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Some remarks on certain invariant geometric properties in Hele–Shaw flows

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ABSTRACT

In this paper we study the time evolution of the free boundary of a viscous fluid for planar flows in Hele–Shaw cells under injection. We discuss the geometrical properties of the moving frontier for bounded and unbounded (with bounded complement) domains under the assumption of zero surface tension. Applying methods from the theory of univalent functions we prove the invariance in time of α-convexity. Moreover, we establish an upper bound for the order of strongly starlikeness of the classical solution in the Hele–Shaw problem which starts with a starlike bounded domain.

1. Introduction and preliminaries

The evolution in time of the free boundary of a viscous fluid for planar flows in Hele–Shaw cells under injection was studied by many authors [10,4,11,20–22,13,2,3,5] etc.

The first results in studying the invariance in time of some geometric properties of the free boundary were obtained by Prokhorov, Vasil'ev and Hohlov in [10]. For the bounded case, they proved that starlike functions are constituting an invariant class while the convex class are not. Since the class of α-convex functions is a one-parameter family class of univalent functions which provides a continuous passage from the starlike functions ($\alpha = 0$) to the convex functions ($\alpha = 1$) it is natural to try to determine the maximum value of $\alpha \in (0,1)$ for which the α-convexity property is preserved. In the second section of this paper we present some results concerning the preserving in time of α-convexity in the case of the inner problem. In the third section we study the invariance in time of the α-convexity of the free boundary in the outer case. In the last part of this paper we establish an upper bound for the order of strongly starlikeness of the classical solution in the Hele–Shaw problem which starts with a starlike bounded domain.

In this section we review certain classical notions regarding the Hele–Shaw problem that are needed later.

We start by presenting the notions regarding the bounded case. We study the flow of a viscous fluid in a planar Hele–Shaw cell under injection through a source of constant strength Q ($Q < 0$), which is located at the origin. Suppose that the initial domain $\Omega(0)$, occupied by the fluid at time $t = 0$, contains the origin, is simply connected and is bounded by an analytic and smooth curve $\partial\Omega(0)$. By using the well-known Riemann mapping theorem, the domain $\Omega(t)$ (occupied by the fluid at the moment t) can be described by an univalent function $f(\cdot,t)$ of the unit disk $U = \{z||z| < 1\}$ onto $\Omega(t)$ normalized by $f(0,t) = 0$, $f'(0,t) > 0$. The function $f(\cdot,0)$ produces a parametrization of the boundary $\partial\Omega(0) = \{f(e^{i\theta},0),\ \theta \in [0,2\pi)\}$, while the moving boundary is parameterized by $\partial\Omega(t) = \{f(e^{i\theta},t),\ \theta \in [0,2\pi)\}$.

The equation satisfied by the free boundary was first derived by Galin and Polubarinova-Kochina (see [6,17,18]) and it has the following form:

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$$\text{Re}\left[\dot{f}(z,t)\overline{zf'(z,t)}\right] = -\frac{Q}{2\pi}, \quad |z| = 1 \tag{1.1}$$

(in the previous equality we have used the notations $f' = \frac{\partial f}{\partial z},\ \dot{f} = \frac{\partial f}{\partial t}$).

A classical solution in the interval $[0,T)$ is a function $f(z,t),\ t \in [0,T)$, that is univalent on $\overline{U}$ and C^1 with respect to t in $[0,T)$. It is known that, starting with an analytic and smooth boundary $\partial\Omega(0)$, the classical solution exists and is unique locally in time [23,19] (see [9, Chapter 1]). Note that T is called the blow-up time.

The case of unbounded domain with bounded complement can be viewed as the dynamics of a contracting bubble in a Hele–Shaw cell, since the fluid occupies a neighborhood of infinity and injection (of constant strength $Q < 0$) is supposed to take place at infinity. Again, we denote by $\Omega(t)$ the domain occupied by the fluid at the moment t. In this case the domain $\Omega(t)$ can be described by an univalent function $F(\zeta,t)$ from the exterior of the unit disk $U^- = \{\zeta||\zeta| > 1\}$ onto $\Omega(t),\ F(\zeta,t) = a\zeta + a_0 + \frac{a_1}{\zeta} + \dots,\ a > 0$.

The equation satisfied by the free boundary is (see [20]):

$$\text{Re}\left[\dot{F}(\zeta,t)\overline{\zeta F'(\zeta,t)}\right] = \frac{Q}{2\pi}, \quad |\zeta| = 1. \tag{1.2}$$

2. The invariance in time of α-convexity (Bounded domains)

In this section we obtain an upper bound for α such that the α-convexity property of the moving boundary in a Hele–Shaw flow problem is preserved in time.

Definition 2.1 [15]. Let f be a holomorphic function on U such that $f(0) = 0$ and $f'(0) \neq 0$ and let $\alpha \in \mathbb{R}$. We say that f is α-convex if

$$\text{Re}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > 0, \quad z \in U. \tag{2.3}$$

It is clear that if $\alpha = 0$ then f is starlike and if $\alpha = 1$ then f is convex. Also, we have.

Remark 2.2 [15].

(i) If $\alpha \in \mathbb{R}$ and f is α-convex then f is starlike.
(ii) If $\alpha,\ \beta \in \mathbb{R}$ such that $0 \leqslant \alpha/\beta < 1$ and f is β-convex then f is α-convex.

The following result is a generalization of Theorem 1, [10] to the case of α-convex functions. The mentioned theorem may be obtained by taking $\alpha = 0$ in Theorem 2.3 below.

Theorem 2.3. *Let $Q < 0$ and let f_0 be an univalent function on a neighborhood U_R $(R > 1)$ of $\overline{U}$. Then there exists $\alpha_0 = \alpha_0(R),\ 0 < \alpha_0 < 1$, such that the following statement is true.*

If f_0 is an α-convex function on $U,\ \alpha < \alpha_0$, then the classical solution of the Polubarinova–Galin equation (1.1) with the initial condition $f(z,0) = f_0(z)$ remains α-convex during the existence time. Moreover, the existence time of the classical solution is infinite.

Proof. Since any α-convex function is a starlike one then f_0 is starlike. By applying Theorem 3.4, [8] we obtain that the lifetime of the classical Hele–Shaw solution is infinite. Moreover, there exists $r = r(R), r > 1$, such that $f(\cdot,t)$, the classical solution of the Polubarinova–Galin equation, is univalent on U_r for every $t \geqslant 0$.

Taking into account that all the functions $f(\cdot,t)$ are univalent on U_r for each $t \geqslant 0$ and in consequence their derivatives $f'(\cdot,t)$ are continuous and do not vanish on $\overline{U}$, we can replace with $\geqslant$ the inequality in the Definition 2.1 of an α-convex function. The equality can be obtained only for $|z| = 1$.

We suppose by contrary that the conclusion of Theorem 2.3 is not true. If we denote by h the function defined by

$$h(z,t) = (1-\alpha)\frac{zf'(z,t)}{f(z,t)} + \alpha\left(1 + \frac{zf''(z,t)}{f'(z,t)}\right),$$

then there exist $t_0 \geqslant 0$ and $z_0 = e^{i\theta_0}$ such that:

$$\text{Re}\, h(z_0,t_0) = 0 \tag{2.4}$$

and for each $\varepsilon > 0$, there are $t > t_0$ and $\theta \in (\theta_0 - \varepsilon,\ \theta_0 + \varepsilon)$ such that

$$\text{Re}\, h(z,t) \leqslant 0, \quad z = e^{i\theta}. \tag{2.5}$$

Let t_0 be the first such point, $t_0 \geqslant 0$.

Since z_0 is a critical point and the image of the unit disk $\overline{U}$ under the mapping h lies in the half-plane $\{z|\mathrm{Re}\, z \geqslant 0\}$ and touches the imaginary axis at the point z_0, by applying Lemma 2.2.f, [14] (see also Lemma 9.25, [16]) we obtain that:

$$\mathrm{Im}\, z_0 h'(z_0, t_0) = 0 \tag{2.6}$$

and

$$\mathrm{Re}\, z_0 h'(z_0, t_0) \leqslant -\frac{1}{2}(1 + \mathrm{Im}^2 h(z_0, t_0)) \leqslant -\frac{1}{2} \tag{2.7}$$

where

$$zh' = (1-\alpha)\frac{zf'}{f}\left(1 + \frac{zf''}{f'} - \frac{zf'}{f}\right) + \alpha\frac{zf''}{f'}\left(1 + \frac{zf'''}{f''} - \frac{zf''}{f'}\right). \tag{2.8}$$

If we denote by p the function defined by $p(z,t) = \frac{\dot{f}(z,t)}{zf'(z,t)},\ z \in \overline{U},\ t \geqslant 0$, then the Polubarinova–Galin equation becomes:

$$\mathrm{Re}\, p(z,t) = -\frac{Q}{2\pi}\cdot\frac{1}{|f'(z,t)|^2}, \quad |z| = 1. \tag{2.9}$$

By straightforward calculations, due to (2.4) and (2.6), we get:

$$\frac{\partial}{\partial t}\mathrm{Re}\, h(z,t)\Big|_{z=z_0, t=t_0} = \mathrm{Re}\, \dot{h}(z_0,t_0) = \mathrm{Re}\, p \cdot \mathrm{Re}\, z_0 h' + \alpha \mathrm{Re}\, (z_0^2 p'' + z_0 p') - \mathrm{Im}\, h \cdot \mathrm{Im}\, z_0 p'. \tag{2.10}$$

By differentiating twice the Polubarinova–Galin equation (2.9) with respect to θ, we obtain:

$$\mathrm{Im}\, zp' = -2\mathrm{Re}\, p \cdot \mathrm{Im}\, \frac{zf''}{f'}, \quad |z| = 1 \tag{2.11}$$

and

$$\mathrm{Re}\, (zp' + z^2 p'') \tag{2.12}$$

$$= -2\mathrm{Re}\, p\left[2\mathrm{Im}^2 \frac{z_0 f''}{f'} + \mathrm{Re}\, \frac{zf''}{f'}\left(1 + \frac{zf'''}{f''} - \frac{zf''}{f'}\right)\right], \quad |z| = 1.$$

If we substitute (2.11), (2.12) and (2.8) in (2.10) and replace z by $z_0 = e^{i\theta_0}$ and t by t_0, we get

$$\frac{\partial}{\partial t}\mathrm{Re}\, h(z,t)\Big|_{z=z_0, t=t_0} = \mathrm{Re}\, p\left[-\mathrm{Re}\, z_0 h' - 2\alpha\left|\frac{z_0 f''}{f'} + 1\right|^2\right.$$

$$\left. -2\frac{\alpha^2}{1-\alpha}\mathrm{Re}^2\left(\frac{z_0 f''}{f'} + 1\right) + 2(1-\alpha)\mathrm{Im}^2 \frac{z_0 f'}{f}\right]$$

$$\geqslant \mathrm{Re}\, p\left[-\mathrm{Re}\, z_0 h' - 2\alpha\left|\frac{z_0 f''}{f'} + 1\right|^2 - 2\frac{\alpha^2}{1-\alpha}\mathrm{Re}^2\left(\frac{z_0 f''}{f'} + 1\right)\right].$$

Now, by taking into account (2.7), after elementary computations, the previous inequality transforms into:

$$\mathrm{Re}\, \dot{h}(z_0, t_0) \geqslant \frac{\mathrm{Re}\, p}{2(1-\alpha)}\left(1 - \alpha - 4\alpha\left|\frac{z_0 f''}{f'} + 1\right|^2\right). \tag{2.13}$$

Due to the fact that the function $f(\cdot, t)$ is univalent on U_r for each $t \geqslant 0$, the following classical estimation holds ([7], see (1.1.9), page 15)

$$\left|\frac{zf''}{f'} + 1\right| \leqslant \frac{1 + r^2 + 4r}{r^2 - 1}, \quad |z| = 1. \tag{2.14}$$

If we take now $\alpha_0 = 1\Big/\left(1 + 4\left(\frac{1+r^2+4r}{r^2-1}\right)^2\right)$, by using (2.9), (2.13) and (2.14) we get that $\mathrm{Re}\, \dot{h}(z_0, t_0) > 0$, for $\alpha < \alpha_0$.

Therefore, $\mathrm{Re}\, h(z,t) > 0$ for $t > t_0$ (close to t_0) in some neighborhood of θ_0. This contradicts the assumption (2.5) and completes the proof.

If in the previous theorem we start with an initial function, f_0 that is univalent only on $\overline{U}$ then we obtain the following result.

Theorem 2.4. *Let $Q < 0$ and let $\alpha \leqslant 0$. If f_0 is an univalent function on $\overline{U}$ which is α-convex function on U then the classical solution of the Polubarinova–Galin equation (1.1) with the initial condition $f(z,0) = f_0(z)$ remains α-convex during the existence time.*

3. The invariance in time of α-convexity (Unbounded domains)

It is known ([4], Theorem 6) that the convex dynamics is preserved for the outer problem. In other words, if the initial domain has a convex complement then the family of domains occupied by the fluid at different moments of time has the same property as long as the solution of the Hele–Shaw problem exists. In this section we generalize the above mentioned result, by studying the invariance in time of α-convexity for α sufficiently close to 1.

Definition 3.1 [16]. *Let F be a holomorphic function on U^-, $F(\zeta) = a\zeta + a_0 + \frac{a_1}{\zeta} + \dots$, $a > 0$, and let $\alpha \in \mathbb{R}$. We say that F is α-convex if*

$$\mathrm{Re}\left[(1-\alpha)\frac{\zeta F'(\zeta)}{F(\zeta)} + \alpha\left(1 + \frac{\zeta F''(\zeta)}{F'(\zeta)}\right)\right] > 0, \quad \zeta \in U^-. \tag{3.15}$$

It is clear that if $\alpha = 1$ then F is convex.

The following result is a generalization of Theorem 6, [4], to the case of α-convex functions. The mentioned theorem may be obtained by taking $\alpha = 1$ in Theorem 3.2 below.

Theorem 3.2. *Let $Q < 0$ and let F_0 be an univalent function on a neighborhood U_R^- $(R < 1)$ of $\overline{U^-}$. Then there exists $\alpha_0 = \alpha_0(R)$, $0 < \alpha_0 < 1$, such that the following statement is true.*

If F_0 is an α-convex function on U^-, $\alpha_0 < \alpha < 1/\alpha_0$, then the classical solution of the Polubarinova–Galin equation (1.2) with the initial condition $F(z,0) = F_0(z)$ remains α-convex during the existence time.

Proof. By considering the function $f(z,t) = \frac{1}{F(1/z,t)}$, the Polubarinova–Galin equation (1.2) can be rewritten in terms of f as follows:

$$\mathrm{Re}\dot{f}(z,t)\overline{zf'(z,t)} = -\frac{Q|f(z,t)|^4}{2\pi}, \quad |z| = 1. \tag{3.16}$$

Elementary computations show that the function $F(\zeta,t)$, $\zeta \in U^-$, is α-convex if and only if $f(z,t)$, $z \in U$, satisfies the following condition:

$$\mathrm{Re}\left[(1+\alpha)\frac{zf'(z)}{f(z)} - \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > 0, \quad z \in U. \tag{3.17}$$

Since the initial function $f_0(z) = f(z,0) = \frac{1}{F(1/z,0)}$ is univalent on $U_{1/R}$, then (by using Theorem 2, [19]) for each $1 < r < 1/R$ there exists $T = T(r)$ such that the classical solution $f(\cdot,t)$ exists for every $t \in [0,T)$. Moreover, all the functions $f(\cdot,t)$, $t \in [0,T)$ are univalent on U_r. Hence, the inequality sign in (3.15) can be replaced with $\geqslant$ where the equality can be obtained only for $|z| = 1$.

We suppose by contrary that the conclusion of Theorem 3.2 is not true. If we denote by h the function defined by

$$h(z,t) = (1+\alpha)\frac{zf'(z,t)}{f(z,t)} - \alpha\left(1 + \frac{zf''(z,t)}{f'(z,t)}\right),$$

then there exist $t_0 \geqslant 0$ and $z_0 = e^{i\theta_0}$ such that (2.4) and (2.5) are true.

If we denote by $t_0 \geqslant 0$ the first such point, then similar arguments to those used in Theorem 2.3 show that (2.6) and (2.7) are fullfilled.

In this case we have

$$zh' = (1+\alpha)\frac{zf'}{f}\left(1 + \frac{zf''}{f'} - \frac{zf'}{f}\right) - \alpha\frac{zf''}{f'}\left(1 + \frac{zf'''}{f''} - \frac{zf''}{f'}\right). \tag{3.18}$$

If we denote by p the function defined by $p(z,t) = \frac{\dot{f}(z,t)}{zf'(z,t)}$, $z \in \overline{U}$, $t \geqslant 0$, then the Polubarinova–Galin equation (3.16) becomes:

$$\mathrm{Re}\, p(z,t) = -\frac{Q}{2\pi}\cdot\frac{|f(z,t)|^4}{|f'(z,t)|^2}, \quad |z| = 1. \tag{3.19}$$

By straightforward calculations, due to (2.4) and (2.6) we get:

$$\frac{\partial}{\partial t}\mathrm{Re}\, h(z,t)\bigg|_{z=z_0,t=t_0} = \mathrm{Re}\, \dot{h}(z_0,t_0) \tag{3.20}$$

$$= \mathrm{Re}\, p \cdot \mathrm{Re}\, z_0h' - \alpha\mathrm{Re}\,(z_0^2p'' + z_0p') - \mathrm{Im}\, h \cdot \mathrm{Im}\, z_0p'.$$

By differentiating twice the Polubarinova–Galin equation (3.16) with respect to θ, we obtain:

$$\mathrm{Im}\ zp' = -\mathrm{Re}\ p\left(2\mathrm{Im}\ \frac{zf''}{f'} - 4\mathrm{Im}\ \frac{zf'}{f}\right), \quad |z| = 1 \tag{3.21}$$

and

$$\mathrm{Re}\ (zp' + z^2p'') \tag{3.22}$$

$$= -\mathrm{Re}\ p\left[\left(2\mathrm{Im}\ \frac{zf''}{f'} - 4\mathrm{Im}\ \frac{zf'}{f}\right)^2 + 2\mathrm{Re}\ \frac{zf''}{f'}\left(1 + \frac{zf'''}{f''} - \frac{zf'}{f}\right)\right.$$

$$\left.-4\mathrm{Re}\ \frac{zf'}{f}\left(1 + \frac{zf''}{f'} - \frac{zf'}{f}\right)\right], \quad |z| = 1.$$

If we substitute (3.21), (3.22) and (3.18) in (3.20) and replace z by $z_0 = e^{i\theta_0}$ and t by t_0, we get

$$\frac{\partial}{\partial t}\mathrm{Re}\ h(z,t)\Big|_{z=z_0,t=t_0} = \mathrm{Re}\ p\left[-\mathrm{Re}\ z_0h' + 2\frac{1-\alpha}{\alpha}\mathrm{Re}\ ^2\frac{z_0f'}{f} - 2(1-\alpha)\mathrm{Im}\ ^2\frac{z_0f'}{f}\right.$$

$$\left.+2\alpha\left(2\mathrm{Im}\ \frac{z_0f''}{f'} - 4\mathrm{Im}\ \frac{z_0f'}{f}\right)^2\right]$$

$$\geqslant \mathrm{Re}\ p\left[-\mathrm{Re}\ z_0h' + 2\frac{1-\alpha}{\alpha}\mathrm{Re}\ ^2\frac{z_0f'}{f} - 2(1-\alpha)\mathrm{Im}\ ^2\frac{z_0f'}{f}\right].$$

Now, by taking into account (2.7), after elementary computations, the previous inequality transforms into:

$$\mathrm{Re}\ \dot{h}(z_0,t_0) \geqslant \frac{\mathrm{Re}\ p}{2}\left[1 - 4\left(1 - \frac{1}{\alpha}\right)\mathrm{Re}\ ^2\frac{z_0f'}{f} - 4(1-\alpha)\mathrm{Im}\ ^2\frac{z_0f'}{f}\right]. \tag{3.23}$$

Due to the fact that the function $f(\cdot,t)$ is univalent on U_r for each $t \geqslant 0$, the following classical estimation holds ([7], Theorem 1.1.6)

$$\left|\frac{zf'}{f}\right| \leqslant \frac{r+1}{r-1}, \quad |z| = 1. \tag{3.24}$$

If we take now $\alpha_0 = 1 - \frac{(r-1)^2}{4(r+1)^2}$, by using 3.19, 3.23 and 3.24 we get for $\alpha_0 < \alpha \leqslant 1$ that $\mathrm{Re}\ \dot{h}(z_0,t_0) \geqslant \frac{\mathrm{Re}\ p}{2}\left[1 - 4(1-\alpha)\mathrm{Im}\ ^2\frac{z_0f'}{f}\right] > 0$. Similarly, if $1 \leqslant \alpha < 1/\alpha_0$ we have $\mathrm{Re}\ \dot{h}(z_0,t_0) \geqslant \frac{\mathrm{Re}\ p}{2}\left[1 - 4(1-\frac{1}{\alpha})\mathrm{Re}\ ^2\frac{z_0f'}{f}\right] > 0$.

In conclusion, we proved that $\mathrm{Re}\ \dot{h}(z_0,t_0) > 0$ for each $\alpha_0 < \alpha < 1/\alpha_0$.

Therefore, $\mathrm{Re}\ h(z,t) > 0$ for $t > t_0$ (close to t_0) in some neighborhood of θ_0. This contradicts the assumption (2.5) and completes the proof.

4. Remarks on the order of strongly starlikeness in Hele–Shaw flows

In this section we present an upper bound for the order of strongly starlikeness of the classical solution in the Hele–Shaw problem which starts with a starlike bounded domain.

Definition 4.1 [1]. Let f be a holomorphic function on the unit disk U such that $f(0) = 0$ and $f'(0) \neq 0$. We say that f is strongly starlike of order α, $0 < \alpha \leqslant 1$, on U, if

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\alpha\pi}{2}, \quad z \in U. \tag{4.25}$$

In this case, $f(U)$ is called a strongly starlike domain of order α.

If $\alpha = 1$ in the previous definition, we obtain the usual notion starlikeness.

Gustafsson, Prokhorov and Vasil'ev proved that (Theorem 3.4, [8]) starting with a starlike domain with an analytic boundary, the lifetime of the classical Hele–Shaw starlike dynamics, $\Omega(t)$, is infinite. Moreover, they showed that (Theorem 2.1, [8]) the domains $\Omega(t)$ are strongly starlike of order $\alpha(t)$ with a strictly decreasing order during the infinite lifetime of existence. We will present next an upper bound for the order of strongly starlikeness $\alpha(t)$.

Theorem 4.2. *Let $Q < 0$ and let f_0 be a function univalent on a neighborhood of $\overline{U}$ which is starlike on U. If $f(z,t)$ is the classical solution of the Polubarinova–Galin equation (1.1) with the initial condition $f(z,0) = f_0(z)$ and $\alpha(t)$is the strongly starlikeness' order of $f(z,t)$ then for each $t > 0$ we have $0 < \alpha(t) < \sqrt{\frac{M(0)}{M(0)-tQ/\pi}}$, where $M(0) = \max_{|z|\leqslant 1}|f_0'(z)|^2$.*

Proof. The functions $f(\cdot,t)$ are starlike on U, $t>0$ (Theorem 1, [10] and Theorem 3.4, [8]).

Therefore, there exists $\alpha(t)$, $0<\alpha(t)\leqslant 1$, such that $f(\cdot,t)$ are strongly starlike of exact order $\alpha(t)$ i.e. for any $\varepsilon>0$ the function $f(\cdot,t)$ is not strongly starlike of order $\alpha(t)-\varepsilon$.

Next, we estimate $\alpha(t_0)$ for a fixed $t_0>0$.

As in the proof of Theorem 2.1, [8] denote by $A^+=\left\{z|\arg\frac{zf'(z,t_0)}{f(z,t_0)}=\alpha(t_0)\frac{\pi}{2},\ |z|=1\right\}$ and $A^-=\left\{z|\arg\frac{zf'(z,t_0)}{f(z,t_0)}=-\alpha(t_0)\frac{\pi}{2},\ |z|=1\right\}$. It is clear that the sets A^+ and A^- are closed and $A^+\cap A^-=\emptyset$. It is possible that one of the sets A^+ and A^- to be empty.

We analyze first the case $A^+\neq\emptyset$. For each $z_0\in A^+$ we have $\mathrm{Im}\,\frac{z_0f'(z_0,t_0)}{f(z_0,t_0)}>0$.

Since any $z_0\in A^+$ is a critical point and the image of $\overline{U}$ under the mapping h, $h(z,t_0)=\left(\frac{zf'(z,t_0)}{f(z,t_0)}\right)^{\frac{1}{\alpha(t_0)}}$ lies in the half-plane $\{z|\mathrm{Re}\,z\geqslant 0\}$ and touches the imaginary axis at the point $z_0\in A^+$, by applying once more Lemma 2.2.f, [14] (see also Lemma 9.25, [16]), we obtain that (2.6) and (2.7) are fulfilled.

After short computations, due to $\mathrm{Re}\,h(z_0,t_0)=0$, (2.6) and (2.7) we obtain that

$$\mathrm{Re}\left(1+\frac{z_0f''}{f'}-\frac{z_0f'}{f}\right)=0,\ \text{and} \tag{4.26}$$

$$\mathrm{Im}\,h(z_0,t_0)\mathrm{Im}\left(\frac{z_0f''}{f'}-\frac{z_0f'}{f}\right)\geqslant\frac{\alpha(t_0)}{2}\left(1+\mathrm{Im}^{\,2}h(z_0,t_0)\right).$$

Since $z_0\in A^+$, then $\mathrm{Im}\,h(z_0,t_0)>0$ and the above inequality becomes

$$\mathrm{Im}\left(\frac{z_0f''}{f'}-\frac{z_0f'}{f}\right)\geqslant\frac{\alpha(t_0)}{2}\left(\frac{1}{\mathrm{Im}\,h}+\mathrm{Im}\,h\right)\geqslant\alpha(t_0). \tag{4.27}$$

By straightforward calculations, using (4.26) and (2.11), (4.27) and the inequality $\mathrm{Im}\,\frac{z_0f'}{f}>0$ we have that:

$$\frac{\partial}{\partial t}\arg\frac{zf'}{f}\Big|_{z=z_0,t=t_0}=\mathrm{Im}\left(\frac{\dot{f}'}{f'}-\frac{\dot{f}}{f}\right)\Big|_{z=z_0,t=t_0}$$

$$=\mathrm{Im}\,z_0p'+\mathrm{Re}\,p\cdot\mathrm{Im}\left(\frac{z_0f''}{f'}-\frac{z_0f'}{f}\right)$$

$$=-\mathrm{Re}\,p\cdot\mathrm{Im}\left(\frac{z_0f''}{f'}-\frac{z_0f'}{f}\right)-2\mathrm{Re}\,p\cdot\mathrm{Im}\left(\frac{z_0f'}{f}\right)$$

$$\leqslant-\mathrm{Re}\,p\cdot\alpha(t_0)=\frac{Q}{2\pi}\cdot\frac{\alpha(t_0)}{|f'(z_0,t_0)|^2}.$$

If we denote by $M(t)=\max_{|z|\leqslant 1}|f'(z,t)|^2$, by applying Theorem 4.1, [12] we obtain that $M(t)\leqslant M(0)-tQ/\pi$, (our estimation differs from the original one $M(t)\leqslant M(0)+2t$, due to their normalization $-Q/2\pi=1$) inequality which yields

$$\frac{\partial}{\partial t}\arg\frac{zf'}{f}\Big|_{z\in A^+,t=t_0}<\frac{Q}{2\pi}\cdot\frac{\alpha(t_0)}{M(0)-tQ/\pi}. \tag{4.28}$$

In our case ($A^+\neq\emptyset$), the desired estimation for the order of strongly starlikeness is obtained by using similar arguments to those used in the proof of Theorem 2.1 [8], applied to the function

$$v(z,t)=\log\left(\arg\frac{zf'}{f}\sqrt{M(0)-tQ/\pi}\right),\quad z\in\overline{U},\quad t>0.$$

From (4.28) we see that $\frac{\partial}{\partial t}v(z,t)\big|_{z\in A^+,t=t_0}=\frac{1}{\alpha(t_0)}\cdot\frac{\partial}{\partial t}\arg\frac{zf'}{f}\Big|_{z\in A^+,t=t_0}-\frac{Q}{2\pi}\cdot\frac{1}{M(0)-t_0Q/\pi}<0$ Therefore, $\max_{z\in A^+}\frac{\partial}{\partial t}v(z,t)\big|_{t=t_0}<0$. We can construct a vicinity on the unit circle of A^+, denoted by $V(A^+)$ such that $\frac{\partial}{\partial t}v(z,t)\big|_{z\in V(A^+),t=t_0}<0$. Also we have $\max_{z\in\partial U\setminus V(A^+)}v(z,t_0)<\max_{|z|=1}v(z,t_0)$.

The continuity with respect to the variable t of the function v provides us an interval $[t_0,t_0+\varepsilon]\subset[0,\infty)$ such that

$$\frac{\partial}{\partial t}v(z,t)\Big|_{z\in V(A^+)}<0,\quad t\in[t_0,t_0+\varepsilon] \tag{4.29}$$

and

$$\max_{z\in\partial U\setminus V(A^+)}v(z,t)<\max_{|z|=1}v(z,t_0),\quad t\in[t_0,t_0+\varepsilon]. \tag{4.30}$$

The conditions (4.29) and (4.30) imply that $v^+(t)=\max_{|z|=1}v(z,t)<\max_{|z|=1}v(z,t_0)$, for all $t\in(t_0,t_0+\varepsilon]$.

This means that v^+ is strictly decreasing on $[0,\infty)$.

By using the monotonicity of v^+ we get first: $\max_{|z|=1} \arg \frac{zf'}{f}\sqrt{M(0)-tQ/\pi} < \max_{|z|=1} \arg \frac{zf_0'}{f_0}\sqrt{M(0)}$ which yields to $\alpha(t) < \sqrt{\frac{M(0)}{M(0)-tQ/\pi}}$, as desired.

If the set $A^- \neq \emptyset$, then similar arguments applied to the function

$$v^-(t) = \max_{|z|=1} \log\left(-\arg \frac{zf'}{f}\sqrt{M(0)-tQ/\pi}\right)$$

conduct us to the fact that v^- is strictly decreasing.

If $A^+ = \emptyset$ (or $A^- = \emptyset$) then $v(t) = v^-(t)$ (or $v^+(t)$), for $t \in [t_0, t_0+\varepsilon]$. If both sets A^+ and A^- are nonempty then $v(t) = \max\{v^+(t), v^-(t)\}$ which is strictly decreasing, too, and the proof is complete.

Remark 4.3. By using the previous result we immediately obtain that $\lim_{t\to\infty}\alpha(t) = 0$. Hence for large values of t the image domains, $\Omega(t)$, can be approximated by disks.

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