Math 501, Fall '03

Counterexample in Riemann Integral Theory

Given a finite interval $[a,b]$, let L be the set of Riemann integrable functions on $[a,b]$. The ultimate goal of these notes is to show that L is not complete with respect to the pseudo-metric

$$d(f,g) \triangleq \int_a^b |f-g|\,dx.$$

This will be accomplished by constructing a sequence of functions in L that is Cauchy with respect to d, but has no limit in L.

The example uses a so-called "fat" Cantor set. Let $0 < \delta < 1$. Let $C_0^\delta \triangleq [0,1]$ and define C_m^δ, for $m \geq 1$, recursively by deleting the middle open interval of length $\delta 3^{-m}$ from each closed component interval of C_{m-1}^δ. It may be verified by induction that for each $m \geq 1$, C_m^δ is the union of 2^m closed subintervals each of length $2^{-m}(1-\delta(1-(2/3)^m))$. The fat Cantor set is then

$$C^\delta \triangleq \bigcap_{m\geq 1} C_m^\delta.$$

It is a non-where dense, perfect set.

For each $m \geq 1$ define $U_m^\delta \triangleq [0,1] - C_m^\delta$. From the results of the preceding paragraph, each U_m^δ is a union of 2^{m-1} disjoint open intervals, and the total length of U_m^δ is $|U_m^\delta| = \delta(1-(2/3)^m)$. Clearly $U_1^\delta \subset U_2^\delta \subset \cdots$. Let

$$U = \bigcup_{m\geq 1} U_m^\delta = [0,1] - C^\delta.$$

Let χ_{U^δ} be the characteristic function of U^δ.

Proposition 1 *χ_{U^δ} is not Riemann-integrable*

Proof. Since, C^δ is nowhere dense, U^δ is dense in $[0,1]$. This implies that for *any* partition P of $[0,1]$, the upper Riemann sum is

$$U_P(\chi_{U^\delta}) = 1$$

For any positive integer m, form the partition P_m as follows. First form the partition of all the endpoints of C_m^δ. The smallest subintervals of this partition are precisely the component intervals of C_m^δ, each with length less than 2^{-m}. To form P_m add in further points, but only to U_m^δ, so that $\|P_m\| \leq 2^{-m}$ Now if, $[x_{i-1}, x_i]$ is a partition of P_m corresponding to one of the component intervals of C_m^δ, this interval will contain points in C^δ and hence $m_i = \inf_{[x_{i-1},x_i]} \chi_{U^\delta}(x) = 0$. It follows that the lower Riemann sum satisfies

$$L_{P_m}(\chi_{U^\delta}) \leq |U_m^\delta| \leq \delta < 1 \quad \forall m \geq 1.$$

But $U_{P_m}(\chi_{U^\delta}) = 1$ for all $m \geq 1$ and $\|P_m\| \to 0$ as $m \to \infty$. Thus χ_{U^δ} is not Riemann integrable.

Proposition 2 *The sequence* $\{\chi_{U_m^\delta}\}$ *is Cauchy with respect to* d*, but there is no Riemann integrable* f *such that* $\lim_{m\to\infty} d(\chi_{U_m^\delta}, f) = 0$*.*

(Intuitively, the limit wants to be χ_{U^δ}, but this is not Riemann integrable.)

Proof. The Cauchy property is easy: if $m > n$

$$d(\chi_{U_m^\delta}, \chi_{U_n^\delta}) = \int_0^1 \left(\chi_{U_m^\delta} - \chi_{U_n^\delta}\right)\, dx = \mid U_m^\delta \mid - \mid U_n^\delta \mid = \delta\left((2/3)^n - (2/3)^m\right),$$

and this tends to 0 as $m, n \to \infty$.

Suppose that a Riemann integrable f, such that $\lim_{m\to\infty} d(\chi_{U_m^\delta}, f) = 0$, did exist. First, observe that

$$\lim_{m\to\infty} \int_0^1 \chi_{U_m^\delta}\, dx = \lim_{m\to\infty} |U_m^\delta| = \delta < 1.$$

Since

$$\mid \int_0^1 f\, dx - \int_0^1 \chi_{U_m^\delta}\, dx \mid \leq \int_0^1 |f - \chi_{U_m^\delta}|\, dx,$$

it follows that

$$\int_0^1 f\, dx = \delta. \tag{1}$$

However, for $m > n$, $U_n^\delta \subset U_m^\delta$, and so for every $m > n$

$$\int_0^1 |f - \chi_{U_m^\delta}|\, dx = \int_{C_n^\delta} |f - \chi_{U_m^\delta}|\, dx + \int_{U_n^\delta} |f - 1|\, dx$$

If this is to approach 0 as $m \to \infty$, it must be true that

$$\int_{U_n^\delta} |f - 1|\, dx = 0, \tag{2}$$

And this must hold for all positive integers n. Now (2) can be true only if every non-trivial subinterval of U_n contains a point y such that $f(y) = 1$, Thus $f(x) = 1$ at least on a dense subset of U_n^δ for every n, Since $U^\delta = \cup_n U_n^\delta$ is dense in $[0, 1]$, f must take the value 1 on at least a dense subset of $[0, 1]$. But this implies that for any partition P, the upper Riemann sum $U_P(f) = 1$, which contradicts (1). Thus a Riemann integrable limit of the Cauchy sequence $\{\chi_{U_m^\delta}\}$ cannot exist.

Reference: This example is taken, in essence, from *An Introduction to Measure and Integration*, Inder K. Rana, Narosa Publishing House, London, 1997.