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Cubic theta functions

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Dedicated to K. Srinivasa Rao on the occasion of his sixtieth birthday

Abstract

Some new identities for the four cubic theta functions $a'(q,z)$, $a(q,z)$, $b(q,z)$ and $c(q,z)$ are given. For example, we show that

$$a'(q,z)^3 = b(q,z)^3 + c(q)^2 c(q,z).$$

This is a counterpart of the identity

$$a(q,z)^3 = b(q)^2 b(q,z^3) + c(q,z)^3,$$

which was found by Hirschhorn et al.

The Laurent series expansions of the four cubic theta functions are given. Their transformation properties are established using an elementary approach due to K. Venkatachaliengar. By applying the modular transformation to the identities given by Hirschhorn et al., several new identities in which $a'(q,z)$ plays the role of $a(q,z)$ are obtained.

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1. Introduction

Let

$$a'(q,z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2} z^n, \tag{1.1}$$

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$$a(q,z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2} z^{m-n}, \tag{1.2}$$

$$b(q,z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2} \omega^{m-n} z^{n}, \tag{1.3}$$

$$c(q,z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2+m+n+1/3} z^{m-n}, \tag{1.4}$$

where $\omega = \exp(2\pi i/3)$ and $|q| < 1$. When $z = 1$, we shall simply write

$$a(q) := a(q,1) = a'(q,1),$$

$$b(q) := b(q,1),$$

$$c(q) := c(q,1).$$

The functions $a'(q,z)$, $a(q,z)$, $b(q,z)$ and $c(q,z)$ were introduced [1] by Hirschhorn et al. [10], as cubic analogues of the classical theta functions [15, p. 464]. They proved several identities, for example:

$$a(q,z)^3 = b(q)^2 b(q,z^3) + c(q,z)^3, \tag{1.5}$$

$$a(q,z)a(q^2,z^2) = b(q^2)b(q,z^3) + c(q,z)c(q^2,z^2), \tag{1.6}$$

$$a(q,z)c(q) - c(q,z)a(q) = q^{1/3}(1-z)(1-z^{-1}) \prod_{n=1}^{\infty} (1-zq^n)^2(1-z^{-1}q^n)^2(1-q^n)^4. \tag{1.7}$$

Eqs. (1.5) and (1.6) are [10, (1.25) and (1.26)], respectively, and (1.7) is [10, (1.21)], rewritten in a slightly different form.

Conspicuous in [10] is the absence of the function $a'(q,z)$ from many of the key formulas. In this article we shall give analogues of (1.5)–(1.7) and other formulas in [10], which involve $a'(q,z)$ in place of $a(q,z)$. Our results are different from those of Bhargava [4], who gave a generalisation of the four cubic theta functions involving a parameter ζ, in addition to z and q.

We begin by giving the basic definitions and notation in Section 2.

In Section 3, we establish the Laurent series representations

$$a'(q,z) = \sum_{m=-\infty}^{\infty} q^{m^2} \sum_{n=-\infty}^{\infty} q^{3n^2} z^{2n} + \sum_{m=-\infty}^{\infty} q^{(m+1/2)^2} \sum_{n=-\infty}^{\infty} q^{3(n+1/2)^2} z^{2n+1},$$

$$a(q,z) = \sum_{m=-\infty}^{\infty} q^{3m^2} \sum_{n=-\infty}^{\infty} q^{n^2} z^{2n} + \sum_{m=-\infty}^{\infty} q^{3(m+1/2)^2} \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} z^{2n+1},$$

[1] The function $c(q,z)$ in [10] differs from the one defined here by a factor of $q^{1/3}$.

$$b(q,z)=\prod_{n=1}^{\infty}\frac{(1-q^n)(1-q^{4n})(1-q^{6n})^2}{(1-q^{2n})(1-q^{3n})(1-q^{12n})}\sum_{n=-\infty}^{\infty}q^{3n^2}z^{2n}$$

$$-q^{1/4}\prod_{n=1}^{\infty}\frac{(1-q^{2n})^2(1-q^{12n})}{(1-q^{4n})(1-q^{6n})}\sum_{n=-\infty}^{\infty}q^{3(n+1/2)^2}z^{2n+1},$$

$$c(q,z)=q^{1/3}\prod_{n=1}^{\infty}\frac{(1-q^{2n})^2(1-q^{3n})(1-q^{12n})}{(1-q^n)(1-q^{4n})(1-q^{6n})}\sum_{n=-\infty}^{\infty}q^{n^2}z^{2n}$$

$$+q^{1/12}\prod_{n=1}^{\infty}\frac{(1-q^{4n})(1-q^{6n})^2}{(1-q^{2n})(1-q^{12n})}\sum_{n=-\infty}^{\infty}q^{(n+1/2)^2}z^{2n+1},$$

valid for $0<|z|<\infty$. These were given, for $a(q,z)$, $b(q,z)$ and $c(q,z)$, by Hirschhorn et al. [10, (7.1), (6.1) and (7.2), resp.].

In Section 4 we give new and simple proofs, based on ideas of Venkatachaliengar [14], of the classical transformation formulas

$$q^{1/24}\prod_{n=1}^{\infty}(1-q^n)=\sqrt{\frac{1}{t}}\,p^{1/24}\prod_{n=1}^{\infty}(1-p^n),$$

$$q^{1/12}\sin\frac{\theta}{2}\prod_{n=1}^{\infty}(1-q^n\mathrm{e}^{\mathrm{i}\theta})(1-q^n\mathrm{e}^{-\mathrm{i}\theta})$$

$$=\exp\left(-\frac{\theta^2}{4\pi t}\right)\;p^{1/12}\sinh\frac{\theta}{2t}\prod_{n=1}^{\infty}(1-p^n\mathrm{e}^{\theta/t})(1-p^n\mathrm{e}^{-\theta/t}),$$

where $q=\mathrm{e}^{-2\pi t}$ and $p=\mathrm{e}^{-2\pi/t}$, and $\mathrm{Re}\,t>0$. We give Ramanujan's elegant form of the transformation formula for theta functions in Corollary 4.17.

In Section 5, we show that the Laurent series expansions, together with Ramanujan's transformation formula for theta functions, readily imply

$$a'(\mathrm{e}^{-2\pi t},\mathrm{e}^{\mathrm{i}\theta})=\frac{1}{t\sqrt{3}}\exp\left(\frac{-\theta^2}{6\pi t}\right)a(\mathrm{e}^{2\pi/3t},\mathrm{e}^{\theta/3t}),$$

$$a(\mathrm{e}^{-2\pi t},\mathrm{e}^{\mathrm{i}\theta})=\frac{1}{t\sqrt{3}}\exp\left(\frac{-\theta^2}{2\pi t}\right)a'(\mathrm{e}^{-2\pi/3t},\mathrm{e}^{\theta/t}),$$

$$b(\mathrm{e}^{-2\pi t},\mathrm{e}^{\mathrm{i}\theta})=\frac{1}{t\sqrt{3}}\exp\left(\frac{-\theta^2}{6\pi t}\right)c(\mathrm{e}^{-2\pi/3t},\mathrm{e}^{\theta/3t}),$$

$$c(\mathrm{e}^{-2\pi t},\mathrm{e}^{\mathrm{i}\theta})=\frac{1}{t\sqrt{3}}\exp\left(\frac{-\theta^2}{2\pi t}\right)b(\mathrm{e}^{-2\pi/3t},\mathrm{e}^{\theta/t}).$$

In Section 6, we use the results in Section 5 and the formulas (1.5)–(1.7) to prove

$$a'(q,z)^3 = b(q,z)^3 + c(q)^2 c(q,z),$$

$$a'(q,z)a'(q^2,z) = b(q,z)b(q^2,z) + c(q)c(q^2,z),$$

$$b(q,z)a(q) - a'(q,z)b(q) = 3q(1-z)(1-z^{-1})\prod_{n=1}^{\infty}(1-zq^{3n})^2(1-z^{-1}q^{3n})^2(1-q^{3n})^4.$$

These are the analogues of (1.5)–(1.7) for which $a'(q,z)$ plays the role of $a(q,z)$. We also establish three representations for $a'(q,z)$ as differences of products, as well as some formulas for $a(q)$ as differences of products. These are obtained by applying the modular transformation to the analogous formulas for $a(q,z)$ given in [10].

2. Definitions and notation

The following notation and conventions will be used throughout.

Let t be a complex number satisfying $\mathrm{Re}\, t > 0$. Let $q = \mathrm{e}^{-2\pi t}$, and observe that $|q| < 1$. Thus t is related to the standard parameter τ from the theory of elliptic functions by $\tau = \mathrm{i}t$.

Let

$$(x;q)_\infty = \prod_{n=1}^{\infty}(1 - xq^{n-1}),$$

$$(x_1, x_2, \ldots, x_m; q)_\infty = (x_1;q)_\infty (x_2;q)_\infty \cdots (x_m;q)_\infty.$$

When $x = q$, we will abbreviate further and write

$$(q)_\infty = (q;q)_\infty = \prod_{n=1}^{\infty}(1-q^n).$$

Following Ramanujan [1], [12, Ch. 16, Entries 18,19], we define

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \tag{2.1}$$

provided $|ab| < 1$. Clearly $f(a,b) = f(b,a)$, and by Jacobi's triple product identity, we have

$$f(a,b) = (-a, -b, ab; ab)_\infty. \tag{2.2}$$

Also, [1], [12, Ch. 16, Entry 22], let

$$\phi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \tag{2.3}$$

$$\psi(q) = f(q,q^3) = \sum_{n=-\infty}^{\infty} q^{n(2n-1)} = \sum_{n=0}^{\infty} q^{n(n+1)/2}. \tag{2.4}$$

Lemma 2.5.

$$(-q;q^2)_\infty = \frac{(q^2)_\infty^2}{(q)_\infty(q^4)_\infty},$$

$$f(\omega q,\omega^2 q) = \frac{(q)_\infty(q^4)_\infty(q^6)_\infty^2}{(q^2)_\infty(q^3)_\infty(q^{12})_\infty},$$

$$\omega^2 f(\omega q^2,\omega^2) = -\frac{(q^2)_\infty^2(q^{12})_\infty}{(q^4)_\infty(q^6)_\infty},$$

$$f(q^5,q) = \frac{(q^2)_\infty^2(q^3)_\infty(q^{12})_\infty}{(q)_\infty(q^4)_\infty(q^6)_\infty},$$

$$f(q^4,q^2) = \frac{(q^4)_\infty(q^6)_\infty^2}{(q^2)_\infty(q^{12})_\infty}.$$

Proof. These follow from the Jacobi triple product identity (2.2), and elementary manipulations of infinite products. □

3. Laurent series expansions

Theorem 3.1.

$$a'(q,z) = f(q,q)f(q^3z^2,q^3z^{-2}) + qzf(1,q^2)f(q^6z^2,z^{-2}) \tag{3.2}$$

$$= \sum_{m=-\infty}^{\infty} q^{m^2} \sum_{n=-\infty}^{\infty} q^{3n^2}z^{2n} + \sum_{m=-\infty}^{\infty} q^{(m+1/2)^2} \sum_{n=-\infty}^{\infty} q^{3(n+1/2)^2}z^{2n+1}, \tag{3.3}$$

$$a(q,z) = f(q^3,q^3)f(qz^2,qz^{-2}) + qzf(1,q^6)f(q^2z^2,z^{-2}) \tag{3.4}$$

$$= \sum_{m=-\infty}^{\infty} q^{3m^2} \sum_{n=-\infty}^{\infty} q^{n^2}z^{2n} + \sum_{m=-\infty}^{\infty} q^{3(m+1/2)^2} \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}z^{2n+1}, \tag{3.5}$$

$$b(q,z) = f(q\omega,q\omega^2)f(q^3z^2,q^3z^{-2}) + \omega^2 qzf(q^2\omega,\omega^2)f(q^6z^2,z^{-2}) \tag{3.6}$$

$$= \frac{(q)_\infty(q^4)_\infty(q^6)_\infty^2}{(q^2)_\infty(q^3)_\infty(q^{12})_\infty} \sum_{n=-\infty}^{\infty} q^{3n^2}z^{2n} - q^{1/4}\frac{(q^2)_\infty^2(q^{12})_\infty}{(q^4)_\infty(q^6)_\infty} \sum_{n=-\infty}^{\infty} q^{3(n+1/2)^2}z^{2n+1}, \tag{3.7}$$

$$c(q,z) = q^{1/3}f(q^5,q)f(qz^2,qz^{-2}) + q^{1/3}zf(q^4,q^2)f(q^2z^2,z^{-2}) \tag{3.8}$$

$$= q^{1/3}\frac{(q^2)_\infty^2(q^3)_\infty(q^{12})_\infty}{(q)_\infty(q^4)_\infty(q^6)_\infty} \sum_{n=-\infty}^{\infty} q^{n^2}z^{2n} + q^{1/12}\frac{(q^4)_\infty(q^6)_\infty^2}{(q^2)_\infty(q^{12})_\infty} \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}z^{2n+1}. \tag{3.9}$$

Proof. Let $[z^n]f(z)$ denote the coefficient of z^n in the Laurent series expansion of f near $z=0$. Then from (1.1), we have

$$[z^{2n}]a'(q,z)=\sum_{m=-\infty}^{\infty} q^{m^2+2mn+4n^2}=\sum_{m=-\infty}^{\infty} q^{(m+n)^2+3n^2}$$
$$=q^{3n^2}\sum_{k=-\infty}^{\infty} q^{k^2}$$

and

$$[z^{2n+1}]a'(q,z)=\sum_{m=-\infty}^{\infty} q^{m^2+m(2n+1)+(2n+1)^2}=\sum_{m=-\infty}^{\infty} q^{(m+n)(m+n+1)+3n^2+3n+1}$$
$$=\sum_{k=-\infty}^{\infty} q^{k(k+1)+3n^2+3n+1}=q^{3(n+1/2)^2}\sum_{k=-\infty}^{\infty} q^{(k+1/2)^2}.$$

This proves (3.3), and (3.2) follows by change of notation using (2.1).

The results (3.4)–(3.9) for $a(q,z)$, $b(q,z)$ and $c(q,z)$ are proved similarly, with the aid of Lemma 2.5. □

Corollary 3.10.

$$a(q)=\phi(q)\phi(q^3)+4q\psi(q^2)\psi(q^6), \tag{3.11}$$

$$b(q)=\frac{(q)_\infty(q^4)_\infty(q^6)_\infty^7}{(q^2)_\infty(q^3)_\infty^3(q^{12})_\infty^3}-2q\,\frac{(q^2)_\infty^2(q^{12})_\infty^3}{(q^4)_\infty(q^6)_\infty^2}, \tag{3.12}$$

$$c(q)=q^{1/3}\,\frac{(q^2)_\infty^7(q^3)_\infty(q^{12})_\infty}{(q)_\infty^3(q^4)_\infty^3(q^6)_\infty}+2q^{1/3}\,\frac{(q^4)_\infty^3(q^6)_\infty^2}{(q^2)_\infty^2(q^{12})_\infty}. \tag{3.13}$$

Proof. Take $z=1$ in Theorem 3.1. □

Remark 3.14. Eq. (3.11) has been given by several authors, for example [2, p. 93]; [3, (2.7)]; [5, (2.2)]; [6, Lemma (2.1)(i)(a)]; [7, p. 111, (60)]; [10, p. 683]; [12, p. 328].

4. The modular transformation

The ideas in this section originate in Venkatachaliengar's beautiful exposition [14]. However some key details were omitted in [14], so the aim of this section is to give a simplified and completed account of [14, pp. 32–35].

Suppose $|\mathrm{Im}\,\theta| < 2\pi\,\mathrm{Re}\,t$, and let

$$\phi(\theta;t)=\frac{1}{4}\cot\frac{\theta}{2}+\sum_{n=1}^{\infty}\frac{\sin n\theta}{e^{2\pi nt}-1}. \tag{4.1}$$

This function plays an important role in Ramanujan's paper [11]. It is essentially the function $\theta_1'(z)/\theta_1(z)$ in [15, p. 489]. Furthermore, if

$$\wp(\theta;t)=\frac{1}{\theta^2}+\sum_{(m,n)\neq(0,0)}\sum\left(\frac{1}{(\theta-2\pi m-2\pi i n t)^2}-\frac{1}{(2\pi m+2\pi i n t)^2}\right)$$

is the Weierstrass $\wp$ function with periods 2π and $2\pi i t$, and

$$E_2(t)=-\frac{1}{24}+\sum_{k=1}^{\infty}\frac{k}{e^{2\pi k t}-1},$$

then

$$\wp(\theta;t)=-2\frac{d}{d\theta}\phi(\theta;t)+2E_2(t).$$

See [8], [9] or [14].

Theorem 4.2. (1) *$\phi(\theta;t)$ can be extended to a meromorphic function on $\mathbb{C}$, with simple poles at $\theta=2\pi m+2\pi i n t$, $m,n\in\mathbb{Z}$, and no other singularities. The residue at each pole is $\frac{1}{2}$.*

(2) $\phi(\theta+2\pi;t)=\phi(\theta;t)$, $\phi(\theta+2\pi i t;t)=\phi(\theta;t)+1/2i$.

(3) $\phi(\theta;t)=1/2\theta+\sum_{n=1}^{\infty}(-1)^{n-1}E_{2n}(t)\theta^{2n-1}/(2n-1)!$ *valid for* $0<|\theta|<\min\{2\pi,|2\pi i t+2\pi k|,k\in\mathbb{Z}\}$, *where*

$$E_{2n}(t)=-\frac{B_{2n}}{4n}+\sum_{k=1}^{\infty}\frac{k^{2n-1}}{e^{2\pi k t}-1}$$

is the normalised Eisenstein series, and B_n, $n=0,1,2,\ldots$, are the Bernoulli numbers defined by

$$\frac{z}{e^z-1}=\sum_{n=0}^{\infty}\frac{B_n}{n!}z^n.$$

In particular,

$$E_2(t)=-\frac{1}{24}+\sum_{k=1}^{\infty}\frac{k}{e^{2\pi k t}-1}, \tag{4.3}$$

$$E_4(t)=\frac{1}{240}+\sum_{k=1}^{\infty}\frac{k^3}{e^{2\pi k t}-1}, \tag{4.4}$$

$$E_6(t)=-\frac{1}{504}+\sum_{k=1}^{\infty}\frac{k^5}{e^{2\pi k t}-1}. \tag{4.5}$$

Proof. Let us write $z=e^{i\theta}$ and $q=e^{-2\pi t}$. Then (4.1) may be rewritten as

$$2i\phi(\theta;t)=\frac{1}{2}+\frac{z}{1-z}+\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}(z^n-z^{-n}),$$

which is readily seen to converge for $|q| < |z| < |q|^{-1}$, $z \neq 1$. This implies that the series in (4.1) converges for $|\mathrm{Im}\,\theta| < 2\pi\,\mathrm{Re}\,t$, $\theta \neq 2\pi m$. The analytic continuation of ϕ is obtained by expanding using geometric series, and reversing the order of summation:

$$\begin{aligned} 2\mathrm{i}\phi(\theta;t) &= \frac{1}{2} + \frac{z}{1-z} + \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}(z^n - z^{-n}) \\ &= \frac{1}{2} + \frac{z}{1-z} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} q^{mn} z^n - \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} q^{mn} z^{-n} \\ &= \frac{1}{2} + \frac{z}{1-z} + \sum_{m=1}^{\infty} \frac{q^m z}{1-q^m z} - \sum_{m=1}^{\infty} \frac{q^m z^{-1}}{1-q^m z^{-1}}. \end{aligned} \tag{4.6}$$

This series converges for all z except $z = q^n$, $n = 0, \pm 1, \pm 2, \ldots$, where there are poles of order 1, and $z = 0$ where there is an essential singularity. Consequently, $\phi(\theta;t)$ has simple poles at $\theta = 2\pi m + 2\pi\mathrm{i}nt$, $m, n \in \mathbb{Z}$, and no other singularities. Using (4.1), we find that the residue of ϕ at $\theta = 0$ is $\frac{1}{2}$; the residues at the other singularities will also be $\frac{1}{2}$ by the periodicity properties, which we now establish.

From (4.1), it is immediate that $\phi(\theta + 2\pi; t) = \phi(\theta; t)$. Using (4.6) we find that

$$\begin{aligned} &2\mathrm{i}\phi(\theta + 2\pi\mathrm{i}t; t) \\ &= \frac{1}{2} + \frac{qz}{1-qz} + \sum_{m=1}^{\infty} \frac{q^{m+1} z}{1-q^{m+1} z} - \sum_{m=1}^{\infty} \frac{q^{m-1} z^{-1}}{1-q^{m-1} z^{-1}} \\ &= \frac{1}{2} + \sum_{m=1}^{\infty} \frac{q^m z}{1-q^m z} - \sum_{m=0}^{\infty} \frac{q^m z^{-1}}{1-q^m z^{-1}} \\ &= 2\mathrm{i}\phi(\theta;t) - \frac{z}{1-z} - \frac{z^{-1}}{1-z^{-1}} \\ &= 2\mathrm{i}\phi(\theta;t) + 1. \end{aligned}$$

This proves Part 2 of the theorem, and completes the proof of Part 1.

Part 3 follows by expanding (4.1) in powers of θ, with the help of the expansion

$$\frac{1}{2}\cot\frac{\theta}{2} = \frac{1}{\theta} + \sum_{n=1}^{\infty} \frac{B_{2n}(-1)^n}{(2n)!}\theta^{2n-1}. \qquad \square$$

Theorem 4.7.

$$\phi(\theta;t) = \frac{1}{\mathrm{i}t}\,\phi\left(\frac{\theta}{\mathrm{i}t}; \frac{1}{t}\right) - \frac{\theta}{4\pi t}. \tag{4.8}$$

Proof. Observe that the function $(1/\mathrm{i}t)\phi(\theta/\mathrm{i}t; 1/t)$ also has simple poles at $\theta = 2\pi m + 2\pi\mathrm{i}nt$, and no other singularities, and the residue at each pole is $\frac{1}{2}$. Therefore the difference

$$g(\theta;t) := \phi(\theta;t) - \frac{1}{\mathrm{i}t}\,\phi\left(\frac{\theta}{\mathrm{i}t}; \frac{1}{t}\right)$$

is entire. Using Part 2 of Theorem 4.2, we see that

$$g(\theta+2\pi;t)=g(\theta;t)-\frac{1}{2t},$$

$$g(\theta+2\pi\mathrm{i}t;t)=g(\theta;t)+\frac{1}{2\mathrm{i}}.$$

It is possible to modify g to a doubly periodic function. In fact

$$\phi(\theta;t)-\frac{1}{\mathrm{i}t}\,\phi\left(\frac{\theta}{\mathrm{i}t};\frac{1}{t}\right)+\frac{\theta}{4\pi t}$$

is doubly periodic. By Liouville's theorem it is a constant function, and since ϕ is odd, the constant is zero. This completes the proof. □

An elementary proof of Theorem 4.7 that does not use Liouville's theorem was given by Venkatachaliengar [14, 33–34]. The elementary proof is easy to follow, but longer.

Using Part 3 of Theorem 4.2 to expand the result of Theorem 4.7 in powers of θ, and comparing coefficients gives

$$E_2(t)=-\frac{1}{t^2}E_2\left(\frac{1}{t}\right)-\frac{1}{4\pi t}, \tag{4.9}$$

$$E_{2n}(t)=\frac{(-1)^n}{t^{2n}}E_{2n}\left(\frac{1}{t}\right),\quad n=2,3,4,\ldots\ . \tag{4.10}$$

Setting $t=1$ in (4.9) gives

$$E_2(1)=-\frac{1}{8\pi},$$

which written out explicitly using (4.3) is

$$\frac{1}{\mathrm{e}^{2\pi}-1}+\frac{2}{\mathrm{e}^{4\pi}-1}+\frac{3}{\mathrm{e}^{6\pi}-1}+\cdots=\frac{1}{24}-\frac{1}{8\pi}.$$

As noted in [14, p. 99], this provides a solution to [13, p. 326, Question 387]; see also [13, pp. 392–393].

We will now use (4.9) to prove the transformation formula for Dedekind's η function, and then use Theorem 4.7 to prove a transformation formula for theta functions.

Theorem 4.11. *Suppose* $\operatorname{Re} t>0$ *and let* $\eta(t)=\mathrm{e}^{-\pi t/12}\prod_{n=1}^{\infty}(1-\mathrm{e}^{-2\pi n t})$. *Then*

$$\eta(t)=\sqrt{\frac{1}{t}}\,\eta\left(\frac{1}{t}\right).$$

The branch of the square root is determined by taking the positive square root when t is a positive real number.

Remark 4.12. This can also be written as

$$q^{1/24}(q)_\infty=\sqrt{\frac{1}{t}}\,p^{1/24}(p)_\infty, \tag{4.13}$$

where $q=\mathrm{e}^{-2\pi t}$ and $p=\mathrm{e}^{-2\pi/t}$.

Proof. Eq. (4.9), written out in full using (4.3), is

$$-\frac{1}{24}+\sum_{n=1}^{\infty}\frac{ne^{-2\pi nt}}{1-e^{-2\pi nt}}=\frac{1}{24t^2}-\sum_{n=1}^{\infty}\frac{nt^{-2}e^{-2\pi n/t}}{1-e^{-2\pi n/t}}-\frac{1}{4\pi t}.$$

Integrate with respect to t to obtain

$$-\frac{t}{24}+\frac{1}{2\pi}\sum_{n=1}^{\infty}\log(1-e^{-2\pi nt})=-\frac{1}{24t}+\frac{1}{2\pi}\sum_{n=1}^{\infty}\log(1-e^{-2\pi n/t})-\frac{1}{4\pi}\log t+C,$$

for some constant C. Multiplying by 4π and exponentiating gives

$$e^{-\pi t/6}(e^{-2\pi t})_\infty^2=\frac{A}{t}e^{-\pi/6t}(e^{-2\pi/t})_\infty^2,$$

for some constant A. Letting $t=1$ implies $A=1$. This completes the proof. □

Theorem 4.14. *Suppose* $\mathrm{Re}\,t>0$ *and let* $q=e^{-2\pi t}$, $p=e^{-2\pi/t}$. *Then*

$$q^{1/12}\sin\frac{\theta}{2}(qe^{i\theta},qe^{-i\theta};q)_\infty=\exp\left(-\frac{\theta^2}{4\pi t}\right)p^{1/12}\sinh\frac{\theta}{2t}(pe^{\theta/t},pe^{-\theta/t};p)_\infty. \tag{4.15}$$

Proof. Using (4.6), Theorem 4.7 becomes

$$\frac{1}{4}\cot\frac{\theta}{2}-\frac{i}{2}\sum_{n=1}^{\infty}\left[\frac{q^ne^{i\theta}}{1-q^ne^{i\theta}}-\frac{q^ne^{-i\theta}}{1-q^ne^{-i\theta}}\right]$$

$$=\frac{1}{4t}\coth\frac{\theta}{2t}-\frac{1}{2t}\sum_{n=1}^{\infty}\left[\frac{p^ne^{\theta/t}}{1-p^ne^{\theta/t}}-\frac{p^ne^{-\theta/t}}{1-p^ne^{-\theta/t}}\right]-\frac{\theta}{4\pi t}.$$

Integrate both sides with respect to θ to get

$$\frac{1}{2}\log\sin\frac{\theta}{2}+\frac{1}{2}\sum_{n=1}^{\infty}[\log(1-q^ne^{i\theta})+\log(1-q^ne^{-i\theta})]$$

$$=\frac{1}{2}\log\sinh\frac{\theta}{2t}+\frac{1}{2}\sum_{n=1}^{\infty}[\log(1-p^ne^{\theta/t})+\log(1-p^ne^{-\theta/t})]-\frac{\theta^2}{8\pi t}+C_t,$$

for some constant C_t, which depends on t but not on θ. Now multiply by 2 and exponentiate both sides to get

$$\sin\frac{\theta}{2}(qe^{i\theta},qe^{-i\theta};q)_\infty=A_t\exp\left(-\frac{\theta^2}{4\pi t}\right)\sinh\frac{\theta}{2t}\left(pe^{\theta/t},pe^{-\theta/t};p\right)_\infty, \tag{4.16}$$

where A_t is independent of θ and depends only on t. To determine A_t, divide by $\theta/2$ and let $\theta\to 0$ to get

$$(q)_\infty^2=\frac{A_t}{t}(p)_\infty^2.$$

Using (4.13), this simplifies to

$$A_t = \left(\frac{p}{q}\right)^{1/12}.$$

Substituting this into (4.16) completes the proof. □

Corollary 4.17 ([12, Ch. 6, Entry 20]). *If* $\alpha\beta = \pi$ *and* $\mathrm{Re}(\alpha^2) > 0$, *then*

$$\sqrt{\alpha} f(\mathrm{e}^{-\alpha^2+n\alpha}, \mathrm{e}^{-\alpha^2-n\alpha}) = \mathrm{e}^{n^2/4}\sqrt{\beta} f(\mathrm{e}^{-\beta^2+\mathrm{i}n\beta}, \mathrm{e}^{-\beta^2-\mathrm{i}n\beta}).$$

Proof. Multiply (4.13) and (4.15) to get

$$q^{1/8} \sin\frac{\theta}{2}(q\mathrm{e}^{\mathrm{i}\theta}, q\mathrm{e}^{-\mathrm{i}\theta}, q; q)_\infty = \sqrt{\frac{1}{t}}\exp\left(\frac{-\theta^2}{4\pi t}\right) p^{1/8} \sinh\frac{\theta}{2t}(p\mathrm{e}^{\theta/t}, p\mathrm{e}^{-\theta/t}, p; p)_\infty.$$

Writing $\sin(\theta/2) = \mathrm{e}^{\mathrm{i}\theta/2}(1-\mathrm{e}^{-\mathrm{i}\theta})/2\mathrm{i}$, $\sinh\theta/2t = \mathrm{e}^{\theta/2t}(1-\mathrm{e}^{-\theta/t})/2$ and rearranging, we get

$$\begin{aligned}
&\sqrt{t}(q\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{-\mathrm{i}\theta}, q; q)_\infty \\
&\quad = \mathrm{i}\left(\frac{p}{q}\right)^{1/8} \exp\left(-\frac{\mathrm{i}\theta}{2}+\frac{\theta}{2t}\right)\exp\left(\frac{-\theta^2}{4\pi t}\right)(p\mathrm{e}^{\theta/t}, \mathrm{e}^{-\theta/t}, p; p)_\infty \\
&\quad = \exp\left(\frac{(\mathrm{i}\theta-\pi t-\pi\mathrm{i})^2}{4\pi t}\right)(p\mathrm{e}^{\theta/t}, \mathrm{e}^{-\theta/t}, p; p)_\infty. \qquad (4.18)
\end{aligned}$$

Now put $n = (\mathrm{i}\theta - \pi t - \pi\mathrm{i})/\sqrt{\pi t}, \alpha = \sqrt{\pi t}, \beta = \sqrt{\pi/t}$ and observe that $\alpha\beta = \pi$. Multiplying (4.18) by $\sqrt{\beta}$ and simplifying using (2.2) completes the proof. □

5. Transformation properties of the cubic theta functions

We shall now apply the results of the previous section to obtain transformation properties of $a'(q,z)$, $a(q,z)$, $b(q,z)$ and $c(q,z)$.

Lemma 5.1.

$$\sum_{m\,\mathrm{even}} a^{m(m+1)/2} b^{m(m-1)/2} = f(a^3b, ab^3),$$

$$\sum_{m\,\mathrm{odd}} a^{m(m+1)/2} b^{m(m-1)/2} = af\left(a^5b^3, \frac{b}{a}\right).$$

Proof. These follow immediately from the definition (2.1). □

Lemma 5.2.

$$f(a,b)f(c,d)+f(-a,-b)f(-c,-d)$$
$$=2f(a^3b,ab^3)f(c^3d,cd^3)+2acf\left(a^5b^3,\frac{b}{a}\right)f\left(c^5d^3,\frac{d}{c}\right),$$

$$f(a,b)f(c,d)-f(-a,-b)f(-c,-d)$$
$$=2af\left(a^5b^3,\frac{b}{a}\right)f(c^3d,cd^3)+2cf(a^3b,ab^3)f\left(c^5d^3,\frac{d}{c}\right).$$

Proof. Using the definition (2.1) and the previous Lemma, we obtain

$$f(a,b)f(c,d)+f(-a,-b)f(-c,-d)$$
$$=\sum_m\sum_n a^{m(m+1)/2}b^{m(m-1)/2}c^{n(n+1)/2}d^{n(n-1)/2}$$
$$+\sum_m\sum_n a^{m(m+1)/2}b^{m(m-1)/2}c^{n(n+1)/2}d^{n(n-1)/2}(-1)^{m^2+n^2}$$
$$=2\sum_{m\,\text{even}}\sum_{n\,\text{even}} a^{m(m+1)/2}b^{m(m-1)/2}c^{n(n+1)/2}d^{n(n-1)/2}$$
$$+2\sum_{m\,\text{odd}}\sum_{n\,\text{odd}} a^{m(m+1)/2}b^{m(m-1)/2}c^{n(n+1)/2}d^{n(n-1)/2}$$
$$=2f(a^3b,ab^3)f(c^3d,cd^3)+2acf\left(a^5b^3,\frac{b}{a}\right)f\left(c^5d^3,\frac{d}{c}\right).$$

The other part of the Lemma is proved similarly. □

Lemma 5.3.

$$f(\mathrm{e}^{-2\pi t},\mathrm{e}^{-2\pi t})=\frac{1}{\sqrt{2t}}f(\mathrm{e}^{-\pi/2t},\mathrm{e}^{-\pi/2t}), \tag{5.4}$$

$$\mathrm{e}^{-\pi t/2}f(1,\mathrm{e}^{-4\pi t})=\frac{1}{\sqrt{2t}}f(-\mathrm{e}^{-\pi/2t},-\mathrm{e}^{-\pi/2t}), \tag{5.5}$$

$$f(\omega\mathrm{e}^{-2\pi t},\omega^2\mathrm{e}^{-2\pi t})=\frac{1}{\sqrt{2t}}\,\mathrm{e}^{-\pi/18t}f(\mathrm{e}^{-5\pi/6t},\mathrm{e}^{-\pi/6t}), \tag{5.6}$$

$$\mathrm{e}^{-\pi t/2}\omega^2f(\omega\mathrm{e}^{-4\pi t},\omega^2)=-\frac{1}{\sqrt{2t}}\mathrm{e}^{-\pi/18t}f(-\mathrm{e}^{-5\pi/6t},-\mathrm{e}^{-\pi/6t}), \tag{5.7}$$

$$\mathrm{e}^{-2\pi t/3}f(\mathrm{e}^{-10\pi t},\mathrm{e}^{-2\pi t})=\frac{1}{\sqrt{6t}}f(\omega\mathrm{e}^{-\pi/6t},\omega^2\mathrm{e}^{-\pi/6t}), \tag{5.8}$$

$$e^{-\pi t/3} f(e^{-8\pi t}, e^{-4\pi t}) = \frac{1}{\sqrt{6t}} f(-\omega e^{-\pi/6t}, -\omega^2 e^{-\pi/6t}), \tag{5.9}$$

$$f(e^{-2\pi t+2i\theta}, e^{-2\pi t-2i\theta}) = \frac{1}{\sqrt{2t}} \exp\left(\frac{-\theta^2}{2\pi t}\right) f(e^{-\pi/2t-\theta/t}, e^{-\pi/2t+\theta/t}), \tag{5.10}$$

$$e^{-\pi t/2+i\theta} f(e^{-4\pi t+2i\theta}, e^{-2i\theta}) = \frac{1}{\sqrt{2t}} \exp\left(\frac{-\theta^2}{2\pi t}\right) f(-e^{-\pi/2t-\theta/t}, -e^{-\pi/2t+\theta/t}). \tag{5.11}$$

Proof. Let $\alpha = \sqrt{2\pi t}, \beta = \sqrt{\pi/2t}$ in Corollary 4.17. Then (5.4)–(5.7), (5.10) and (5.11) follow on setting $n=0, \sqrt{2\pi t}, (i/3)\sqrt{2\pi/t}, -\sqrt{2\pi t}+(i/3)\sqrt{2\pi/t}, 2i\theta/\sqrt{2\pi t}$ and $-\sqrt{2\pi t}+2i\theta/\sqrt{2\pi t}$, respectively. Eqs. (5.8) and (5.9) follow from (5.6) and (5.7), respectively, by replacing t with $1/12t$. □

Theorem 5.12.

$$a'(e^{-2\pi t}, e^{i\theta}) = \frac{1}{t\sqrt{3}} \exp\left(\frac{-\theta^2}{6\pi t}\right) a(e^{-2\pi/3t}, e^{\theta/3t}), \tag{5.13}$$

$$a(e^{-2\pi t}, e^{i\theta}) = \frac{1}{t\sqrt{3}} \exp\left(\frac{-\theta^2}{2\pi t}\right) a'(e^{-2\pi/3t}, e^{\theta/t}), \tag{5.14}$$

$$b(e^{-2\pi t}, e^{i\theta}) = \frac{1}{t\sqrt{3}} \exp\left(\frac{-\theta^2}{6\pi t}\right) c(e^{-2\pi/3t}, e^{\theta/3t}), \tag{5.15}$$

$$c(e^{-2\pi t}, e^{i\theta}) = \frac{1}{t\sqrt{3}} \exp\left(\frac{-\theta^2}{2\pi t}\right) b(e^{-2\pi/3t}, e^{\theta/t}). \tag{5.16}$$

Proof. Successively applying Theorem 3.1, Lemma 5.3, Lemma 5.2 and then Theorem 3.1 again, we obtain

$$\begin{aligned}
a'(e^{-2\pi t}, e^{i\theta}) &= f(e^{-2\pi t}, e^{-2\pi t}) f(e^{-6\pi t+2i\theta}, e^{-6\pi t-2i\theta}) \\
&\quad + e^{-2\pi t+i\theta} f(1, e^{-4\pi t}) f(e^{-12\pi t+2i\theta}, e^{-2i\theta}), \\
&= \frac{1}{t\sqrt{12}} \exp\left(\frac{-\theta^2}{6\pi t}\right) [f(e^{-\pi/2t}, e^{-\pi/2t}) f(e^{-\pi/6t-\theta/3t}, e^{-\pi/6t+\theta/3t}) \\
&\quad + f(-e^{-\pi/2t}, -e^{-\pi/2t}) f(-e^{-\pi/6t-\theta/3t}, -e^{-\pi/6t+\theta/3t})], \\
&= \frac{1}{t\sqrt{3}} \exp\left(\frac{-\theta^2}{6\pi t}\right) [f(e^{-2\pi/t}, e^{-2\pi/t}) f(e^{-2\pi/3t-2\theta/3t}, e^{-2\pi/3t+2\theta/3t}) \\
&\quad + e^{-2\pi/3t-\theta/3t} f(e^{-4\pi/t}, 1) f(e^{-4\pi/3t-2\theta/3t}, e^{2\theta/3t})], \\
&= \frac{1}{t\sqrt{3}} \exp\left(\frac{-\theta^2}{6\pi t}\right) a(e^{-2\pi/3t}, e^{-\theta/3t}).
\end{aligned}$$

Finally, observing that

$$a(q,z)=a(q,z^{-1}) \tag{5.17}$$

by using (3.5) for example, we complete the proof of (5.13).

To prove (5.14), first replace t with $1/3t$ in (5.13). Then replace θ with $-\mathrm{i}\theta/t$ and rearrange to get

$$a(\mathrm{e}^{-2\pi t},\mathrm{e}^{-\mathrm{i}\theta})=\frac{1}{t\sqrt{3}}\exp\left(\frac{-\theta^2}{2\pi t}\right)a'(\mathrm{e}^{-2\pi/3t},\mathrm{e}^{\theta/t}).$$

Applying (5.17) to the left-hand side completes the proof of (5.14).

To prove (5.15), we apply Theorem 3.1, Lemma 5.3, Lemma 5.2 and then Theorem 3.1 once more, in succession, to obtain

$$\begin{aligned}
b(\mathrm{e}^{-2\pi t},\mathrm{e}^{\mathrm{i}\theta}) &= f(\omega\mathrm{e}^{-2\pi t},\omega^2\mathrm{e}^{-2\pi t})f(\mathrm{e}^{-6\pi t+2\mathrm{i}\theta},\mathrm{e}^{-6\pi t-2\mathrm{i}\theta})\\
&\quad+\omega^2\mathrm{e}^{-2\pi t+\mathrm{i}\theta}f(\omega\mathrm{e}^{-4\pi t},\omega^2)f(\mathrm{e}^{-12\pi t+2\mathrm{i}\theta},\mathrm{e}^{-2\mathrm{i}\theta})\\
&=\frac{\mathrm{e}^{-\pi/18t}}{t\sqrt{12}}\exp\left(\frac{-\theta^2}{6\pi t}\right)[f(\mathrm{e}^{-\pi/6t},\mathrm{e}^{-5\pi/6t})f(\mathrm{e}^{-\pi/6t+\theta/3t},\mathrm{e}^{-\pi/6t-\theta/3t})\\
&\quad-f(-\mathrm{e}^{-\pi/6t},-\mathrm{e}^{-5\pi/6t})f(-\mathrm{e}^{-\pi/6t+\theta/3t},-\mathrm{e}^{-\pi/6t-\theta/3t})]\\
&=\frac{\mathrm{e}^{-\pi/18t}}{t\sqrt{3}}\exp\left(\frac{-\theta^2}{6\pi t}\right)[\mathrm{e}^{-\pi/6t}f(\mathrm{e}^{-10\pi/3t},\mathrm{e}^{-2\pi/3t})f(\mathrm{e}^{-2\pi/3t+2\theta/3t},\mathrm{e}^{-2\pi/3t-2\theta/3t})\\
&\quad+\mathrm{e}^{-\pi/6t+\theta/3t}f(\mathrm{e}^{-4\pi/3t},\mathrm{e}^{-8\pi/3t})f(\mathrm{e}^{-4\pi/3t+2\theta/3t},\mathrm{e}^{2\theta/3t})]\\
&=\frac{1}{t\sqrt{3}}\exp\left(\frac{-\theta^2}{6\pi t}\right)c(\mathrm{e}^{-2\pi/3t},\mathrm{e}^{\theta/3t}).
\end{aligned}$$

This completes the proof of (5.15).

To prove (5.16), first replace t with $1/3t$ in (5.15). Then replace θ with $-\mathrm{i}\theta/t$ and rearrange to get

$$c(\mathrm{e}^{-2\pi t},\mathrm{e}^{-\mathrm{i}\theta})=\frac{1}{t\sqrt{3}}\exp\left(\frac{-\theta^2}{2\pi t}\right)b\left(\mathrm{e}^{-2\pi/3t},\mathrm{e}^{\theta/t}\right).$$

Since $c(\mathrm{e}^{-2\pi t},\mathrm{e}^{-\mathrm{i}\theta})=c(\mathrm{e}^{-2\pi t},\mathrm{e}^{\mathrm{i}\theta})$, this completes the proof of (5.16). □

Remark 5.18. In [10, (1.22), (1.23)] the product formulas

$$b(q,z)=(q)_\infty(q^3)_\infty\frac{(qz,qz^{-1};q)_\infty}{(q^3z,q^3z^{-1};q^3)_\infty},$$

$$c(q,z)=q^{1/3}(q)_\infty(q^3)_\infty(1+z+z^{-1})\frac{(q^3z^3,q^3z^{-3};q^3)_\infty}{(qz,qz^{-1};q)_\infty},$$

were proved. These, together with Theorems 4.11 and 4.14, can be used to give alternate proofs of (5.15) and (5.16).

The following results are due to J.M. and P.B. Borwein [5, (2.2)].

Corollary 5.19.

$$a(\mathrm{e}^{-2\pi t}) = \frac{1}{t\sqrt{3}} a(\mathrm{e}^{-2\pi/3t}),$$
$$b(\mathrm{e}^{-2\pi t}) = \frac{1}{t\sqrt{3}} c(\mathrm{e}^{-2\pi/3t}),$$
$$c(\mathrm{e}^{-2\pi t}) = \frac{1}{t\sqrt{3}} b(\mathrm{e}^{-2\pi/3t}).$$

Proof. Let $\theta = 0$ in Theorem 5.12. □

6. New identities involving $a'(q, z)$

We are now ready to prove analogues of (1.5)–(1.7) with $a'(q,z)$ playing the role of $a(q,z)$.

Theorem 6.1.

$$a'(q,z)^3 = b(q,z)^3 + c(q)^2 c(q,z), \tag{6.2}$$

$$a'(q,z)a'(q^2,z) = b(q,z)b(q^2,z) + c(q)c(q^2,z), \tag{6.3}$$

$$b(q,z)a(q) - a'(q,z)b(q) = 3q(1-z)(1-z^{-1})\prod_{n=1}^{\infty}(1-zq^{3n})^2(1-z^{-1}q^{3n})^2(1-q^{3n})^4. \tag{6.4}$$

Proof. Put $q = \mathrm{e}^{-2\pi t}, z = \mathrm{e}^{\mathrm{i}\theta}$ in Eq. (1.5), to get

$$a(\mathrm{e}^{-2\pi t}, \mathrm{e}^{\mathrm{i}\theta})^3 = b(\mathrm{e}^{-2\pi t})^2 b(\mathrm{e}^{-2\pi t}, \mathrm{e}^{3\mathrm{i}\theta}) + c(\mathrm{e}^{-2\pi t}, \mathrm{e}^{\mathrm{i}\theta})^3.$$

Apply Theorem 5.12 to this and simplify to get

$$a'(\mathrm{e}^{-2\pi/3t}, \mathrm{e}^{\theta/t})^3 = c(\mathrm{e}^{-2\pi/3t})^2 c(\mathrm{e}^{-2\pi/3t}, \mathrm{e}^{\theta/t}) + b(\mathrm{e}^{-2\pi/3t}, \mathrm{e}^{\theta/t})^3.$$

Now replace $\mathrm{e}^{-2\pi/3t}$ with q and $\mathrm{e}^{\theta/t}$ with z, to obtain (6.2).

Similarly, putting $q = \mathrm{e}^{-2\pi t}, z = \mathrm{e}^{\mathrm{i}\theta}$ in Eq. (1.6) gives

$$a(\mathrm{e}^{-2\pi t}, \mathrm{e}^{\mathrm{i}\theta})a(\mathrm{e}^{-4\pi t}, \mathrm{e}^{2\mathrm{i}\theta}) = b(\mathrm{e}^{-4\pi t})b(\mathrm{e}^{-2\pi t}, \mathrm{e}^{3\mathrm{i}\theta}) + c(\mathrm{e}^{-2\pi t}, \mathrm{e}^{\mathrm{i}\theta})c(\mathrm{e}^{-4\pi t}, \mathrm{e}^{2\mathrm{i}\theta}).$$

Apply Theorem 5.12 to this and simplify to get

$$a'(\mathrm{e}^{-2\pi/3t}, \mathrm{e}^{\theta/t})a'(\mathrm{e}^{-\pi/3t}, \mathrm{e}^{\theta/t}) = c(\mathrm{e}^{-\pi/3t})c(\mathrm{e}^{-2\pi/3t}, \mathrm{e}^{\theta/t}) + b(\mathrm{e}^{-2\pi/3t}, \mathrm{e}^{\theta/t})b(\mathrm{e}^{-\pi/3t}, \mathrm{e}^{\theta/t}).$$

Now replace $\mathrm{e}^{-\pi/3t}$ with q and $\mathrm{e}^{\theta/t}$ with z, to obtain (6.3).

Lastly, putting $q = e^{-2\pi t}, z = e^{i\theta}$ in Eq. (1.7) gives

$$a(e^{-2\pi t}, e^{i\theta})c(e^{-2\pi t}) - c(e^{-2\pi t}, e^{i\theta})a(e^{-2\pi t})$$
$$= 4e^{-2\pi t/3}\sin^2\frac{\theta}{2}(e^{-2\pi t}e^{i\theta}, e^{-2\pi t}e^{-i\theta}; e^{-2\pi t})_\infty^2(e^{-2\pi t})_\infty^4.$$

Apply Theorems 4.11, 4.14 and 5.12 to this, to obtain

$$\frac{1}{3t^2}\exp\left(\frac{-\theta^2}{2\pi t}\right)[a'(e^{-2\pi/3t}, e^{\theta/t})b(e^{-2\pi/3t}) - b(e^{-2\pi/3t}, e^{\theta/t})a(e^{-2\pi/3t})]$$
$$= \frac{4}{t^2}e^{-2\pi/3t}\exp\left(\frac{-\theta^2}{2\pi t}\right)\sinh^2\frac{\theta}{2t}(e^{-2\pi/t}e^{\theta/t}, e^{-2\pi/t}e^{-\theta/t}; e^{-2\pi/t})_\infty^2(e^{-2\pi/t})_\infty^4.$$

Now multiply by $3t^2\exp(\theta^2/2\pi t)$ and replace $e^{-2\pi/3t}$ with q, and $e^{\theta/t}$ with z, to obtain (6.4). □

Three representations of $a(q,z)$ as a difference of products were given in [10, (1.31)–(1.33)]:

$$a(q,z) = \frac{1}{3}(2+z+z^{-1})\frac{(q)_\infty(q^2)_\infty^2(q^3)_\infty}{(q^6)_\infty^2}(-zq, -z^{-1}q; q)_\infty^2$$
$$+\frac{1}{3}(1-z-z^{-1})\frac{(q)_\infty^4}{(q^2)_\infty(q^6)_\infty}\frac{(-z^3q^3, -z^{-3}q^3; q^3)_\infty}{(-zq, -z^{-1}q; q)_\infty}, \tag{6.5}$$

$$a(q,z) = (2+z+z^{-1})\frac{(q)_\infty(q^2)_\infty^2(q^3)_\infty}{(q^6)_\infty^2}(-zq, -z^{-1}q; q)_\infty^2$$
$$-(1+z+z^{-1})\frac{(q^2)_\infty(q^3)_\infty^4}{(q^6)_\infty^3}\frac{(z^3q^3, z^{-3}q^3; q^3)_\infty}{(zq, z^{-1}q; q)_\infty}, \tag{6.6}$$

$$a(q,z) = \frac{1}{2}(1+z+z^{-1})\frac{(q^2)_\infty(q^3)_\infty^4}{(q^6)_\infty^3}\frac{(z^3q^3, z^{-3}q^3; q^3)_\infty}{(zq, z^{-1}q; q)_\infty}$$
$$+\frac{1}{2}(1-z-z^{-1})\frac{(q)_\infty^4}{(q^2)_\infty(q^6)_\infty}\frac{(-z^3q^3, -z^{-3}q^3; q^3)_\infty}{(-zq, -z^{-1}q; q)_\infty}. \tag{6.7}$$

The counterparts of these for which $a'(q,z)$ plays the role of $a(q,z)$ are:

Theorem 6.8.

$$a'(q^2,z) = \frac{(q^2)_\infty(q^3)_\infty^2(q^6)_\infty}{(q)_\infty^2}(q^3z, q^3z^{-1}; q^6)_\infty^2$$
$$-2q\frac{(q^6)_\infty^4}{(q)_\infty(q^3)_\infty}\frac{(qz, qz^{-1}; q^2)_\infty}{(q^3z, q^3z^{-1}; q^6)_\infty}, \tag{6.9}$$

$$a'(q^2,z) = 3\frac{(q^2)_\infty(q^3)_\infty^2(q^6)_\infty}{(q)_\infty^2}(q^3z, q^3z^{-1}; q^6)_\infty^2$$
$$-2\frac{(q^2)_\infty^4(q^3)_\infty}{(q)_\infty^3}\frac{(q^2z, q^2z^{-1}; q^2)_\infty}{(q^6z, q^6z^{-1}; q^6)_\infty}, \tag{6.10}$$

$$a'(q^2,z) = \frac{(q^2)_\infty^4 (q^3)_\infty}{(q)_\infty^3} \frac{(q^2z, q^2z^{-1}; q)_\infty}{(q^6z, q^6z^{-1}; q^6)_\infty} - 3q \frac{(q^6)_\infty^4}{(q)_\infty (q^3)_\infty} \frac{(qz, qz^{-1}; q)_\infty}{(q^3z, q^3z^{-1}; q^6)_\infty}. \tag{6.11}$$

Proof. Eqs. (6.9)–(6.11) follow by applying Theorems 4.11, 4.14 and 5.12 to Eqs. (6.5)–(6.7), and then replacing q with q^2. The details are similar to those in the proof of Theorem 6.1, and hence we omit them. □

We conclude by giving the analogues of [10, (1.27)–(1.30)].

Theorem 6.12.

$$a(q^2) = \frac{1}{3}\frac{c(q)^2}{c(q^2)} - \frac{2}{3}\frac{c(q^2)^2}{c(q)} \tag{6.13}$$

$$= \frac{c(q)^2}{c(q^2)} - 2\frac{b(q^2)^2}{b(q)} \tag{6.14}$$

$$= \frac{b(q^2)^2}{b(q)} - \frac{c(q^2)^2}{c(q)} \tag{6.15}$$

$$= \frac{b(q)^2}{b(q^{1/2})} - \frac{c(q^{1/2})c(q^2)}{c(q)}. \tag{6.16}$$

Proof. These are obtained by applying Corollary 5.19 to [10, (1.27)–(1.30)] and then replacing q with q^2. Eqs. (6.13)–(6.15) can also be proved by letting $z = 1$ in Theorem 6.8. □

References

[1] B.C. Berndt, Ramanujan's Notebooks, Part III, Springer, New York, 1991.

[2] B.C. Berndt, Ramanujan's Notebooks, Part V, Springer, New York, 1998.

[3] B.C. Berndt, S. Bhargava, F.G. Garvan, Ramanujan's theories of elliptic functions to alternative bases, Trans. Amer. Math. Soc. 347 (11) (1995) 4163–4244.

[4] S. Bhargava, Unification of the cubic analogues of the Jacobian theta-function, J. Math. Anal. Appl. 193 (2) (1995) 543–558.

[5] J.M. Borwein, P.B. Borwein, A cubic counterpart of Jacobi's identity and the AGM, Trans. Amer. Math. Soc. 323 (2) (1991) 691–701.

[6] J.M. Borwein, P.B. Borwein, F.G. Garvan, Some cubic modular identities of Ramanujan, Trans. Amer. Math. Soc. 343 (1) (1994) 35–47.

[7] J.H. Conway, N.J.A. Sloane, Sphere Packings, Lattices and Groups, Springer, New York, 1988.

[8] S. Cooper, The development of elliptic functions according to Ramanujan and Venkatachaliengar, in: C. Adiga, D.D. Somashekara (Eds.), Proceedings of The International Conference on the Works of Srinivasa Ramanujan, University of Mysore, India, 2001, pp. 81–99. Available online at http://www.massey.ac.nz/~wwiims/research/letters/volume1number1/

[9] S. Cooper, On sums of an even number of squares, and an even number of triangular numbers: an elementary approach based on Ramanujan's ${}_1\psi_1$ summation formula, q-Series with Applications to Combinatorics, Number Theory and Physics, in: B.C. Berndt, K. Ono (Eds.), Contemporary Mathematics, No. 291, American Mathematical Society, Providence, RI, 2001, pp. 115–137.

[10] M. Hirschhorn, F. Garvan, J. Borwein, Cubic analogues of the Jacobian theta functions $\theta(z,q)$, Canad. J. Math. 45 (4) (1993) 673–694.

[11] S. Ramanujan, On certain arithmetical functions, Trans. Camb. Phil. Soc. 22 (9) (1916) 159–184 (Reprinted in S. Ramanujan, Collected Papers, AMS Chelsea Publishing, Providence, Rhode Island, 2000, pp. 136–162).

[12] S. Ramanujan, Notebooks, (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957.

[13] S. Ramanujan, Collected Papers, AMS Chelsea Publishing, Providence, Rhode Island, 2000.

[14] K. Venkatachaliengar, Development of Elliptic Functions according to Ramanujan, Department of Mathematics, Madurai Kamaraj University, Technical Report 2, 1988.

[15] E. Whittaker, G. Watson, A Course of Modern Analysis, 4th Edition, Cambridge University Press, Cambridge, 1927.