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DISTORTION PROPERTIES OF p-FOLD SYMMETRIC ALPHA-STARLIKE FUNCTIONS

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ABSTRACT. Starlike functions f which are of Mocanu type α and have power series of the form

$$f(z) = z + a_{p+1}z^{p+1} + a_{2p+1}z^{2p+1} + \cdots,$$

where $p=1, 2, 3, \cdots$, are shown to satisfy the relation $f(z)=[g(z^p)]^{1/p}$ where g is of Mocanu type $p\alpha$ with power series $g(z)=z+b_2z^2+b_3z^3+\cdots$. Distortion results dealing with the $\frac{1}{4}$-theorem and bounds on $|f(z)|$ are obtained.

1. **Introduction.** In a recent paper [**1**] S. S. Miller obtained distortion theorems for the class of alpha-starlike functions. In this paper we look at functions which are alpha-starlike and p-fold symmetric. Specifically we look at functions f which are alpha-starlike with power series of the form

$$f(z) = z + a_{p+1}z^{p+1} + a_{2p+1}z^{2p+1} + \cdots, \tag{1.1}$$

where $p=1, 2, 3, \cdots$.

For completeness we recall the pertinent definitions.

DEFINITION 1. Let α be real and suppose $f(z)=z+b_2z^2+b_3z^3+\cdots$ is regular in $D=\{z:|z|<1\}$ with $f(z)f'(z)\neq 0$ in $0<|z|<1$. If

$$\operatorname{Re}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf''(z)}{f'(z)} + 1\right)\right] > 0$$

for $z \in D$ then f is an α-starlike function. We write $f \in \mathcal{M}_\alpha$.

DEFINITION 2. If f is starlike and $\alpha=\sup\{\beta : f \in \mathcal{M}_\beta\}$ then f is of Mocanu type α ($f \in \mathcal{M}(\alpha)$).

The above definitions may be found in [**1**], [**2**] and [**3**].

We now introduce some notation.

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DEFINITION 3. If $f \in \mathcal{M}_\alpha$ and $f(z)$ has a power series of the form (1.1) we write $f \in \mathcal{M}_{\alpha,p}$. If $f \in \mathcal{M}(\alpha)$ with power series of the form (1.1) we write $f \in \mathcal{M}_p(\alpha)$.

The results of this paper will depend upon the theorem (proven in §2) that $f \in \mathcal{M}_p(\alpha)$ iff $g \in \mathcal{M}_1(p\alpha)$ where $f(z) = [g(z^p)]^{1/p}$. The subsequent distortion theorems (proven in §3) will follow from results in [**1**].

2. **The basic relation.** In this section we consider the following result.

THEOREM 1. *$f \in \mathcal{M}_p(\alpha)$ iff $g \in \mathcal{M}_1(p\alpha)$, where $f(z) = [g(z^p)]^{1/p}$.*

PROOF. Let $f \in \mathcal{M}_{\alpha,p}$, α real, thus

$$\text{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf''(z)}{f'(z)} + 1\right)\right\} > 0. \tag{2.1}$$

Setting $f(z) = [g(z^p)]^{1/p}$ and computing $f'(z)/f(z)$ and $f''(z)/f'(z)$ we notice the left-hand side of (2.1) is equal to

$$\text{Re}\left\{(1-p\alpha)\frac{z^p g'(z^p)}{g(z^p)} + p\alpha\left(\frac{z^p g''(z^p)}{g'(z^p)} + 1\right)\right\}. \tag{2.2}$$

But the condition that this quantity is positive is equivalent to $g \in \mathcal{M}_{p\alpha,1}$. Since the computations are reversible it follows that $f \in \mathcal{M}_{\alpha,p}$ iff $g \in \mathcal{M}_{p\alpha,1}$. Furthermore since the correspondence of α and $p\alpha$ is monotone increasing it follows that $f \in \mathcal{M}_p(\alpha)$ iff $g \in \mathcal{M}_1(p\alpha)$.

Note that an alternate proof to Theorem 1 can be obtained by using the integral representation for functions in $\mathcal{M}_\alpha$ (see [**1**] or [**2**]) plus the fact that if $g(z)$ is a starlike function then $[g(z^n)]^{1/n}$ is also a starlike function.

3. **Distortion theorems.** In the following theorems we will need the functions

$$g_0(p\alpha, z) = \left[\frac{1}{p\alpha}\int_0^z \zeta^{1/p\alpha - 1}(1-\zeta)^{-2/p\alpha}\, d\zeta\right]^{p\alpha}, \tag{3.1}$$

$$f_0(\alpha, z) = [g_0(p\alpha, z^p)]^{1/p} \tag{3.2}$$

and

$$K(\alpha, r) = r\left[G\left(\frac{1}{\alpha}, \frac{2}{\alpha}, \frac{1}{\alpha} + 1; r\right)\right]^\alpha, \tag{3.3}$$

where $G(a, b, c; z)$ is the hypergeometric function.

THEOREM 2. *If $f(z)$ is a p-fold symmetric alpha-starlike function,* $\alpha>0$, *then for* $|z|=r$ $(0<r<1)$

$$[-K(p\alpha, -r^p)]^{1/p} \leqq |f(z)| \leqq [K(p\alpha, r^p)]^{1/p}. \tag{3.4}$$

PROOF. In [**1**] it is shown that for $g \in \mathcal{M}_1(p\alpha)$,

$$-K(p\alpha, -r) \leqq |g(z)| \leqq K(p\alpha, r). \tag{3.5}$$

By Theorem 1 $f(z)=[g(z^p)]^{1/p}$ and (3.4) follows. Since (3.5) is sharp for $g_0(p\alpha, z)$, we have equality for $f_0(\alpha, z)$.

REMARKS. If $\alpha=1$ and $p=2$ we have for odd convex functions

$$\tan^{-1} r \leqq |f(z)| \leqq \tfrac{1}{2} \log \frac{1+r}{1-r},$$

whereas if α approaches zero we have the known result for all odd starlike functions

$$\frac{r}{1+r^2} \leqq |f(z)| \leqq \frac{r}{1-r^2}.$$

Furthermore, since $g \in \mathcal{M}_1(\alpha)$ for $\alpha>2$ implies g is a bounded convex function [**1**] we have that $f \in \mathcal{M}_p(\alpha)$, for $\alpha>2/p$, is a bounded convex function. In particular all odd alpha-starlike functions are bounded if $\alpha>1$.

THEOREM 3. *If* $f \in \mathcal{M}_p(\alpha)$, $\alpha>0$, *with power series* (1.1) *then* $|a_{p+1}| \leqq 2/p(1+p\alpha)$ *and this bound is sharp.*

PROOF. In [**1**] it is shown that if $g \in \mathcal{M}_1(p\alpha)$, $p\alpha>0$, the coefficient $b_2=g''(0)/2$ satisfies $|b_2| \leqq 2/(1+p\alpha)$. Since $f(z)=[g(z^p)]^{1/p}$, a straightforward calculation shows $|a_{p+1}| \leqq 2/p(1+p\alpha)$. This inequality is sharp for f_0. Notice that for $\alpha=0$ or 1 and $p=2$ this reduces to the familiar bounds 1 and $\frac{1}{3}$ respectively.

THEOREM 4. *If* $f \in \mathcal{M}_p(\alpha)$, $\alpha>0$, *then the image of D under the mapping* $w=f(z)$ *always contains the disc* $|w|<\hat{d}(\alpha)$ *where*

$$\hat{d}(\alpha) = (\tfrac{1}{2})^{2/p} \qquad \text{when } \alpha = 0,$$
$$= \left[\frac{1}{2p\alpha} \frac{[\Gamma(1/p\alpha)]^2}{\Gamma(2/p\alpha)}\right]^\alpha \qquad \text{when } \alpha > 0.$$

These results are sharp with equality for f_0.

PROOF. Clearly $\hat{d}(\alpha)=[d(p\alpha)]^{1/p}$ where d is the radius of the largest disc always contained in the image $w=g(z)$ where $f(z)=[g(z^p)]^{1/p}$. But by [1],

$$d(p\alpha) = \left[\frac{1}{2p\alpha}\frac{\Gamma(1/p\alpha)^2}{\Gamma(2/p\alpha)}\right]^{p\alpha},$$

which proves the result for $\alpha>0$. For $\alpha=0$, the Koebe function gives us $(\frac{1}{2})^{2/p}$ and in fact $\lim_{\alpha\to 0^+}\hat{d}(\alpha)=(\frac{1}{2})^{2/p}$, thus establishing the result.

Notice that for $\alpha=0$ or 1 and $p=1$ or 2, we have $\hat{d}(\alpha)$ given by

$p \backslash \alpha$	0	1
1	$\frac{1}{4}$	$\frac{1}{2}$
2	$\frac{1}{2}$	$\frac{\pi}{4}$

If we let $p\to\infty$ we notice $\lim_{p\to\infty}\hat{d}(\alpha)=1$, $\alpha\geqq 0$ thus providing another proof of the well-known fact that $\lim_{p\to\infty}[g(z^p)]^{1/p}$ is the function $h(z)=z$.

THEOREM 5. *If $f\in\mathcal{M}_p(\alpha)$, $\alpha>0$, and $M(r)=\max_\theta|f(re^{i\theta})|$, then*

$$\begin{aligned} M(r) &= O(1/(1-r))^{(2-p\alpha)/p} \quad \textit{for } 0\leqq\alpha<2/p, \\ &= O\log(1/(1-r)) \quad \textit{for } \alpha=2/p, \end{aligned}$$

as $r\to 1^-$. If $\alpha>2/p$, then

$$M(r) \leqq \left[\frac{1}{p\alpha}\Gamma(1/p\alpha)\frac{\Gamma(1-2/p\alpha)}{\Gamma(1-1/p\alpha)}\right]^{\alpha}.$$

These bounds are best possible with equality for f_0.

PROOF. From [1] we see that if $g\in\mathcal{M}_1(p\alpha)$, then

$$\begin{aligned} \max_\theta|g(r^pe^{i\theta})| &= O(1/(1-r^p))^{2-p\alpha} \quad \text{for } 0\leqq\alpha<2, \\ &= O(\log 1/(1-r^p)) \quad \text{for } \alpha=2, \end{aligned}$$

as $r\to 1^-$. If $\alpha>2$, then

$$\max_\theta|g(r^pe^{i\theta})| \leqq \left[\frac{1}{p\alpha}\frac{\Gamma(1/p\alpha)\Gamma(1-2/p\alpha)}{\Gamma(1-1/p\alpha)}\right]^{p\alpha}.$$

Letting $f(z)=[g(z^p)]^{1/p}$ and taking pth roots of the above we obtain the desired result.

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