Costing Non-Classical Solutions to the Paradoxes of Self-Reference

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The paradoxes of self-reference are *genuinely* paradoxical. The liar paradox, Russell's paradox and their cousins pose enormous difficulties to anyone who seeks to give a comprehensive theory of semantics, or of sets, or of any other domain which allows a modicum of self-reference and a modest number of logical principles.

One approach to the paradoxes of self-reference takes these paradoxes as motivating a *non-classical* theory of logical consequence. Similar logical principles are used in the paradoxical inferences, and one or other of these problematic inferences are rejected, to give a consistent (or at least, a coherent) theory.

I am not going to show that non-classical accounts of the paradoxes are misguided or wrong-headed. (On the contrary, I think that the general approach is quite sane, and have argued as much in print [16].) However, I am going to show that such approaches do come at a serious cost. The general approach of using the paradoxes to restrict the class of allowable inferences places severe constraints on the domain of possible propositional logics. Non-classical solutions are not cheap.

1 Non-Classical Solutions

In this section I will sketch the kinds of non-classical solutions I have in mind. They have the same general features: Firstly, we keep whatever semantic, or set-theoretic principles are in dispute. For example, if it is the liar paradox in question, we keep the naïve T-scheme, to the effect that

$$T\langle A\rangle \leftrightarrow A$$

where $\langle - \rangle$ is some name-forming functor, taking propositions to names, and where $\leftrightarrow$ is some form of biconditional. This scheme says, in effect, that $T\langle A\rangle$ is true under the same circumstances as A. To say that A is true, is saying no more and no less than saying A.[1]

Not only do we keep the naïve T-scheme, but we ensure that our language has a degree of self-reference. As a result, we can express sentences such as the liar: "This very sentence is not true." If the language in question is a natural language, then indexicals will do the trick. If the language is a formal language

[1] This is rather naïve, of course, for more must be said about propositions. If we take the bearers of truth to be *sentences* then $T\langle A\rangle$ says something not only about whatever A talks about, but it also says something about *language*. Still, the T-scheme in this form is the cornerstone of many sophisticated theories of truth, such as Horwich's *minimalism* [10].

without indexicals, some other trick will be needed to construct sentences analogous to the liar. A Gödel numbering and a means of diagonalisation will do nicely to give the required results.[2]

With such machinery at hand, we can reason as follows: Use a means of diagonalisation to construct a proposition λ such that λ is equivalent to $\sim T\langle\lambda\rangle$. Then reason as follows: $\lambda \leftrightarrow \sim T\langle\lambda\rangle$, but by the T-scheme, $\lambda \leftrightarrow T\langle\lambda\rangle$. Therefore $T\langle\lambda\rangle \leftrightarrow \sim T\langle\lambda\rangle$, and equivalently, $\lambda \leftrightarrow \sim\lambda$. We can then deduce $\lambda \wedge \sim\lambda$ (from inferences such as $p \rightarrow \sim p \vdash \sim p$) and we have a contradiction.

If your favourite paradox is Russell's, instead of the liar, the non-classical approach will keep the naïve class abstraction scheme

$$x \in \{y : \phi(y)\} \leftrightarrow \phi(x)$$

and you reason similarly, from the definition of the Russell class r as $\{x : x \notin x\}$. If $r \in r$ then $r \notin r$, and if $r \notin r$ then $r \in r$. The same sort of thing holds for Berry's paradox, the Burali-Forti paradox, and many others.[3]

The non-classical response to these paradoxes is to find fault with the logical principles involved in the deduction. Typical approaches to the paradoxes take them to be important lessons in the behaviour of negation. There are two different lessons we might learn about negation. One is that the inference from $A \leftrightarrow \sim A$ to $A \wedge \sim A$ fails, since A might be (speaking crudely) neither true nor false. Another possible lesson is that the inference from $A \wedge \sim A$ to an arbitrary B fails, since A can be (speaking less crudely this time) both true and false. However negation works, it cannot be *Boolean*. Boolean negation allows *both* inferences. It gives us just too much logical machinery [5, 6, 14, 15].

If you wish to define negation non-classically, there are plenty of options available to you. You can define negation inferentially[4] or directly by way of the equivalence between the *truth* of $\sim A$ and the *falsity* of A.[5] The former account takes truth as primary, and defines negation in terms of a rejected proposition and other logical machinery. In the context of the semantics of relevant logics, this approach is sometimes called the *Australian Plan*. The account which takes truth and falsity as on a par is sometimes called the *American Plan*.

I sketch this general typology of negation merely to indicate that I need not take a stand on it here. Negation is not my focus. The paradoxes cause problems in a language completely free of negation. We do not need to worry about how to treat negation in order to see how restricted our choice of logic must be.

2 A Problematic Inference

The paradox I have in mind can be found in a logic independently of its stand on negation. The deduction appeals to no particular principles of negation, as

[2] See Boolos and Jeffrey [4] for a nice review of the standard approach, and see Smullyan [18] for more on what a language must contain to feature self-reference.

[3] A compendium of such paradoxes is given by Graham Priest [15].

[4] See, for example, Meyer and Martin's account of negation as implying falsehood, and its idiosyncracies when combined with a *relevant* notion of implication [11]

[5] Three examples are four valued semantics of relevant logics, used by Dunn and Belnap [2, 3, 8, 9], the semantics of Priest's *In Contradiction* [15] and the semantics of Nelson's constructible falsity [13] and its extensions by Heinrich Wansing [20].

it is negation-free. Therefore, the conclusions apply equally to paraconsistent theories as to incomplete theories.

Any deduction must use some inferential principles. Here are the principles I need.

A TRANSITIVE RELATION OF CONSEQUENCE: We write this by '$\vdash$'. I take $\vdash$ to be a relation between propositions, and I require that it be transitive: if $A \vdash B$ and $B \vdash C$ then $A \vdash C$.

CONJUNCTION AND IMPLICATION: I require that the conjunction operator $\wedge$ be a lattice infimum with respect to $\vdash$. That is, $A \vdash B$ and $A \vdash C$ if and only if $A \vdash B \wedge C$. Furthermore, I require that there be a *residual* for conjunction: a connective $\supset$ such that

$$A \wedge B \vdash C \text{ if and only if } A \vdash B \supset C$$

This is our connective of implication. (You may wonder how I might come across such a connective. There are many ways to construct it. See the next section for a few.)

A PARADOX GENERATOR: I need only a very weak one. I will select the T scheme in the following enthymematic form:

$$T\langle A\rangle \wedge C \vdash A \qquad A \wedge C \vdash T\langle A\rangle$$

for some true proposition C. The idea is simple: $T\langle A\rangle$ need not *entail* A. It must simply give A under some background constraints (such as some facts about language) we can infer A. And we can reason similarly in reverse. Take C to be the conjunction of all required background constraints. (Under one scenario, C can be taken to be the conjunction of *all* truths: a maximally specific proposition. Then we need simply that there is no case in which $T\langle A\rangle$ is true and A fails to be true, or vice versa.)

To generate the paradox we use a technique of diagonalisation to construct a proposition λ such that λ is equivalent to $T\langle\lambda\rangle \supset A$, where A is any proposition you please. Then we reason as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{C \wedge T\langle\lambda\rangle \vdash \lambda \quad \lambda \vdash T\langle\lambda\rangle \supset A}{C \wedge T\langle\lambda\rangle \vdash T\langle\lambda\rangle \supset A}}{C \wedge T\langle\lambda\rangle \wedge T\langle\lambda\rangle \vdash A}}{C \wedge T\langle\lambda\rangle \vdash A}\,(*)}{C \vdash T\langle\lambda\rangle \supset A} \quad T\langle\lambda\rangle \supset A \vdash \lambda}{C \vdash \lambda} \quad C \wedge \lambda \vdash T\langle\lambda\rangle}{C \vdash T\langle\lambda\rangle} \quad \dfrac{\dfrac{\text{from } (*)}{C \wedge T\langle\lambda\rangle \vdash A}}{T\langle\lambda\rangle \vdash C \supset A}}{C \vdash C \supset A}}{C \wedge C \vdash A}}{C \vdash A}$$

This is a problem. Our true C entails an arbitrary A.

This inference arises independently of any particular treatment of negation. The form of the inference is reasonably well known. It is *Curry's paradox*, and it is known to give a great deal of trouble to any non-classical approach to the paradoxes [12]. In the next section I show how the tools for Curry's paradox are closer to hand than you might think. Avoiding this paradox severely constrains the non-classical theorist.

3 How the Problem Arises

There are many different ways to get the logical tools necessary for our problematic deduction. In particular, there are many ways to get a connective $\supset$ which residuates conjunction. We will examine them one at a time.

BOOLEAN NEGATION: If Boolean negation is present (write it "$\sim$") then we can define $A \supset B$ to be $\sim A \vee B$. However, the non-classical theorist has explicitly rejected Boolean negation, so we need not tarry here. This is not a problem by itself.

INTUITIONISTIC LOGIC: The rule for the residual is satisfied by the conditional of intuitionistic logic. Any semantic account which motivates intuitionism motivates the residual of conjunction. Now no non-classical theorist of the paradoxes is going to *explicitly* use the intuitionistic conditional, for it is well known to suffer from Curry-style paradoxes. Our point in the rest of the paper is to show that the *implicit* acceptance of this conditional is deeply embedded in our practices of logic.

INFINITARY DISJUNCTION: There are ways, however, to motivate a residual for conjunction without explicitly motivating intuitionistic implication. If I have infinitary disjunction at hand, such that a (finite) conjunction distributes over infinitary disjunction, we can define $B \supset C$ to be

$$\bigvee\{A : A \wedge B \vdash C\}$$

This will satisfy the definition of $\supset$. If $A' \wedge B \vdash C$ then $A' \vdash \bigvee\{A : A \wedge B \vdash C\}$, since $A' \in \{A : A \wedge B \vdash C\}$. Conversely, if $A' \vdash B \supset C$, we have $A' \vdash \bigvee\{A : A \wedge B \vdash C\}$. Then $A' \wedge B \vdash B \wedge \bigvee\{A : A \wedge B \vdash C\}$ and by the distribution of conjunction over disjunction, $A' \wedge B \vdash \bigvee\{A \wedge B : A \wedge B \vdash C\}$ and clearly $\bigvee\{A \wedge B : A \wedge B \vdash C\} \vdash C$, so $A' \wedge B \vdash C$ by the transitivity of entailment. Therefore, *any* theory which motivates infinitary disjunction and distributive lattice logic motivates the residual $\supset$ of conjunction, and our problematic inference. This seriously constrains non-classical solutions to the paradoxes, for infinitary disjunction can be motivated in many different ways.

Proof Theory: If your favourite way to introduce connectives is by way of natural deduction (introduction and elimination rules) then infinitary disjunction is no less motivated than ordinary disjunction. To infer $\bigvee X$ from a proposition A, it is sufficient to infer a member of X.

$$\frac{A \vdash B_i}{A \vdash \bigvee\{B_i : i \in I\}}$$

If you can infer A from each element of X, then you can infer A From $\bigvee X$ too.

$$\frac{A_i \vdash B \text{ (each } i \in I)}{\bigvee\{A_i : i \in I\} \vdash B}$$

This rule is the left-hand Gentzen rule. For a traditional elimination rule for a natural deduction system, you use

$$\frac{C \vdash \bigvee\{A_i : i \in I\} \quad A_i \vdash B \text{ (each } i \in I)}{C \vdash B}$$

which is equivalent, given the transitivity of entailment. These rules seem to motivate the connective straightforwardly. However, a non-classical theorist of the paradoxes must do one of two things. One response is to allow the connective but to deny the distribution of conjunction over disjunction: that is, we do not have

$$A \wedge \bigvee\{B_i : i \in I\} \vdash \bigvee\{A \wedge B_i : i \in I\}$$

Such an approach has its own difficulties: however, it may be attempted. The second response is to reject the definition of $\bigvee$ in some way. It must be argued that this does not define a connective. The problem does not end here, however. The non-classical theorist must also have something to say in areas other than proof theory, for we can define disjunction in many different ways.

The Algebra of Propositions: Some logicians like treating the class of propositions as an *algebra*. This algebra is closed under various operations, which have different algebraic properties. The algebra of propositions is *complete* if it is closed under arbitrary conjunctions and disjunctions. The non-classical theorist (who accepts the distribution of conjunction over disjunction) must hold that the "intended" algebra of propositions is incomplete. This is not a particularly great burden in and of itself. However, it becomes a burden when we consider the constructions available which naturally *complete* incomplete lattices [7, 17].

Here is a result of this nature. If your lattice of propositions is incomplete, then define a *new* lattice of propositions like this. The new propositions are *ideals* of the old lattice. A set I of propositions is an ideal iff it is closed under converse entailment (if a entails b and $b \in I$ then $a \in I$ too) and disjunction (if $a \in I$ and $b \in I$ then $a \vee b \in I$ too). You can think of I as a set of propositions such that you'd like *one* of them to be true. Our conditions ensure that we add into the set any other proposition such that making it true will be enough to make one of our original choices true. (The smallest ideal containing $\{a, b, c\}$ is the set of all propositions entailing $a \vee b \vee c$. If you make any of these propositions true, you make $a \vee b \vee c$ true, which ensures that either a or b or c is true.)

Now the ideals are *just like propositions*. The conjunction of a class of ideals is the intersection of that class. The disjunction of a class of ideals is the smallest ideal containing that class. The entailment relation among ideals is just the relation of inclusion. Ideals form a *complete* lattice. Every set of ideals has a disjunction and a conjunction. The logic of the set of ideals is very similar to the logic of propositions out of which it was constructed. However, it is complete.[6]

This is not merely a mathematical construction with no interpretive power. Given an algebra of propositions, any ideal in the structure can be treated as a proposition, with sensible truth conditions: I is true just when one member of I is true. Given a class I of propositions, it makes sense to say "one member of I is true", and this claim ought to be true just when one member of I is true.

[6] The proof is not difficult, but I will not rehearse it here [7, 17].

The non-classical theorist must say that this reasoning breaks down. There is some ideal in the structure such that the truth of a member of I is not expressible in the domain of propositions. This is a strange result indeed, and it is a cost to the non-classical theory.

State Models: Perhaps the simplest way to construct infinitary disjunction is by way of what we might call "state models" of our logics. In a state model, each proposition is modelled by the set of states in which that proposition is true. Possible worlds models for modal logics are one form of state model.

Given a state model, it seems that infinitary disjunction is close at hand. Take a class of propositions. Their disjunction is true at the union of the class of sets of states at which each proposition in that class is true. The disjunction is true at a state just when one member of the disjunction is true at that state. This will define a proposition, which is the infinitary disjunction in the language. This construction relies on the notion that this class of states gives rise to a proposition. The non-classical theorist is free to reject this. However, to do so would require an explanation of which classes of states *do* give rise to propositions and which do not. No explanation like this has been given, as yet.

4 Choices

Here are the choices for any theory which seeks to give an account of the paradoxes of self-reference.

BE FINITE: This requires formulating responses to each of the arguments of the previous section. This has not been done, as yet, and it is unclear what a non-classical theory which takes those arguments seriously might look like. In particular, it would aid the cause remarkably to be able to point to a class of propositions and to have a clear explanation for why *that* class has no disjunction.

LIVE WITHOUT DISTRIBUTION: A crucial step in each argument has been the distribution of conjunction over disjunction. This inference has been under question for a number of reasons; primarily in quantum logic and in substructural logics. It is unclear how to motivate the failure of distribution in *this* context. It would be very nice to be able to point to a particular case of distribution and to have an explanation of why the premise is true but the conclusion fails. Such explanations are forthcoming in quantum logic (even if they are not always convincing). We need one to motivate the failure of distribution for non-quantum reasons. Uwe Petersen has the most fully developed non-classical theory of the paradoxes which lives without distribution [14]. However, I have not found an explanation of why distribution fails, other than as an artefact of the proof theory. J. L. Bell has developed a semantics for quantum logic which motivates the failure of distribution [1]. It would be a great advance to see if such a semantics could help *explain* a failure of distribution in the case of the paradoxes.

LIVE WITHOUT TRANSITIVITY OF ENTAILMENT: This may be seen to be cutting off one's nose to spite one's face, but this approach has its proponents. Neil Tennant gives one theory of consequence which abandons the transitivity of entailment [19]. Tennant does not do this for reasons of the paradoxes, so it would be interesting to see whether his approach can fit neatly into this area, or whether it needs modification.

LIVE WITHOUT THE STRONG LAWS: To live without the T-scheme or the naïve class comprehension scheme is to give up the goal of giving a non-classical account of the paradoxes. If the fault *isn't* with the logic but is with the semantics or the mathematics or whatever else we used, then the paradoxes do not motivate a non-classical logical theory. A classical one will do.

Each approach has its own cost. None are simple and straightforward. There is much work left to do, if we wish to give a non-classical account of the account of the paradoxes.[7]

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