap_{n \in \mathbb{N}} \widehat{\mathcal{C}}_n$ and each $\widehat{\mathcal{C}}_n$ is internal. The middle set is squeezed between the outer ones since $\mathcal{C} \subseteq \mathcal{C}_n \subseteq \widehat{\mathcal{C}}_n$ for all n. □

Some crucial properties of the set $\mathcal{C}$ are now gathered together.

Theorem 9.4

(a) $\mathcal{C} \subseteq \mathcal{B} (= \mathcal{X}_\rho)$.

(b) *$\mathcal{C}$ is a countable intersection of internal sets.*

(c) **(Invariance of $\mathcal{C}$)** *For finite τ,*

$$T_\tau \mathcal{C} = \mathcal{C}.$$

(d) **($\mathcal{C}$ is nearstandard)**

$$\mathcal{C} \subseteq \mathrm{NS}.$$

Proof (a) $\mathcal{C} \subseteq T_0\mathcal{B} = \mathcal{B}$.

(b) Obvious from Proposition 9.2.

(c) For any finite τ and σ we have

$$\mathcal{C} \subseteq T_{\sigma+r_0}\mathcal{B} \quad \text{and} \quad T_{\tau+r_0}\mathcal{B} \subseteq \mathcal{B},$$

so

$$T_\tau\mathcal{C} \subseteq T_\tau T_{\sigma+r_0}\mathcal{B} = T_\sigma T_{\tau+r_0}\mathcal{B} \subseteq T_\sigma\mathcal{B}.$$

Since this holds for all finite $\sigma \geq 0$, it follows that $T_\tau\mathcal{C} \subseteq \mathcal{C}$.

For the opposite inclusion, from Proposition 9.3 it is sufficient to show that $\mathcal{C} \subseteq T_\tau\mathcal{C}_n$ for each n. Take any $U \in \mathcal{C}$; then $U = T_{\tau+r_0+n}V = T_\tau T_{r_0+n}V$ for some $V \in \mathcal{B}$. Now for each $\sigma \leq n$ we have $T_{r_0+n}V = T_\sigma T_{r_0+n-\sigma}V \in T_\sigma\mathcal{B}$, so $T_{r_0+n}V \in \mathcal{C}_n$ and $U \in T_\tau\mathcal{C}_n$.

(d) Let $U \in \mathcal{C}$. By Proposition 9.2 (b), $U \in \bigcap_{n\in\mathbb{N}} \widehat{\mathcal{C}}_n$. Recall that the sets $\mathcal{B}_n$ and hence $\widehat{\mathcal{C}}_n$ are already defined for all $n \in {}^*\mathbb{N}$. By overflow, there is an infinite J with $U \in \widehat{\mathcal{C}}_J$. By the definition of $\widehat{\mathcal{C}}_J$, $U = T_JV$ for some $V \in \mathcal{B}_J$. Using ($\mathcal{X}$6), there is a $\tau_0 \in {}^*[0,1]$ such that $\mathbb{E}(|V(\tau_0)|^2)$ is finite, and (25) holds for V with $n = J$. That is,

$$\mathbb{E}(\varphi_m(V_{\tau_1})) \leq \mathbb{E}(\varphi_m(V_{\tau_0}))\exp(-k_3(\tau_1-\tau_0)) + m^{-\frac{1}{2}}(\alpha\mathbb{E}(|V_{\tau_0}|^2)+\beta) + \frac{1}{J} \quad (35)$$

for all $\tau_1 \in {}^*[\tau_0,\infty)$ and each $m \leq J$. Since $\psi(x) \leq 1$ for all x, $\varphi_m(V_{\tau_0}) \leq |V_{\tau_0}|^2$, so $\mathbb{E}(\varphi_m(\mathbb{V}_{\tau_0}))$ is also finite for all m.

Consider any infinite $K \leq J$. For each $\sigma \in {}^*[0,1]$ using (35) with $\tau_1 = J+\sigma$ we have

$$\mathbb{E}(\varphi_K(U(\sigma))) = \mathbb{E}(\varphi_K(V_{J+\sigma})) \leq$$
$$\mathbb{E}(\varphi_K(V_{\tau_0}))\exp(-k_3(J+\sigma-\tau_0)) + K^{-\frac{1}{2}}(\alpha\mathbb{E}(|V_{\tau_0}|^2)+\beta) + \frac{1}{J},$$

so $\mathbb{E}(\varphi_K(U(\sigma))) \approx 0$. Therefore, by Corollary 13.8 in the Appendix, $|U_\sigma(\omega)|^2$ is S-integrable for all $\sigma \in {}^*[0,1]$. Then by Proposition 13.6 in the Appendix, $|U|^2$ is S-integrable in the product ${}^*[0,1]\times\Omega$, that is, $U \in SL^2[0,1]$. Since $U \in \mathcal{X}$, we have $U \in \mathrm{NS}$ by Theorem 7.7. □

It is of interest to note the following seemingly weaker characterization of $\mathcal{C}$

Proposition 9.5 *For any sequence (n_k) from $\mathbb{N}$ such that $n_k \to \infty$, we have*

$$\mathcal{C} = \bigcap_{k\in\mathbb{N}} T_{n_k}\mathcal{B}.$$

Proof One inclusion is obvious. For the other, if $\tau \geq 0$ take n_k with $n_k \geq r_0 + \tau$. Then

$$T_{n_k}\mathcal{B} = T_\tau T_{n_k-\tau}\mathcal{B} \subseteq T_\tau\mathcal{B},$$

so that $\bigcap_k T_{n_k}\mathcal{B} \subseteq T_\tau\mathcal{B}$ for all τ. □

We also need the following property of $\mathcal{C}$.

Proposition 9.6 ($\mathcal{C}$ attracts) *For every $n, k \in \mathbb{N}$,*

$$T_\tau\mathcal{X}_k \subseteq \mathcal{C}_n$$

for all finite $\tau \geq n + r_0 + r(k)$. In particular, taking $k = \rho$ so that $r(k) = r_0$

$$T_\tau\mathcal{B} \subseteq \mathcal{C}_n$$

for all finite $\tau \geq n + 2r_0$.

Proof We have $\tau = n + \sigma + r(k)$ for some finite $\sigma \geq r_0$. Then

$$T_\tau\mathcal{X}_k = T_n T_\sigma T_{r(k)}\mathcal{X}_k \subseteq T_n T_\sigma\mathcal{B} \subseteq T_n\mathcal{B} \subseteq \mathcal{C}_n.$$

□

Remark 9.7 The above attraction property together with the invariance and the fact that $\mathcal{C} \subseteq \mathrm{NS}$ (Theorem 9.4) is the reason for calling $\mathcal{C}$ the S-attractor for the internal set of processes $\mathcal{X}$.

The attractor for the semiflow S_t can now be defined.

Definition 9.8 Define the sets A_n, A by

$$A_n = {}^\circ(\mathcal{C}_n \cap \mathrm{NS}), \qquad A = {}^\circ\mathcal{C}.$$

Immediate properties of A are as follows (where we write $B = X_\rho$).

Theorem 9.9 (a) *$S_t A = A$ for all $t \geq 0$.*

(b) *$A \subseteq B$, where $B = X_\rho$.*

(c) *$A \subseteq Y$.*

(d) *$A = \bigcap_{n\in\mathbb{N}} A_n$.*

(e) *$A = \bigcap_{t\geq 0} S_t B = \bigcap_k S_{t_k} B$ for any sequence $t_k \to \infty$.*

(f) *A is closed in the space* $L^2(\Omega, M)$.

Proof (a) follows from Theorem 9.4 and Proposition 8.3(b).

(b) follows from the fact that $\mathcal{C} \subseteq \mathcal{B}$.

(c) By part (b), $A = S_1 A \subseteq S_1 X \subseteq Y$.

For (d), it is clear that $A \subseteq \bigcap A_n$. Consider an element u in the righthand set. For each n we have $u = {}^\circ U_n$ for some $U_n \in \mathcal{C}_n \cap \mathrm{NS}$. Then $U_n \in \widehat{\mathcal{C}}_n$ for all $n \in \mathbb{N}$. Let $|\cdot|_k$ denotes the norm $|\cdot|$ restricted to the time interval $[0, k]$, that is,

$$|U|_k = \left(\mathbb{E}\int_0^k |U_\tau|^2 \mu(d\tau)\right)^{\frac{1}{2}}.$$

Then for all n and $m \leq n$ we have

$$|U_m - U_n|_n \leq \frac{1}{n}.$$

By $\aleph_1$-saturation there is an infinite J with $U_J \in \widehat{\mathcal{C}}_J \subseteq \mathcal{C}$ and

$$|U_m - U_J|_J \approx 0$$

for all $m \in \mathbb{N}$. Thus $u = {}^\circ U_J \in A$.

(e) Since $\mathcal{C} \subseteq T_\tau \mathcal{B}$ we have $A \subseteq S_t B$ for all $t \geq 0$, so it is enough to prove that the third set is contained in A. For any n we have

$$S_t B \subseteq {}^\circ(T_t \mathcal{B} \cap \mathrm{NS}) \subseteq {}^\circ(\mathcal{C}_n \cap \mathrm{NS}) = A_n$$

for all $t \geq n + 2r_0$, by Proposition 9.6. The result follows.

(f) follows from Proposition 13.5 in the Appendix. □

In order to show that A has the required attracting properties, we first show:

Theorem 9.10 (a) *For every* $n, k \in \mathbb{N}$,

$$S_t X_k \subseteq A_n$$

for all $t \geq n + r_0 + r(k)$.

(b) *Let* Z *be a bounded subset of* X *(with the norm of* $L^2(\Omega, M)$*). Then for every* $n \in \mathbb{N}$ *there is finite* $t_0(n, Z)$ *such that*

$$S_t Z \subseteq A_n$$

for all $t \geq t_0(n, Z)$.

Proof (a) is immediate from Proposition 9.6 after observing that

$$S_t X_k \subseteq {}^\circ(T_t \mathcal{X}_k).$$

(b) By Lemma 4.2, $Z \subseteq X_k$ for some $k \in \mathbb{N}$. The result now follows from (a). In fact, Lemma 4.2 (a) shows that one can take $t_0(n, Z) = n + r_0 + r(k)$ where $k \geq (\sup\{|u| : u \in Z\})^2 e$. □

Lemma 9.11 *A has the attracting property of Definition 3.7(b)(iii)*

Proof Let K be compact with $\rho(A, K) = \varepsilon > 0$ (otherwise there is nothing to prove). Then $K^{\leq\varepsilon} \cap A = \emptyset$. By Theorem 9.10 it is sufficient to show that $K^{\leq\varepsilon} \cap A_n = \emptyset$ for some $n \in \mathbb{N}$.

For each n there is a finite set $\{v_{n,i} : 1 \leq i \leq k_n\} \subseteq K$ such that K is covered by the open balls $\{v_{n,i}\}^{<\frac{1}{n}}$, $1 \leq i \leq k_n$. ($\{v\}^{<r}$ denotes the open ball of radius r with centre v.) Taking $v_{n,i} = {}^\circ V_{n,i}$ with $V_{n,i} \in \mathrm{NS}$ we have

$$K^{\leq\varepsilon} = {}^\circ\left(\bigcap_{n\in\mathbb{N}} \bigcup_{i=1}^{k_n} \{U : |U - V_{n,i}| \leq \varepsilon + 1/n\}\right) = {}^\circ\left(\bigcap_{n\in\mathbb{N}} K_n\right)$$

where the sets K_n are all internal. Similar reasoning gives $K^{\leq\varepsilon} = {}^\circ\left(\bigcap_{n\in\mathbb{N}} K_n^{\leq 1/n}\right)$.

$\aleph_1$-saturation shows that there is $n \in \mathbb{N}$ with

$$\bigcap_{m\leq n} K_m^{\leq\frac{1}{m}} \cap \hat{\mathcal{C}}_n = \emptyset \tag{36}$$

(for otherwise there is $U \in \bigcap_{n\in\mathbb{N}}(K_n^{\leq\frac{1}{n}} \cap \hat{\mathcal{C}}_n)$, and such a U belongs to $\mathcal{C}$ and is thus nearstandard with ${}^\circ U \in K^{\leq\varepsilon} \cap A$).

It follows that $K^{\leq\varepsilon} \cap A_n = \emptyset$, for if not there is $u = {}^\circ U = {}^\circ V$ with $U \in \bigcap_{m\leq n} K_m$ and $V \in \hat{\mathcal{C}}_n$. But then $V \in \bigcap_{m\leq n} K_m^{\leq\frac{1}{m}}$ contradicting (36). □

Now we can prove the main theorem of the paper:

Theorem 9.12 *The set A is an attractor for the set of solutions X.*

Proof We have seen in Theorem 9.9 and Lemma 9.11 that A is closed and invariant and has the required attracting property, so it remains to prove that $\mathrm{law}_w(A)$ is a law-attractor. Write $\mathcal{A} = \mathrm{law}_w(A)$.

For invariance, using (4) gives

$$\widehat{S}_t\mathcal{A} = \mathrm{law}_w(S_t A) = \mathrm{law}_w(A) = \mathcal{A}$$

.

Using Proposition 13.9(b) in the Appendix), we have

$$\mathcal{A} = \mathrm{law}_w(A) = \mathrm{law}_w({}^\circ\mathcal{C}) = {}^{\circ*}\mathrm{law}_w(\mathcal{C}) = {}^\circ\bigcap\nolimits_{n\in\mathbb{N}} {}^*\mathrm{law}_w(\widehat{\mathcal{C}}_n).$$

Since ${}^*\mathrm{law}_w(\mathcal{C}) \subseteq \mathrm{ns}(\mathcal{M}_{1,2}(M \times C_0))$, the compactness of $\mathcal{A}$ follows from Proposition 13.2 in the Appendix with $E_n = {}^*\mathrm{law}_w(\widehat{\mathcal{C}}_n)$.

For the attraction property, suppose that $\mathcal{Z} \subseteq \mathrm{law}_w(X)$ is d-bounded, and $\mathcal{O} \supset \mathcal{A}$ is open. We will prove first that there is some $n \in \mathbb{N}$ with

$$\mathcal{A}_n = \mathrm{law}_w(A_n) \subseteq \mathcal{O} \tag{37}$$

and then that $\widehat{S}_t\mathcal{Z} \subseteq \mathcal{A}_n$ eventually.

For (37), let $F = \mathcal{M}_{1,2}(M \times C_0) \setminus \mathcal{O}$, so the hypothesis is that $\mathcal{A} \cap F = \emptyset$. This means that

$$\bigcap_{n\in\mathbb{N}} {}^*\mathrm{law}_w(\widehat{\mathcal{C}}_n) \ \cap \ \bigcap_n {}^*(F^{\leq 1/n}) = \emptyset,$$

since any member of this set has the form $\Lambda = \mathrm{law}_w(U)$ with $U \in \mathcal{C}$ and so Λ is nearstandard in $\mathcal{M}_{1,2}(M \times C_0)$ with ${}^\circ\Lambda \in \mathcal{A} \cap F$. Saturation gives an $m \in \mathbb{N}$ with

$$\bigcap_{n\leq m} {}^*\mathrm{law}_w(\widehat{\mathcal{C}}_n) \ \cap \ \bigcap_{n\leq m} {}^*(F^{\leq 1/n}) = \emptyset,$$

and so $\mathcal{A}_n \cap F = \emptyset$, which is (37).

Since $\mathcal{Z}$ is d-bounded, $\mathcal{Z} \subseteq \mathrm{law}_w(X_k)$ for some k. By Theorem 9.10, $S_tX_k \subseteq A_n$ eventually. Hence, eventually

$$\widehat{S}_t\mathcal{Z} \subseteq \widehat{S}_t(\mathrm{law}_w(X_k)) = \mathrm{law}_w(S_tX_k) \subseteq \mathrm{law}_w(A_n) = \mathcal{A}_n,$$

as required. □

The attraction property in Lemma 9.11 and its proof generalizes easily to the following, from which the law-attraction property of A can also be deduced.

Theorem 9.13 *If $\mathcal{O} \supseteq A$ is an open set whose complement has the form $\mathcal{O}^c = {}^\circ(\bigcap_{n\in\mathbb{N}} \mathcal{E}_n)$ with each $\mathcal{E}_n$ internal, then $A_n \subseteq \mathcal{O}$ eventually, and so $\mathcal{O}$ attracts bounded subsets of X.*

Remark This result will be generalized and explored further in the sequel to this paper, [18].

10 Two-sided solutions

In this section we formulate the notion of a *two-sided* solution to the Navier-Stokes equation (3) – that is, a solution defined for all time, negative and positive. It will be shown that the set $\bar{X}$ of two-sided solutions is non-empty and that the attractor A identified in the previous section is simply the set of restrictions to non-negative times of the solutions in $\bar{X}$. The corresponding fact for the deterministic equations is noted by Sell in [32].

We must first modify the space $\mathbf{\Omega}$ so that it can accommodate two-sided processes $u(t, \omega)$ with $t \in \mathbb{R}$. Recall that the underlying internal filtered probability space is

$$\bar{\mathbf{\Omega}} = (\Omega, \mathcal{G}, (\mathcal{G}_\tau)_{\tau \in {}^*\mathbb{R}}, Q),$$

where $\mathcal{G}_\tau = {}^*\sigma(\{\mathcal{W}(\tau') : \tau' \leq \tau\})$ and $\mathcal{G} = \bigvee_{\tau \in {}^*\mathbb{R}} \mathcal{G}_\tau$. So we simply extend the filtration $\mathcal{F}_t$ to negative times using the same recipe as before: for any $t \in \mathbb{R}$ define

$$\mathcal{F}_t = \bigcap_{t <^\circ \tau} \sigma(\mathcal{G}_\tau) \vee \mathcal{N}.$$

Thus, the space we now work with is

$$\mathbf{\Omega} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, P).$$

The Wiener process $w(t, \omega)$ is extended to negative times in the same way:

$$w(t, \omega) = {}^\circ W(t, \omega) \tag{38}$$

for all real times.

Definition 10.1 Denote by $\bar{X}$ the set of bounded two-sided solutions to the stochastic Navier-Stokes equations as follows. The members of $\bar{X}$ are adapted stochastic processes $u : \mathbb{R} \times \Omega \to \mathbf{H}$ with the following properties.

($\bar{\text{X}}$1) For a.a. ω the path $u(\cdot, \omega)$ belongs to the following spaces:

$$L^\infty_{\text{loc}}(-\infty, \infty; \mathbf{H}) \cap \mathbf{L}^{\mathbf{2}}_{\text{loc}}(-\infty,\infty;\mathbf{V}) \cap \mathbf{C}(-\infty,\infty;\mathbf{H}_{\text{weak}}).$$

($\bar{\text{X}}$2) For all $t_1 \geq t_0 \in \mathbb{R}$,

$$u(t_1) = u(t_0) + \int_{t_0}^{t_1} [-\nu A u(t) - B(u(t)) + f(u(t))]dt + \int_{t_0}^{t_1} g(u(t)) dw_t.$$

($\bar{\text{X}}$3) For a.a. $t_0 \in \mathbb{R}$ and all $t_1 \geq t_0$,

$$\mathbb{E}(|u(t_1)|^2) \leq \mathbb{E}(|u(t_0)|^2) \exp(-k_1(t_1 - t_0)) + k_2. \tag{39}$$

($\bar{\mathrm{X}}$4) For a.a. $t_0 \in \mathbb{R}$ and all $t_1 \geq t_0$,

$$\mathbb{E}\left(\sup_{t_0 \leq s \leq t_1} |u(s)|^2 + 2\nu \int_{t_0}^{t_1} \|u(s)\|^2 ds\right) \leq \alpha \mathbb{E}(|u(t_0)|^2) + \beta(t_1 - t_0). \tag{40}$$

($\bar{\mathrm{X}}$5) For a.a. $t_0 \in \mathbb{R}$ and all $t_1 \geq t_0$ and $n \in \mathbb{N}$,

$$\mathbb{E}(\varphi_n(u(t_1))) \leq \mathbb{E}(\varphi_n(u(t_0)) \exp(-k_3(t_1 - t_0)) + n^{-\frac{1}{2}}(\alpha \mathbb{E}(|u(t_0)^2|) + \beta). \tag{41}$$

($\bar{\mathrm{X}}$6) $\mathbb{E}(|u(t)|^2) \leq k_2$ for all $t \in \mathbb{R}$.

Remark It is immediate from the definitions that

$$\bar{X} \restriction [0, \infty) \subseteq Y \cap X_{k_2} \subseteq X.$$

Corresponding to the space M of paths for one-sided solutions, the natural space that contains paths of two-sided solutions to the stochastic Navier–Stokes equations is the space $\bar{M}$ defined as follows.

Definition 10.2 (a) For a measurable (deterministic) function $\xi : \mathbb{R} \to \mathbf{H}$ define a norm

$$|\xi| = \left(\int_{-\infty}^{\infty} \xi(t)^2 \exp(-|t|)dt\right)^{\frac{1}{2}} = \left(\int_{-\infty}^{\infty} \xi(t)^2 \mu(dt)\right)^{\frac{1}{2}},$$

where $\mu(dt) = \exp(-|t|)dt$, and write

$$\bar{M} = \{\xi : |\xi| < \infty\}$$

for this space of paths, which is a separable Hilbert space.

(b) For a process $u(t, \omega)$ with paths in $\bar{M}$ define

$$|u| = \left(\mathbb{E}(|u(\cdot, \omega)|^2)\right)^{\frac{1}{2}} = \left(\mathbb{E} \int_{-\infty}^{\infty} |u(t, \omega)|^2 \exp(-|t|)dt\right)^{\frac{1}{2}},$$

which is simply the norm of $L^2(\Omega, \bar{M})$.

Remark It is clear from ($\bar{\mathrm{X}}$6) that $\bar{X} \subseteq L^2(\Omega, \bar{M})$. In fact, $|u|^2 \leq 2k_2$ for all $u \in \bar{X}$.

It would be possible to proceed by adapting the basic existence result Theorem 6 to show that $\bar{X} \neq \emptyset$. However, this follows from the more detailed analysis of $\bar{X}$ to come, and in particular from the main theorem of this section:

Theorem 10.3

$$A = \bar{X} \restriction [0, \infty).$$

The proof of this requires some additional analysis involving approximate two-sided solutions, which we now define (the counterpart of Definition 7.1 for one-sided solutions).

Definition 10.4 (a) For each $n \in \mathbb{N}$ denote by $\bar{\mathcal{X}}_n$ the internal class of *-adapted (with respect to $(\mathcal{G}_\tau)_{\tau \in {}^*\mathbb{R}}$) processes $U : {}^*\mathbb{R} \times \Omega \to \mathbf{H}_N$ with the following properties:

($\bar{\mathcal{X}}1$) $U_\tau(\omega)$ has paths *a.s. in $^*\bar{M}$ and $U \in {}^*L^2(\Omega, {}^*\bar{M})$, i.e.

$$\mathbb{E}\left(\int_{-{}^*\infty}^{{}^*\infty} |U_\tau(\omega)|^2 \exp(-|\tau|) d\tau\right) < {}^*\infty.$$

($\bar{\mathcal{X}}2_n$) With Q-probability $\geq 1 - \frac{1}{n}$ on Ω, for all $\tau_1 \in {}^*[-n, n]$ and all $m \leq n$,

$$\left| U(\tau_1)_{(m)} - U(-n)_{(m)} - \int_{-n}^{\tau_1} [-(\nu A U_\tau)_{(m)} - B(U_\tau)_{(m)} + F(U_\tau)_{(m)}] d\tau - \int_{-n}^{\tau_1} G(U_\tau)_{(m)} dW_\tau \right| \leq 2^{-n}. \tag{42}$$

($\bar{\mathcal{X}}3_n$) For all $\tau_0 \in {}^*[-n, \infty)$ except for a set of *Lebesgue measure $\frac{1}{n}$, and for all $\tau_1 \geq \tau_0$,

$$\mathbb{E}(|U_{\tau_1}|^2) \leq \mathbb{E}(|U_{\tau_0}|^2) \exp(-k_1(\tau_1 - \tau_0)) + k_2 + \tfrac{1}{n}. \tag{43}$$

($\bar{\mathcal{X}}4_n$) For all $\tau_0 \in {}^*[-n, \infty)$ except for a set of *Lebesgue measure $\frac{1}{n}$, for all $\tau_1 \geq \tau_0$,

$$\mathbb{E}\left(\sup_{\tau_0 \leq \sigma \leq \tau_1} |U_\sigma| + \int_{\tau_0}^{\tau_1} \|U_\sigma\|^2 d\sigma\right) \leq \alpha \mathbb{E}(|U_{\tau_0}|^2) + \beta(\tau_1 - \tau_0) + \tfrac{1}{n}. \tag{44}$$

($\bar{\mathcal{X}}5_n$) For all $\tau_0 \in {}^*[-n, \infty)$ except for a set of *Lebesgue measure $\frac{1}{n}$, for all $\tau_1 \geq \tau_0$ and all $m \leq n$,

$$\mathbb{E}(\varphi_m(U_{\tau_1})) \leq \mathbb{E}(\varphi_m(U_{\tau_0})) \exp(-k_3(\tau_1 - \tau_0)) + m^{-\frac{1}{2}}(\alpha \mathbb{E}(|U_{\tau_0}|^2) + \beta) + \tfrac{1}{n}. \tag{45}$$

($\bar{\mathcal{X}}6_n$) $$\mathbb{E}(|U_\tau|^2) \leq k_2 + \frac{1}{n} \text{ for } \tau \in {}^*[-n, \infty).$$

(b) Define

$$\bar{\mathcal{X}} = \bigcap_{n \in \mathbb{N}} \bar{\mathcal{X}}_n.$$

Remark It is immediate from the definitions that

$$\bar{\mathcal{X}}_n \restriction {}^*[0,\infty) \subseteq \mathcal{X}_{k_2,n} \subseteq \mathcal{X}_n, \bar{\mathcal{X}} \restriction {}^*[0,\infty) \subseteq \mathcal{X}_{k_2} \subseteq \mathcal{X}.$$

The notion of the standard part of an internal process from earlier sections extends naturally to two-sided internal processes. Corresponding to Theorem 7.6 (a) we have:

Theorem 10.5 *If* $U \in \bar{\mathcal{X}}$ *then*

(a) *For* P*-almost all* ω *the path* $U(\cdot,\omega)$ *has* $|U(\tau,\omega)|$ *finite and* $U(\tau,\omega)_{(m)}$ *S-continuous for* $m \in \mathbb{N}$ *and all finite* τ *(positive and negative).*

(b) *The process* $u = {}^\circ U$ *defined by*

$$u(t,\omega) = {}^\circ U(\tau,\omega)$$

for $\tau \approx t$ *(where* ${}^\circ U$ *denotes the weak standard part in* $\mathbf{H}$*) belongs to* $\bar{X}$*.*

Proof We first show that, unlike Theorem 7.6, it is not necessary to restrict to internal processes U that are S-integrable in some sense.

Take $U \in \bar{\mathcal{X}}$; then $U \in \bar{\mathcal{X}}_J$ for some infinite J. From ($\bar{\mathcal{X}}6$) we have that $\mathbb{E}(|U_\tau|^2) < k_2 + \frac{1}{J}$ for all finite τ. Then ($\bar{\mathcal{X}}5_J$) ensures that $|U_\tau(\omega)|^2$ is S-integrable for all finite τ, using the criterion of Corollary 13.8(b) of the Appendix.

Now simply follow the proof of Theorem 7.6 (a) with the origin $t = 0$ moved to $t = -n$ to construct $u = {}^\circ U$ with u fulfilling the requirements of a two-sided solution on each time interval $[-n,\infty)$. Consequently $u \in \bar{X}$. □

We also have the converse of Theorem 10.5:

Theorem 10.6

$$\bar{X} = {}^\circ\bar{\mathcal{X}}.$$

Proof Let $u \in \bar{\mathrm{X}}$. Simply adapt the proof of Theorem 7.8 to obtain $U^n \in \bar{\mathcal{X}}_n$ for each n with ${}^\circ U^n \restriction {}^*[-n,\infty) = u \restriction [-n,\infty)$. Then for all sufficiently small infinite $J \in {}^*\mathbb{N}$ we have $U = U^J \in \bar{\mathcal{X}}$ and ${}^\circ U = u$. □

To continue we must introduce a group of shift operators on ${}^*L^2(\Omega,\bar{M})$ as follows:

Definition 10.7 For $V \in {}^*L^2(\Omega,\bar{M})$ define the left shift operator L_τ for $\tau \in {}^*\mathbb{R}$ by

$$(L_\tau V)(\sigma,\omega) = V(\tau+\sigma,\Theta_\tau\omega).$$

Clearly $L_\tau \circ L_{-\tau} =$ identity for all τ, and $L_\sigma \circ L_\tau = L_{\sigma+\tau}$, so the left shift operators L_τ form an internal group of mappings on ${}^*L^2(\Omega, \bar{M})$ under composition.

It is also clear from the definitions that $L_\tau \bar{\mathcal{X}} = \bar{\mathcal{X}}$ for every finite τ.

The connection with the internal semigroup T_τ is given by:

Lemma 10.8 *For any* $V \in {}^*L^2(\Omega, \bar{M})$,

$$T_\tau(V \restriction {}^*[0,\infty)) = (L_\tau V) \restriction {}^*[0,\infty)). \tag{46}$$

Proof From the definitions. □

To proceed it is necessary to make one further definition, the counterpart of $\mathcal{C}$ for two-sided solutions. Recall the set $\mathcal{B} = \mathcal{X}_\rho$ of one-sided approximate solutions.

Definition 10.9

(a) Define $\bar{\mathcal{B}}$ by

$$\bar{\mathcal{B}} = \{U \in {}^*L^2(\Omega, \bar{M}) : U \restriction {}^*[0,\infty) \in \mathcal{B}\},$$

(i.e. the processes in $\mathcal{B}$ are extended to negative times in an arbitrary manner) and in the same way for each $k \in \mathbb{N}$ define

$$\bar{\mathcal{B}}_k = \{U \in {}^*L^2(\Omega, \bar{M}) : U \restriction {}^*[0,\infty) \in \mathcal{B}_k\},$$

(b) Define

$$\mathcal{D} = \bigcap \{L_\tau \bar{\mathcal{B}} : \tau \geq 0, \tau \text{ finite}\}.$$

Lemma 10.10 $\bar{\mathcal{X}} \subseteq \bar{\mathcal{B}}$.

Proof Let $U \in \bar{\mathcal{X}}$ and let $r = r(k_2)$. Since $\bar{\mathcal{X}} = L_r(\bar{\mathcal{X}})$, $U = L_r V$ for some $V \in \bar{\mathcal{X}}$. Then $V \restriction {}^*[0,\infty) \in \mathcal{X}_{k_2}$, and by (46) we have

$$U \restriction {}^*[0,\infty) = (L_r V) \restriction {}^*[0,\infty) = T_r(V \restriction {}^*[0,\infty)) \in T_r \mathcal{X}_{k_2} \subseteq \mathcal{B}.$$

Therefore $U \in \bar{\mathcal{B}}$. □

The main result will follow from its counterpart at the level of approximate solutions:

Theorem 10.11 (a) $\mathcal{D} = \bar{\mathcal{X}}$.

(b) $\mathcal{C} = \mathcal{D} \restriction {}^*[0,\infty)$.

Proof (a) We have $\bar{\mathcal{X}} \subseteq \bar{\mathcal{B}}$ and $L_\tau \bar{\mathcal{X}} = \bar{\mathcal{X}}$ for finite τ, so

$$\bar{\mathcal{X}} = L_\tau \bar{\mathcal{X}} \subseteq L_\tau \bar{\mathcal{B}}$$

for finite $\tau \geq 0$. Hence $\bar{\mathcal{X}} \subseteq \mathcal{D}$.

Conversely, let $V \in \mathcal{D}$. Then for each $n \in \mathbb{N}$ we have $V = L_n V_n$ for some $V_n \in \bar{\mathcal{B}}_{2n}$. Overflow gives an infinite $J \in {}^*\mathbb{N}$ with $V = L_J V_J$ and $V_J \in \bar{\mathcal{B}}_{2J}$. Thus $V_J \restriction {}^*[0,\infty) = U_J$ is in $\mathcal{B}_{2J}$. From the definition of $\mathcal{B}_{2J}$ we can read off the properties of V on the interval ${}^*[-J,\infty)$ and check that $V \in \bar{\mathcal{X}}$. For example, the property $(\mathcal{X}2_{2J})$ for U_J gives the approximate equation (22) on ${}^*[1/(2J), 2J]$, and so V satisfies the same equation on ${}^*[-J+1/(2J), J]$, which ensures that V satisfies condition $(\bar{\mathcal{X}}2_n)$ for all $n \in \mathbb{N}$.

The rest of the properties needed for $V \in \bar{\mathcal{X}}$ are easily checked.

(b) Writing $q(V) = V \restriction {}^*[0,\infty)$ for a two-sided process V we have by definition

$$\bar{\mathcal{B}} = q^{-1}(\mathcal{B}) \quad \text{and} \quad \mathcal{D} = \bigcap_{0 \leq \tau < \infty} L_\tau \bar{\mathcal{B}}.$$

Thus

$$\begin{aligned}
\mathcal{D} \restriction {}^*[0,\infty) &= q(\mathcal{D}) \\
&= q\left(\bigcap_{0 \leq \tau < \infty} L_\tau(q^{-1}(\mathcal{B}))\right) \\
&\subseteq \bigcap_{0 \leq \tau < \infty} q(L_\tau(q^{-1}(\mathcal{B}))) \\
&= \bigcap_{0 \leq \tau < \infty} T_\tau(q(q^{-1}(\mathcal{B}))) \\
&= \bigcap_{0 \leq \tau < \infty} T_\tau \mathcal{B} \\
&= \mathcal{C},
\end{aligned}$$

giving one inclusion. For the other direction, let $U \in \mathcal{C}$. So $U = T_n U_n$ with $U_n \in \mathcal{B}_{2n}$ for each $n \in \mathbb{N}$. By overflow we have $U = T_J U_J$ for some infinite $J \in {}^*\mathbb{N}$, with $U_J \in \mathcal{B}_{2J}$. Take any V_J such that $V_J \restriction {}^*[0,\infty) = U_J$, so that $(L_J V_J) \restriction {}^*[0,\infty) = T_J U_J = U$, using (46). It now suffices to show that $L_J V_J \in \mathcal{D}$.

For any finite $\tau \geq 0$,

$$L_J V_J = L_\tau \circ L_{J-\tau} V_J$$

Now

$$(L_{J-\tau} V_J) \restriction {}^*[0,\infty) = T_{J-\tau}(V_J \restriction {}^*[0,\infty)) = T_{J-\tau} U_J \in \mathcal{B}_J,$$

using (46) again, and Proposition 8.8. This gives $L_{J-\tau} V_J \in \bar{\mathcal{B}}_J \subseteq \bar{\mathcal{B}}$, and so $L_J V_J \in L_\tau \bar{\mathcal{B}}$. □

As a simple corollary we now have the main result of this section.

Theorem 10.12 (=Theorem 10.3)

$$A = \bar{X} \restriction [0, \infty).$$

Proof

$$A = {}^\circ\mathcal{C} = {}^\circ(\mathcal{D} \restriction {}^*[0,\infty)) = {}^\circ(\bar{\mathcal{X}} \restriction {}^*[0,\infty)) = \bar{X} \restriction [0,\infty)$$

using Theorems 10.6 and 10.11. □

The next theorem is proved in the same way as the corresponding results for A:

Theorem 10.13 $\mathrm{law}_w(\bar{X})$ *is compact.*

Proof Modify the proof that $\mathrm{law}_w(A)$ is compact in Theorem 9.12. □

11 Noncompactness Results

In this section we prove results showing that one usually cannot expect the attractor for the stochastic Navier-Stokes equation to be a compact set.

Theorem 11.1 *Suppose that $\mathbf{\Omega} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, P)$ is an arbitrary filtered probability space carrying a two-sided Brownian motion and suppose there is an attractor A for the stochastic Navier-Stokes equations satisfying Theorem 10.12. Suppose further that A has at least two distinct elements u, v, and there are sets $Z_n, n \in \mathbb{N}$ in $\mathcal{F}_{-\infty}$ of measure $1/2$ that are independent of each other and of w, u, v. Then A is not compact in the topology of convergence in probability (a fortiori, A is not compact in $L^2(\Omega, M)$).*

Proof We have $P[|u - v| \geq r] \geq r$ for some $r > 0$. By definition of A, there exist two-sided solutions $u', v' \in X$ such that $u_t = u'_t$ and $v_t = v'_t$ for all $t \geq 0$. For each n, define the process x_n on $\mathbf{\Omega}$ by

$$x_n(\omega) = \begin{cases} u(\omega) & \text{if } \omega \in Z_n \\ v(\omega) & \text{otherwise.} \end{cases}$$

and define x'_n analogously. Since Z_n is $\mathcal{F}_{-\infty}$-measurable, it follows that each x'_n is a two-sided solution of the Navier-Stokes equation belonging to $\bar{X}$, and therefore each x_n belongs to A. For each $m \neq n$,

$$P[|x_m - x_n| \geq r] \geq r \cdot P[Z_m \Delta Z_n] = r/2,$$

since $\{|x_m - x_n| \geq r\} \supseteq \{|u - v| \geq r]\} \cap (Z_m \Delta Z_n)$. Therefore A is not compact in the topology of convergence in probability. □

Remark It is well-known that when the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}}$ is generated by the Wiener process w_t $(t \in \mathbb{R})$, $\mathcal{F}_{-\infty}$ is the trivial algebra (e.g. see [11], page 583), so the above theorem does not apply. In all known proofs of existence for solutions to the stochastic Navier-Stokes equations in 3-dimensions, however, the space required is richer than the Wiener space - for example the Loeb space that we have used in the present paper. Here the non-compactness of A can be verified, due to the following consequence of the richness of the space.

Lemma 11.2 *In the filtered Loeb space* $\mathbf{\Omega}$ *of Section 10, for any countable collection of Loeb measurable sets* $y_0, y_1, \ldots$ *there exists an* $\mathcal{F}_{-\infty}$*-measurable set* Z *of measure* $1/2$ *which is independent of each* y_n*.*

Proof Let Q be the internal measure on Ω which generates the Loeb space $\mathbf{\Omega}$. For each $n \in \mathbb{N}$, let Y_n be an internal approximation of y_n. For each $h \in \mathbb{N}$ there is a finite internal partition of Ω into $\mathcal{G}_{-h}$-measurable sets of Q-measure $1/h$. By saturation, there is a hyperfinite partition $X_i, i \leq H$ of Ω into $\mathcal{G}_{-H}$-measurable sets of Q-measure $1/H$. Let P_H be the internal counting measure on 2^H. For each $\gamma \in 2^H$ let $Z(\gamma)$ be the internal $\mathcal{F}_{-\infty}$-measurable set

$$Z(\gamma) = \bigcup \{X_i : i \leq H, \gamma(i) = 1\}.$$

Fix $n \in \mathbb{N}$ for the moment and let $Y = Y_n$. It is sufficient to show that for a.a. γ we have $Q(Z(\gamma)) \approx \frac{1}{2}$ and $Z(\gamma)$ is independent of Y with respect to the Loeb measure, – i.e. $Q(Z(\gamma) \cap Y) \approx \frac{1}{2}Q(Y)$.

For each $1 \leq i \leq H$ and $\gamma \in 2^H$, define the random variable $\eta_i(\gamma)$ by

$$\begin{aligned}\eta_i(\gamma) &= \left(Q(Y \cap Z(\gamma) \cap X_i) - \frac{1}{2}Q(Y \cap X_i)\right) \\ &= \begin{cases} \frac{1}{2}Q(Y \cap X_i) & \text{if } \gamma(i) = 1 \\ -\frac{1}{2}Q(Y \cap X_i) & \text{if } \gamma(i) = 0. \end{cases}\end{aligned}$$

With respect to P_H, the random variables $\eta_i, i \leq H$, are mutually independent and have expected values $\mathbb{E}\eta_i = 0$. Moreover, $|\eta_i(\gamma)| \leq (2H)^{-1}$, so η_i has variance $\leq (2H)^{-2}$ with respect to P_H.

Define a random variable

$$\xi(\gamma) = \sum_{i=1}^{H} \eta_i(\gamma).$$

Then with respect to P_H,

$$\mathbb{E}(\xi^2) = \sum_{i=1}^{H} \mathbb{E}(\eta_i^2) \leq \frac{1}{4H} \approx 0.$$

Since the sets X_i partition Ω, we have

$$\sum_{i=1}^{H} Q(Y \cap Z(\gamma) \cap X_i) = Q\left(\bigcup_{i=1}^{H}(Y \cap Z(\gamma) \cap X_i)\right) = Q(Y \cap Z(\gamma))$$

and

$$\sum_{i=1}^{H} Q(Y \cap X_i) = Q\left(\bigcup_{i=1}^{H}(Y \cap X_i)\right) = Q(Y).$$

Therefore

$$\xi(\gamma) = Q(Y \cap Z(\gamma)) - \frac{1}{2}Q(Y).$$

Thus, for $(P_H)_L$-a.a. γ we have $Q(Y \cap Z(\gamma)) - \frac{1}{2}Q(Y) \approx 0$, and so $Q(Z(\gamma) \cap Y) \approx \frac{1}{2}Q(Y)$ as required. Applying this to $Y = \Omega$ also gives that $Q(Z(\gamma)) \approx \frac{1}{2}$ for a.a. γ. □

Corollary 11.3 *In the filtered Loeb space* $\mathbf{\Omega}$ *of Section 10, if the Navier-Stokes attractor has more than one element, then it is not compact in the topology of convergence in probability.*

To see how, in a general setting, simply enlarging the space eliminates compactness of an attractor, we have the following.

Let $\mathbf{\Omega} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, P)$ be a filtered probability space equipped with a family of measure preserving maps $(\theta_t)_{t\geq 0}$ satisfying properties $(\theta 1, \theta 2, \theta 3)$ of Section 3. Let $(\Omega', \mathcal{F}', P')$ be another probability space. The **product** $\bar{\mathbf{\Omega}} = \mathbf{\Omega}\times\mathbf{\Omega}'$ is the filtered probability space $(\Omega\times\Omega', \mathcal{F}\times\mathcal{F}', (\mathcal{F}_t\times\mathcal{F}')_{t\in\mathbb{R}}, P\times P')$ with the measure preserving maps $(\bar{\theta})_t(\omega,\gamma) = (\theta_t(\omega), \gamma)$.

For a stochastic process u on $\mathbf{\Omega}$, let $\bar{u}$ be the process defined on $\bar{\Omega}$ by $\bar{u}_t(\omega,\gamma) = u_t(\omega)$.

If w is a two-sided Brownian motion on $\mathbf{\Omega}$, then $\bar{w}$ is a two-sided Brownian motion on $\bar{\mathbf{\Omega}}$.

Note that for each set $D \in \mathcal{F}'$, $\Omega \times D$ is $\mathcal{F}_{-\infty} \times \mathcal{F}'$-measurable.

With this notation we have the following non-compactness result.

Corollary 11.4 *Let* w *be a two-sided Brownian motion in* $\mathbf{\Omega}$*, and suppose the stochastic Navier-Stokes equations have an attractor* A *satisfying Theorem 10.12, and* A *has at least two distinct elements. Let* $\mathbf{\Omega}'$ *be an atomless probability space, and let* $\bar{A}$ *be the stochastic Navier-Stokes attractor for* $(\bar{\mathbf{\Omega}}, \bar{w})$*. Then* $\bar{A}$ *is not compact in the topology of convergence in probability.*

12 Final Remarks

On any filtered probability space $\mathbf{\Omega}$ with a family of measure preserving maps θ_t as in section 3 we can define the sets X_k, X and $\bar{X}$ of solutions and a set A by

$$A = \bigcap S_n B$$

with $B = X_\rho$. A natural question is to isolate the properties of $\mathbf{\Omega}$ that are needed to give the main results above – namely that $A = \bar{X} \restriction [0, \infty)$ and that A is an attractor in some sense (and the above results about compactness of families of laws hold).

A further natural question is to isolate the particular properties of the Navier-Stokes equations and/or the families of solutions X that are needed to develop a theory such as this.

We will present one approach to these questions in the sequel [18] to this paper.

13 Appendix: Nonstandard preliminaries

We work in an $\aleph_1$-saturated nonstandard universe that contains a nonstandard extension ${}^*\mathcal{J}$ for every mathematical object $\mathcal{J}$ involved in our theory. In particular we have ${}^*\mathbf{H}, {}^*M, {}^*C_0(\mathbb{R}), {}^*$Wiener measure, etc.

Given a standard Hausdorff space S, we identify each point $x \in S$ with ${}^*x$, so that $S \subseteq {}^*S$. If $x \in S$ and $X \in {}^*S$, we say that x is the *standard part of* X, in symbols $x = {}^\circ X$, if $X \in {}^*O$ for every open neighborhood O of x. Since S is Hausdorff, each $X \in {}^*S$ has at most one standard part. An element $X \in {}^*S$ is said to be *near-standard*, in symbols $X \in \mathrm{ns}(S)$, if X has a standard part. Thus the standard part function maps $\mathrm{ns}(S)$ onto S and is the identity on S. The *standard part* of a set $B \subseteq \mathrm{ns}(S)$ is the set ${}^\circ B = \{{}^\circ X : X \in B\}$. Here is a useful immediate consequence of the definition of standard part.

Remark 13.1 Suppose $x = {}^\circ X$ in a Hausdorff space S.
(a) If O is open in S, $x \in O$ implies $X \in {}^*O$.
(b) If C is closed in S, $X \in {}^*C$ implies $x \in C$.

In the particular case of a standard metric space (S, ρ), $x = {}^\circ X$ if and only if ${}^*\rho(X, x) \approx 0$, and two points $X, Y \in {}^*S$ are said to be *infinitely close*, in symbols $X \approx Y$, if ${}^*\rho(X, Y) \approx 0$.

We need the following compactness criterion ([20], Lemma 4.9).

Proposition 13.2 *Let S be a standard metric space, let E_n be an internal subset of ${}^*S$ for each n, and let $E = \bigcap_n E_n$.*
(a) *${}^\circ(E \cap \mathrm{ns}(S))$ is closed.*
(b) *If $E \subseteq \mathrm{ns}(S)$, then ${}^\circ E$ is compact.*

Proof We give a proof here for the sake of completeness, and so we can refer to the proof later on. We may assume without loss of generality that $E_1 \supseteq E_2 \supseteq \cdots$.

(a) Suppose $x_n \in {}^\circ(E \cap \mathrm{ns}(S))$ and $x_n \to x \in S$. Take a subsequence x_m such that $\rho(x_m, x) < 1/m$ for each $m \in \mathbb{N}$. Choose $X_m \in E \subseteq E_m$ with

${}^\circ X_m = x_m$, and choose $X \in {}^*S$ with ${}^\circ X = x$. Then ${}^*\rho(X_m, X) < 1/m$ for each m. By the triangle inequality, ${}^*\rho(X_m, X_n) < 2/m$ whenever $m \leq n$. By $\aleph_1$-saturation, there is a $Y \in {}^*S$ such that $Y \in E_n$ for each n, and ${}^*\rho(X_m, Y) < 2/m$ for each m. Then $Y \in E$ and $Y \approx X$, so $x = {}^\circ Y$ and $Y \in E \cap \mathrm{ns}(S)$. Thus $x \in {}^\circ(E \cap \mathrm{ns}(S))$, so ${}^\circ(E \cap \mathrm{ns}(S))$ is closed.

(b) Since S is a metric space, to show that ${}^\circ E$ is compact it suffices to show that every countable open cover $\{O_n\}$ of ${}^\circ E$ has a finite subcover. Let $A_{mn} = \{x \in S : \rho(x, S \setminus O_m) > \frac{1}{n}\}$. Then A_{mn} is open, $\overline{A_{mn}} \subseteq O_m$, and $\{A_{mn}\}$ covers ${}^\circ E$. By 13.1 (a), $\{{}^*A_{mn}\}$ covers E. By $\aleph_1$-saturation, there is a k such that $\{{}^*A_{mn} : m, n \leq k\}$ covers E_k. Then by 13.1 (b), $\{\overline{A_{mn}} : m, n \leq k\}$ covers ${}^\circ E$. Therefore $\{O_m : m \leq k\}$ covers ${}^\circ E$. □

The book [7] gives information about the standard part mapping for various topologies on the standard set $\mathbf{H}$. The most important are as follows. Here, ${}^*\mathbf{H}$ has an internal *basis $\{{}^*e_n\}_{n \in {}^*\mathbb{N}}$, and we write $E_n = {}^*e_n$. Thus for each $N \in {}^*\mathbb{N}$, $\mathbf{H}_N = {}^*\mathrm{span}\{E_1, \ldots, E_N\} \subseteq {}^*\mathbf{H}$. We also write $u_{(n)} = (u, e_n)$ and $U_{(n)} = (U, E_n)$.

Lemma 13.3 *Let $U \in \mathbf{H}_N$. Then:*

(a) *If $|U| < \infty$ (i.e. $|U|$ is finite) then U is weakly nearstandard in $\mathbf{H}$, and the weak standard part $u = \mathrm{st}_{\mathrm{weak}}(U)$ is defined by*

$$u_{(n)} = {}^\circ(U_{(n)}), n \in \mathbb{N}.$$

(b) *If U is nearstandard in the strong topology of $\mathbf{H}$ then $|U| < \infty$ and*

$$\mathrm{st}_{\mathrm{weak}}(U) = \mathrm{st}(U).$$

(c) *If $\|U\| < \infty$ then U is (strongly) nearstandard in $\mathbf{H}$.*

In view of the consistency (b) above we use ${}^\circ U$ to denote the standard part of U whenever $|U|$ is finite.

Near-standard points and standard parts also appear in the setting of Loeb spaces. For an internal probability space $\mathbf{\Omega} = (\Omega, \mathcal{G}, Q)$ we write Q_L for the corresponding Loeb measure. The theory of Loeb measure and Loeb integration is assumed (see [7, 16, 17] for example). For convenience, we assume here that $\mathbf{\Omega}$ is *countably additive, although most of the general theory carries over to the *finitely additive case. Recall that a *measurable function $U : \Omega \to {}^*\mathbb{R}$ is S-*integrable* if $\mathbb{E}(|U|)$ is finite and $\int_{|U(\omega)| \geq J} |U| dQ(\omega) \approx 0$ for all infinite $J \in {}^*\mathbb{N}$. The following notation, taken from [19, 20], is used.

Definition 13.4 Suppose that (S, ρ) is a separable metric space (we will mainly have $S = M$ and $S = M \times C_0$).

(a) $L^0(\Omega, S)$ is the space of all random variables on $v : \Omega \to S$ with the topology of convergence in probability, and for $0 < p \le \infty$ the spaces $L^p(\Omega, S)$ are defined as usual.

(b) $SL^0(\Omega, S) = {}^*L^0(\Omega, {}^*S) = \{V : \Omega \to {}^*S; V \text{ is } \mathcal{G}\text{-measurable}\}$.

(c) $\mathrm{ns}^0(\Omega, S) = \{V \in SL^0(\Omega, S) : V(\omega) \in \mathrm{ns}(S) \quad \text{for } Q_L \text{ a.a. } \quad \omega \in \Omega\}$. If $V : \Omega \to {}^*S$ is in $\mathrm{ns}^0(\Omega, S)$ then we write $v = {}^\circ V$ for the member of $L^0(\Omega, S)$ given by

$$({}^\circ V)(\omega) = {}^\circ(V(\omega)),$$

and we say that V is a *lifting* of v and that v is the *standard part* of V.

(d) For $p \in [1, \infty)$, $SL^p(\Omega, S)$ is the set of all $V \in {}^*L^0(\Omega, S)$ such that $\rho(V(\omega), z)^p$ is S-integrable for some (equivalently, any) $z \in S$. If $V \in SL^p(\Omega, S)$, we say that V is SL^p on Ω.

(e) For $p \in [1, \infty)$,

$$\mathrm{ns}^p(\Omega, S) = \mathrm{ns}^0(\Omega, S) \cap SL^p(\Omega, S).$$

(Note that $\mathrm{ns}^p(\Omega, S) \subseteq {}^*L^p(\Omega, {}^*S)$, and if $V \in \mathrm{ns}^p(\Omega, S)$ then ${}^\circ V \in L^p(\Omega, S)$)

(f) Since $p = 2$ occurs frequently, we write $\mathrm{NS} = \mathrm{ns}^2(\Omega, S)$ with the particular Ω and S being clear from the context.

This notation is actually a generalization of the standard part in stars of Hausdorff spaces. In the case that $\Omega = \{\omega\}$ is a one-point space, the points of $SL^0(\{\omega\}, S)$ can be identified with the points of ${}^*S$ in the obvious way, so that

$$SL^0(\{\omega\}, S) = SL^p(\{\omega\}, S) = {}^*S, \mathrm{ns}^0(\{\omega\}, S) = \mathrm{ns}^p(\{\omega\}, S) = \mathrm{ns}(S),$$

and the standard part mapping for $\mathrm{ns}^0(\{\omega\}, S)$ is the same as for $\mathrm{ns}(S)$.

Given a set $B \subseteq SL^p(\Omega, S)$, where $p \in \{0\} \cup [1, \infty)$, the *standard part* of B is the set ${}^\circ B = \{{}^\circ V : V \in B\}$. The following result is needed:

Proposition 13.5 *Let* $E = \bigcap_{n \in \mathbb{N}} E_n$ *where each set* $E_n \subseteq SL^0(\Omega, S)$ *is internal. Then for each* $p \in \{0\} \cup [1, \infty)$*, the set*

$${}^\circ(E \cap \mathrm{ns}^p(\Omega, S))$$

is closed in $L^p(\Omega, S)$*.*

Proof Exactly the same as the proof of Proposition 13.2 (a). □

We need the following fact about S-integrability on a product space.

Proposition 13.6 *Suppose* $V : {}^*[0, 1] \times \Omega \to {}^*[0, \infty)$ *is* **-measurable on the product, and* $V(\tau, \cdot)$ *is S-integrable on* Ω *for each* τ*. Then* V *is S-integrable on the product.*

Proof This result is well-known, but we include a proof for completeness. Take any infinite $J \in {}^*\mathbb{N}$. For each $\tau \in {}^*[0,1]$, $\int V(\tau,\omega)dQ(\omega)$ is finite, and by overspill, $\int V(\tau,\omega)dQ(\omega)$ has a uniform finite bound. Therefore $\int_0^1 \int V(\tau,\omega)dQ(\omega)d\tau$ is finite. Moreover, for each infinite $J \in \mathbb{N}$,

$$\int_{V(\tau,\omega)\geq J} V(\tau,\omega)dQ(\omega) \approx 0.$$

Therefore

$$\int_0^1 \int_{V(\tau,\omega)\geq J} V(\tau,\omega)dQ(\omega)d\tau \approx 0.$$

By the transfer of the Fubini Theorem, it follows that V is S-integrable on the product. □

The following proposition and corollary explain how the "truncation" functions ψ_n and φ_n are used to characterize S-integrability.

Proposition 13.7 *Let* $V : \Omega \to {}^*[0,\infty)$ *be an internal random variable. Then the following are equivalent.*

(a) V *is S-integrable.*

(b) $\mathbb{E}(V\psi_J(V)) \approx 0$ *for all sufficiently small infinite* J.

(c) ${}^\circ\mathbb{E}(V\psi_n(V)) \to 0$ *as* $n \to \infty$, *(*$n \in \mathbb{N}$*).*

Proof Assume (a), so that $\int_{V\geq K} V \approx 0$ for all infinite $K \in {}^*\mathbb{R}$. Writing

$$\mathbb{E}(V\psi_J(V)) = \int_{V<K} V\psi_J(V) + \int_{V\geq K} V\psi_J(V)$$

for infinite K we see that the second term on the right is infinitesimal since $0 \leq \psi_J \leq 1$. For the first term, if we now set $K = J^{\frac{1}{2}}$ we have

$$\int_{V<K} V\psi_J(V) \leq K\psi_J(K) = K\left(\left(\frac{K}{J^2}-1\right)^3+1\right) \approx 0,$$

which establishes (b). The converse implication (assuming (b)) is trivial since $\mathbb{E}(V\psi_J(V)) \geq \int_{V\geq J} V$. The equivalence of (b) and (c) is routine. □

Recalling that $\varphi_n(u) = |u|^2\psi_n(|u|^2)$ in any Hilbert space, we have

Corollary 13.8 *Let* $U : \Omega \to {}^*S$ *be an internal random vector where* S *is a Hilbert space. The following are equivalent.*

(a) $|U|^2$ *is S-integrable, that is,* $U \in SL^2(\Omega, S)$.

(b) $\mathbb{E}(\varphi_J(U)) \approx 0$ *for all sufficiently small infinite* J.

(c) $^\circ\mathbb{E}(\varphi_n(U)) \to 0$ *as* $n \to \infty$, $(n \in \mathbb{N})$.

As indicated in Section 3.2, the space of all Borel probability measures on a separable metric space S is denoted by $\mathcal{M}_1(S)$, and for $v \in L^0(\Omega, S)$, $\mathrm{law}(v) \in \mathcal{M}_1(S)$ is the probability induced by v. The Prohorov metric on $\mathcal{M}_1(S)$ gives weak convergence of measures, and part (a) of the following is well-known (see [20] Prop 5.7 for example).

Proposition 13.9 *Let* $V \in SL^0(\Omega, S)$.

(a)

$$V \in \mathrm{ns}^0(\Omega, S) \Leftrightarrow {}^*\mathrm{law}(V) \in \mathrm{ns}(\mathcal{M}_1(S)),$$

and if either side holds then $\mathrm{law}(^\circ V) = {}^\circ({}^*\mathrm{law}(V))$.

(b) *For the particular case* $S = M$

$$V \in \mathrm{NS} = \mathrm{ns}^2(\Omega, M) \Leftrightarrow {}^*\mathrm{law}(V) \in \mathrm{ns}(\mathcal{M}_{1,2}(M)),$$

and if either side holds then $\mathrm{law}(^\circ V) = {}^\circ({}^*\mathrm{law}(V))$.

Proof (b) follows from (a) together with the observation that for $\Lambda = {}^*\mathrm{law}(V) \in \mathrm{ns}(\mathcal{M}_{1,2}(M))$ there is the additional requirement that

$$\mathbb{E}_\Lambda |U|^2 = \int |V(\omega)|^2 dP(\omega) \approx \int |^\circ V(\omega)|^2 dP_L = \mathbb{E}_\lambda |u|^2,$$

where $\lambda = \mathrm{law}(^\circ V)$. This condition is precisely that $V(\omega)$ is SL^2 on Ω, which ensures that $V \in \mathrm{NS}$. □

Finally, we need the following result from [26], Theorem 8.1, on liftings of continuous adapted stochastic processes. (The equivalence of (a) and (b) is proved there, and it is an easy exercise to show that (b) is equivalent to (c)).

Proposition 13.10 *Let* $x : [0, \infty) \times \Omega \to \mathbb{R}$ *be a stochastic process in the filtered Loeb space* $\boldsymbol{\Omega}$ *given in Definition 5.1. The following are equivalent.*

(a) x *is adapted and almost surely continuous.*

(b) *There is an internal process* $X : {}^*[0, \infty) \times \Omega \to {}^*\mathbb{R}$ *and a positive* $\delta \approx 0$ *such that* $X(\cdot, \tau)$ *is* $\mathcal{G}_\tau$*-measurable for all* $\tau \geq \delta$*, and for almost all* ω*,* $^\circ X(\tau, \omega) = x(^\circ\tau, \omega)$ *for all finite* τ.

(c) *There is an internal* $\mathcal{G}_\tau$*-adapted process* $X : {}^*[0, \infty) \times \Omega \to {}^*\mathbb{R}$ *and a positive* $\delta \approx 0$ *such that* $X(\omega, \tau) = 0$ *for* $\tau < \delta$*, and for almost all* ω*,* $^\circ X(\tau, \omega) = x(^\circ\tau, \omega)$ *for all finite* $\tau \geq \delta$.

References

[1] S. Albeverio, J.-E. Fenstad, R. Høegh-Krohn, and T.L. Lindstrøm, *Nonstandard Methods in Stochastic Analysis and Mathematical Physics*, Academic Press, New York 1986.

[2] J.M. Ball, Continuity properties and global attractors of generalised semiflows and the Navier–Stokes equations, *J. Nonlinear Science,* **7**(1997), 475-502.

[3] M. Capiński and N.J. Cutland, A simple proof of existence of weak and statistical solutions of Navier–Stokes equations, *Proc. Royal. Soc. London* A **436** (1992), 1-11.

[4] M. Capiński & N.J. Cutland, Stochastic Navier-Stokes equations, *Applicandae Mathematicae* **25**(1991), 59-85.

[5] M. Capiński & N.J. Cutland, Statistical soutions of stochastic Navier-Stokes equations, *Indiana University Math Journal* **43**(1994), 927-940.

[6] M. Capiński & N.J. Cutland, Measure attractors for stochastic Navier–Stokes equations, *Electronic J. Prob.* **3**(1998), Paper 8, 1-15.

[7] M.Capiński, N.J.Cutland, *Nonstandard Methods for Stochastic Fluid Mechanics*, Series on Advances in Mathematics for Applied Sciences, Vol 27, World Scientific, Singapore 1995.

[8] M. Capiński & N.J. Cutland, Attractors for 3-D Navier–Stokes equations, *Proc. Royal Society London A* **453**(1997), 2413-2426.

[9] M. Capiński & N.J. Cutland, Existence of global stochastic flow and attractors for Navier–Stokes equations, *Probab. Theory Relat. Fields* **115**(1999), 121-151.

[10] H.Crauel and F.Flandoli, Attractors for random dynamical systems, *Probability Theory and Related Fields* **100** (1994), 365–393.

[11] J.L.Doob, *Classical Potential Theory and its Probabilistic Counterpart*, Springer-Verlag 1984.

[12] F. Flandoli and B. Schmalfuss, Random attractors for the 3D stochastic Navier–Stokes equation with multiplicative white noise, *Stochastic & Stochastics Reports,* **59**(1996), 21-45.

[13] F. Flandoli and B. Schmalfuss, Weak solutions and attractors for three-dimensioanl Navier–Stokes equations with nonregular force, *J. Dynamics and Differential Equations*, **11**(1999), 355-398.

[14] N.J. Cutland, Nonstandard measure theory and its applications, *Bull. Lond. Math. Soc.* **15**(1983), 529-589.

[15] N.J. Cutland (ed.), *Nonstandard Analysis and its Applications*, Cambridge University Press, Cambridge 1988.

[16] N.J. Cutland, Loeb measure theory, in *Developments in Nonstandard Mathematics*, Eds. N.J.Cutland, F.Oliveira, V.Neves, J.Sousa-Pinto, Pitman Research Notes in Mathematics Series Vol. **336**, Longman 1995, 151-177.

[17] N.J. Cutland, *Loeb Measures in Practice: Recent Advances*, Lecture Notes in Mathematics **1751**(2000), Springer.

[18] N.J. Cutland and H.J. Keisler, *Neometric Attractors for Navier-Stokes Equations*, to appear.

[19] S. Fajardo and H.J. Keisler, Existence theorems in probability theory, *Advances in Mathematics,* **120**(1996), 191-257.

[20] S. Fajardo and H.J.Keisler, Neometric spaces, *Advances in Mathematics,* **118**(1996), 134-175.

[21] W. Feller, *An Introduction to Probability Theory and its Applications, vol. 1*, Wiley 1950.

[22] C.Foias, Statistical study of Navier-Stokes equations I, *Rend. Sem. Mat. Univ. Padova* **48**(1973), 219-348.

[23] C. Foias & R. Temam, The connection between the Navier–Stokes equations, dynamical systems, and turbulence theory, in *Directions in Partial Differential Equations,* Academic Press, NY, 1987, 55-73.

[24] A.E.Hurd and P.A.Loeb, *An Introduction to Nonstandard Real Analysis*, Academic Press, New York 1985.

[25] A.Ichikawa, Stability of semilinear stochastic evolution equations, *J. Math. Anal. Appl.* **90**(1982), 12-44.

[26] H.J.Keisler, An infinitesimal approach to stochastic analysis, *Mem. Amer. Math. Soc.* **48** No. 297 (1984).

[27] H. J. Keisler, From Discrete to Continuous Time, *Ann. Pure and Applied Logic* 52 (1991), 99-141.

[28] H. J. Keisler, Rich and saturated adapted spaces, *Advances in Mathematics,* **128**(199), 242-288.

[29] T.L.Lindstrøm, An invitation to nonstandard analysis, in [15], 1-105.

[30] D.A.Ross, Loeb measure and probability, in *Nonstandard Analysis: Theory and Applications,* Eds. L.O. Arkeryd, N.J. Cutland, C.W. Henson, NATO ASI Series C: Vol **493**, Kluwer, Dordrecht, 1997, 91-120.

[31] B. Schmallfuss, Measure attractors of the stochastic Navier–Stokes equation, Bremen Report No.258, 1991.

[32] G.R. Sell, Global attractors for the three-dimensional Navier–Stokes equations, *J.Dynamics and Diff. Equations* **8**(1996), 1-33.

[33] R.Temam, *Navier-Stokes equations*, North-Holland, Amsterdam 1979.

[34] R.Temam, *Navier-Stokes Equations and Nonlinear Functional Analysis,* SIAM, 1983.