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Boundary behavior of the iterates of a self-map of the unit disk

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ABSTRACT

We show that there is a proper boundary Denjoy–Wolff theorem for those parabolic self-maps of $\mathbb{D}$ of zero hyperbolic step whose Koenigs function has an angular limit almost everywhere on $\partial\mathbb{D}$. We also provide some quantitative information about this convergence.

1. Introduction

The study of the dynamics of an arbitrary analytic self-map φ of the (open) unit disk $\mathbb{D}$ is a classical and well-established branch of complex iteration. Probably, the central result in the area is the celebrated Denjoy–Wolff theorem, which states that, if φ is different from an elliptic automorphism, the iterates (φ_n) converge to a certain point $\tau \in \overline{\mathbb{D}}$ uniformly on compacta of $\mathbb{D}$. This point is clearly unique and it is called the *Denjoy–Wolff point* of φ. Moreover, if $\tau \in \partial\mathbb{D}$, it is the unique angular fixed point whose angular derivative belongs to $(0, 1]$.

By Fatou's theorem, the above iterates (φ_n) are indeed well-defined (in the angular sense) in the boundary of the unit disk $\partial\mathbb{D}$, up to a set of Lebesgue measure zero. In other words, there is a set $A \subseteq \partial\mathbb{D}$, with Lebesgue measure zero, such that the angular limit $\varphi_n(\xi) := \angle\lim_{z\to\xi} \varphi_n(z)$ exists, for all $\xi \in \partial\mathbb{D} \setminus A$ and for all $n \in \mathbb{N}$. This fact opens the possibility of considering (almost everywhere) the iteration (φ_n) in the *closed* unit disk and poses the natural question about how the classical Denjoy–Wolff theorem can be extended to this situation. Also, this problem has been treated by several authors (see Section 3), but a definitive boundary version of the Denjoy–Wolff theorem is still open, mainly due to the difficulties that appeared in the so-called parabolic case. In this paper, we approach this problem from a geometrical–dynamical point of view. Namely, we prove:

Theorem 1.1. *Let φ be a parabolic self-map of $\mathbb{D}$ of zero hyperbolic step with Denjoy–Wolff point τ. If the associated Koenigs function σ has an angular limit almost everywhere on $\partial\mathbb{D}$, then $(\varphi_n(\xi))$ converges to τ for almost all $\xi \in \partial\mathbb{D}$.*

The assumption about σ is satisfied if for instance σ belongs to some Hardy class. It is also satisfied if $\mathbb{C}\setminus\sigma(\mathbb{D})$ has positive logarithmic capacity, for instance if σ omits a continuum as it does for univalent φ.

In Section 2, we give a quick overview of the iteration of the unit disk, fixing notation and recalling some notions. In view of the fact that the material is relatively recent and spread over several papers, in Section 3 we present a quick but detailed review of the state of the art for the boundary version of the Denjoy–Wolff theorem. In Section 4, we give the proof of our main result (Theorem 1.1) and, in the final section, we provide new quantitative information about the uniform convergence to $\tau \in \partial\mathbb{D}$ of the iterates (φ_n) in $\partial\mathbb{D}$, for most of the cases where the boundary Denjoy–Wolff theorem is known.

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2. Discrete iteration in the unit disk

As is often done in the literature, we classify, according to their behavior near the Denjoy–Wolff point, the non-identity holomorphic self-maps of the disk into three categories:

(a) *elliptic*: the ones with a fixed point inside the unit disk (also called the Denjoy–Wolff point of φ);
(b) *hyperbolic*: the ones with the Denjoy–Wolff point $\tau \in \partial\mathbb{D}$ such that $\varphi'(\tau) < 1$;
(c) *parabolic*: the ones with the Denjoy–Wolff point $\tau \in \partial\mathbb{D}$ such that $\varphi'(\tau) = 1$.

Moreover, it is customary to say that φ is of *zero hyperbolic step* if, for some point $z_0 \in \mathbb{D}$, the associated forward orbit $(z_n) := (\varphi_n(z_0))$ satisfies that $\lim_{n\to\infty} \rho_{\mathbb{D}}(z_n, z_{n+1}) = 0$, where $\rho_{\mathbb{D}}$ denotes the hyperbolic metric in $\mathbb{D}$. It is well-known that the words "some point" here can be replaced by "all points". Using the Schwarz–Pick Lemma, one can prove that the maps which are not of zero hyperbolic step are precisely those holomorphic self-maps φ for which

$$\lim_{n\to\infty} \rho_{\mathbb{D}}(z_n, z_{n+1}) > 0,$$

for some (or all) forward orbits (z_n) of φ. This is why they are called *maps of positive hyperbolic step*. If φ is hyperbolic, then it is always of positive hyperbolic step. However, there exist parabolic maps of zero as well as positive hyperbolic step (see [1] for a wide variety of examples and more information).

One of the most striking results concerning self-maps of the unit disk is the so-called Linear Fractional Model Theorem which essentially says that the iteration properties of an analytic self-map of the unit disk can be understood, via conjugation, by means of the iteration properties of a certain linear fractional map. In the elliptic case, such conjugation was provided by Koenigs as early as 1884 (see [2]). In 1931, Valiron treated the hyperbolic case, showing the existence of a holomorphic map σ from $\mathbb{D}$ into the right half-plane $\mathbb{H}$ such that

$$\sigma \circ \varphi = \frac{1}{\varphi'(\tau)}\sigma. \tag{2.1}$$

In addition, if we assume that $|\sigma(0)| = 1$, then the map σ satisfying (2.1) is unique (see [3]) and it is called the Koenigs function associated with φ.

Over more than fifty years, the problem of finding a model for the remaining parabolic case was a challenging problem. Anyway, this problem was solved by the third author [4] in 1979 for the case of positive hyperbolic step, and by Baker and the third author [5], also in 1979, for the case of zero hyperbolic step. In both cases, it was proved that if φ is a parabolic self-map of the unit disk, then there exists a holomorphic map $\sigma : \mathbb{D} \to \mathbb{C}$ such that

$$\sigma \circ \varphi = \sigma + 1. \tag{2.2}$$

Eq. (2.2) is known as the Abel equation and the solution built in [4,5] will be called *the Koenigs intertwining map associated with* φ (see [6,1] for an analysis of the properties of such map).

3. A bit of history

Thinking about elliptic inner functions (recall that an inner function is a holomorphic self-map of $\mathbb{D}$ such that their radial limits have modulus 1 almost everywhere on $\partial\mathbb{D}$), it is clear that we have to impose some restrictions in order to obtain a proper boundary Denjoy–Wolff theorem (i.e., $(\varphi_n(\xi))$ converges to the Denjoy–Wolff point for almost every $\xi \in \partial\mathbb{D}$). Such a theorem is almost settled nowadays thanks to the efforts of Bourdon, Matache, and Shapiro and also, independently and with completely different techniques, of Poggi-Corradini.

Theorem 3.1 (*[7,8]*). *Assume that φ is a holomorphic self-map of $\mathbb{D}$.*

(1) *If φ is elliptic and other than the identity, then $(\varphi_n(\xi))$ converges to the Denjoy–Wolff point of φ, for almost every $\xi \in \partial\mathbb{D}$, if and only if φ is not an inner function.*
(2) *If φ is either hyperbolic or parabolic of positive hyperbolic step, then $(\varphi_n(\xi))$ converges to the Denjoy–Wolff point of φ, for almost every $\xi \in \partial\mathbb{D}$.*

As in the elliptic case, for parabolic functions of zero hyperbolic step we must consider in turn two different subclasses of functions: the inner and the non-inner ones. The problem for the first group is also completely understood. To state the corresponding result we need to recall that a sequence (b_n) in the unit disk is a Blaschke sequence if

$$\sum_{n=1}^{\infty}(1 - |b_n|) < \infty.$$

A Blaschke orbit is an orbit which is a Blaschke sequence.

The next theorem is due to Doering and Mañé [9, Theorems E and F] (see also [7, Theorem 4.2]).

Theorem 3.2. *Suppose that φ is an inner parabolic self-map of the unit disk of zero hyperbolic step and with Denjoy–Wolff point τ. Then $(\varphi_n(\xi))$ converges to τ, for almost every $\xi \in \partial\mathbb{D}$, if and only if some (or all) forward orbit(s) of φ is a (are) Blaschke orbit(s).*

It must be pointed out that there are inner parabolic maps of zero hyperbolic step whose orbits are not Blaschke orbits (see, e.g., [1, Example 8.3] and also [10, Lemma 4.5]). At the same time, there are even parabolic Blaschke products of zero hyperbolic step whose orbits satisfy the Blaschke condition (see, e.g., [7, Corollary 5.4]).

In the literature, one can also find another way of coping with the existence of a boundary Denjoy–Wolff theorem. Namely, the idea is just to establish *general* criteria which guarantee such a theorem, instead of dealing with subcases. The state of the art is, more or less, as follows:

Theorem 3.3 *([7, Theorems 4.2 and 4.4]). Let φ be an analytic self-map of the unit disk with Denjoy–Wolff point $\tau \in \partial\mathbb{D}$.*

(1) *If some forward orbit of φ is a Blaschke sequence, then $(\varphi_n(\xi))$ converges to the Denjoy–Wolff point of φ, for almost every $\xi \in \partial\mathbb{D}$.*
(2) *If φ is hyperbolic or parabolic with positive hyperbolic step, then every forward orbit of φ is a Blaschke sequence.*

In the final section of this paper we give a more explicit form of the first part of the above theorem (see Theorem 5.3). Certainly, our main result fits in this line of research.

Note this approach could be considered closed if one settles the following conjecture: If a parabolic map with zero hyperbolic step satisfies the boundary Denjoy–Wolff theorem, then there exists a forward orbit of φ which is a Blaschke orbit. Apparently, this conjecture is still unsolved.

4. Proof of our main theorem

It is not difficult to show that Theorem 1.1 is a consequence of the following one:

Theorem 4.1. *Let φ be a parabolic self-map of $\mathbb{D}$ of zero hyperbolic step with Denjoy–Wolff point 1 and associated Koenigs function σ. Let A be the set of points $\xi \in \partial\mathbb{D} \setminus \{1\}$ with the following property: there exists a semi-open curve $\Gamma \subset \mathbb{D}$ such that $\overline{\Gamma} = \Gamma \cup \{\xi\}$ and*

$$\inf\{\operatorname{Re}\sigma(z) : z \in \Gamma \cap \mathbb{D}\} > -\infty. \tag{4.1}$$

Assume that there are two sequences (ξ_k^+) and (ξ_k^-) in A such that

$$\arg \xi_k^+ > 0, \qquad \arg \xi_k^- < 0 \quad \text{for all } k \in \mathbb{N} \quad \text{and} \quad \lim_{k\to\infty} \xi_k^{\pm} = 1. \tag{4.2}$$

If $\xi \in A$, then

$$\sup\{|\varphi_n(z) - 1| : z \in \Gamma \cap \mathbb{D}\} \to 0 \quad (n \to \infty). \tag{4.3}$$

Proof. Fix $\xi \in A$ and a curve Γ such that $\overline{\Gamma} = \Gamma \cup \{\xi\}$ and $\inf\{\operatorname{Re}\sigma(z) : z \in \Gamma \cap \mathbb{D}\} > -\infty$. Denote by z_0 the starting point of the curve Γ. Since 1 is the Denjoy–Wolff point of the function φ, we know that $\varphi_n(z_0) \to 1$.

Write $\Gamma_n := \varphi_n(\Gamma)$. Then we claim that

$$\inf\{|z| : z \in \Gamma_n\} \to 1 \quad \text{as } n \to \infty. \tag{4.4}$$

Otherwise, there are $a \in \mathbb{D}$ and two sequences $n_k \in \mathbb{N}$ and $z_k \in \Gamma$ such that $\varphi_{n_k}(z_k) \to a$. Thus $\sigma(\varphi_{n_k}(z_k)) \to \sigma(a) \in \mathbb{C}$ and

$$\operatorname{Re}\sigma(z_k) = \operatorname{Re}\sigma(\varphi_{n_k}(z_k)) - n_k \to -\infty,$$

a contradiction with (4.1).

By (4.4), there are positive numbers $\rho_n \to 1$ such that $\Gamma_n \subset \{z : \rho_n < |z| < 1\}$.

Assume that (4.3) does not hold. Then there are a point $b \in \partial\mathbb{D}$, $b \neq 1$, and two sequences $n_k \in \mathbb{N}$ and $z_k \in \Gamma$ such that $\varphi_{n_k}(z_k) \to b$. Let $B^{\pm}$ denote the two arcs of $\partial\mathbb{D} \setminus \{1, b\}$. By (4.2), there are

$$\xi_j^{\pm} \in A \cap B^{\pm} \quad (j = 1, 2), \qquad \xi_1^{\pm} \neq \xi_2^{\pm}. \tag{4.5}$$

Let $C_j^{\pm}$ be the curves associated with the points $\xi_j^{\pm}$ according to the definition of the set A. Since $\varphi_n(z_0) \to 1$ and (4.4) holds, then there is a subsequence $(\Gamma_{n_k}^*)$ of closed subarcs of (Γ_{n_k}) such that either

$$\Gamma_{n_k}^* \cap C_j^+ \neq \emptyset \quad \text{for all } k \ (j = 1, 2) \tag{4.6}$$

or

$$\Gamma_{n_k}^* \cap C_j^- \neq \emptyset \quad \text{for all } k \ (j = 1, 2). \tag{4.7}$$

Let us assume that (4.6) holds. Let L be a closed arc in B^+ between ξ_1^+ and ξ_2^+ with $\xi_j^+ \cap L = \emptyset$ $(j = 1, 2)$ and let $S = \{rz : 0 \le r < 1,\ z \in L\}$. Shortening the arcs C_j^+ we may assume that $S \cap C_j^+ = \emptyset$ for $j = 1, 2$.

We claim that

$$\inf_{w \in \Gamma_{n_k}^*} \operatorname{Re}\sigma(w) \to +\infty \quad (k \to \infty). \tag{4.8}$$

Otherwise, up to taking another subsequence that we denote again by (w_k), there are points $w_k \in \Gamma^*_{n_k}$ and a real constant c such that $\operatorname{Re}\sigma(w_k) < c$ for all k. Take $z^*_k \in \Gamma$ such that $w_k = \varphi_{n_k}(z^*_k)$. Thus

$$\operatorname{Re}\sigma(z^*_k) = \operatorname{Re}\sigma(w - K) - n_k \to -\infty \quad (k \to \infty).$$

Thus (4.8) is satisfied.

There are two possible cases for the behavior of σ in the set L:

(1) If the angular limit of σ exists on a subset of L of positive measure, by the Privalov Uniqueness Theorem, there is a point $\xi' \in L$ where the angular limit $\sigma(\xi')$ is finite. Thus there are points $z'_k \in (\rho_{n_k}\xi', \xi') \cap \Gamma^*_{n_k}$ such that, by (4.8),

$$|\sigma(z'_k)| \geq \operatorname{Re}\sigma(z'_k) \to \infty \quad (k \to \infty).$$

Therefore, $\sigma(\xi')$ is not finite, a contradiction!

(2) If the subset of L where the angular limit of σ does exist has measure zero, by Plessner's Theorem, there is a point $\xi' \in L$ such that $\sigma(\Delta)$ is dense in $\mathbb{C}$ for every Stolz angle Δ at ξ'. Thus there are points $z'_k \in S$ such that

$$|z'_k| > \max\{|w|: w \in \Gamma^*_{n_k}\}, \tag{4.9}$$

and

$$\operatorname{Re}\sigma(z'_k) \to -\infty \quad (k \to \infty). \tag{4.10}$$

For each k, there is $\mu(k) > k$ such that

$$\min\{|w|: w \in \Gamma^*_{n_{\mu(k)}}\} > \rho_{n_{\mu(k)}} > |z'_k|. \tag{4.11}$$

Let $V_{\mu(k)}$ be the connected component of

$$\mathbb{D} \setminus (C^+_1 \cup C^+_2 \cup \Gamma^*_{n_k} \cup \Gamma^*_{n_{\mu(k)}})$$

that contains the point z'_k. Denote by $w_{\mu(k)} \in \partial V_{\mu(k)}$ a point where the function $\operatorname{Re}\sigma$ attains its minimum on the set $\overline{V_{\mu(k)}}$. Since $\operatorname{Re}\sigma(w_{\mu(k)}) \leq \operatorname{Re}\sigma(z'_k)$, by (4.10), we have that $\operatorname{Re}\sigma(w_{\mu(k)}) \to -\infty$ as k goes to ∞. Since $\operatorname{Re}\sigma$ is bounded below on C^+_j (because ξ^+_k belongs to A), we deduce that

$$w_{\mu(k)} \in \Gamma^*_{n_k} \cup \Gamma^*_{n_{\mu(k)}}$$

for k large enough, a contradiction to (4.8).

In any case, we get a contradiction. Therefore (4.3) is satisfied. □

5. When the orbits are Blaschke sequences

In this section we present a slight improvement of Theorem 3.3 that is based on the following two previous results:

Lemma 5.1 (*Bourdon, Matache and Shapiro; See the Proof of Theorem 4.2 of [7]*)**.** *Let φ be an analytic self-map of the unit disk with Denjoy–Wolff point 1. Then*

$$\sum_{n=m}^{\infty} \frac{1}{2\pi} \int_{\partial\mathbb{D}} |\varphi_n(z) - 1|^2 |dz| \leq 4 \sum_{n=m}^{\infty} (1 - |\varphi_n(0)|), \tag{5.1}$$

for all m.

Lemma 5.2 (*Hardy and Littlewood; See Chapter VII, Theorem 7.36 of [11]*)**.** *Let $f \in H^1$ and $0 < \rho < 1$. There is a constant $c = c(\rho) < \infty$ such that*

$$\int_{\partial\mathbb{D}} \sup_{z \in \Omega_\rho(\zeta)} |f(z)||d\zeta| \leq c \int_{\partial\mathbb{D}} |f(\zeta)||d\zeta|, \tag{5.2}$$

where $\Omega_\rho(\zeta)$ is the convex hull of the set $\{|z| < \rho\} \cup \{\zeta\}$.

Theorem 5.3. *Let φ be an analytic self-map of the unit disk with Denjoy–Wolff point $\tau \in \partial\mathbb{D}$ and such that some orbit is a Blaschke sequence. Then there exist compact sets $A_k \in \overline{\mathbb{D}}$, natural numbers m_k and a null sequence of positive numbers (δ_k) such that for all k:*

(i) $|\varphi_n(z) - \tau| \leq 2^{-k}$ *for all $z \in A_k$ and $n \geq n_k$;*
(ii) $|\partial\mathbb{D} \cap A_k| > 2\pi - 2^{-k}$;
(iii) $A_k \subset A_{k+1}$;

(iv) $\{|z| \le 1 - 2^{-k}\} \subset A_k$;
(v) *for all* $\zeta \in \partial\mathbb{D} \cap A_k$ *the Stolz angle* $\{z \in \mathbb{D} : |\arg(1 - \overline{\zeta} z)| < (\pi - \delta_k)/2\}$ *is in* A_k;
(vi) *the iterates* φ_n *tend to* τ *uniformly on* A_k.

Proof. To simplify we assume that $\tau = 1$. We claim that, with our hypothesis, the series $\sum_{n=1}^{\infty}(1 - |\varphi_n(0)|)$ does converge. On the one hand, if the map is of positive hyperbolic step, we apply Theorem 3.3. On the other hand, if the map is of zero hyperbolic step, take z_0 such that the series $\sum_{n=1}^{\infty}(1-|\varphi_n(z_0)|)$ converges. Then, by Contreras et al. [6, Proof of (iii) implies (i) in Proposition 3.3], we have that $\rho_{\mathbb{D}}(\varphi_n(z_0), \varphi_n(0)) \to 0$. Thus, $\frac{|\varphi_n(z_0)-\varphi_n(0)|}{1-|\varphi_n(0)|} \to 0$ as $n \to \infty$. Therefore, for n large enough, we have that

$$1 - |\varphi_n(z_0)| \ge 1 - |\varphi_n(0)| - |\varphi_n(z_0) - \varphi_n(0)| \ge \frac{1}{2}\,(1 - |\varphi_n(0)|)\,.$$

And we conclude that the series $\sum_{n=1}^{\infty}(1 - |\varphi_n(0)|)$ converges.

By Lemma 5.2, with $f = (\varphi_n - 1)^2$ and $\rho_k = 1 - 2^{-k}$, there is a constant $c(\rho_k)$ such that

$$\sum_{n=m}^{\infty} \int_{\partial\mathbb{D}} \sup_{z \in \Omega_\rho(\zeta)} |\varphi_n(z) - 1|^2 |d\zeta| \le c(\rho_k) \sum_{n=m}^{\infty} \int_{\partial\mathbb{D}} |\varphi_n(\zeta) - 1|^2 |d\zeta|. \tag{5.3}$$

Writing $\eta_m = \sum_{n=m}^{\infty}(1 - |\varphi_n(0)|)$, Lemma 5.1 gives

$$\sum_{n=m}^{\infty} \int_{\partial\mathbb{D}} \sup_{z \in \Omega_\rho(\zeta)} |\varphi_n(z) - 1|^2 |d\zeta| \le 8\pi c(\rho_k)\eta_m. \tag{5.4}$$

By hypothesis, the sequence (η_m) goes to zero. Thus there is m_k such that $8\pi c(\rho_k)\eta_{m_k} < 2^{-3k-2}$.

Let E_k be the set of all $\zeta \in \partial\mathbb{D}$ such that

$$\sum_{n=m_k}^{\infty} \sup\{|\varphi_n(z) - 1| : \, z \in \Omega_{\rho_k}(\zeta)\} \ge 2^{-k}. \tag{5.5}$$

Then $2^{-2k}|E_k| < 2^{-3k-2}$. That is, $|E_k| < 2^{-k-2}$. The sets $E_k^* := \cup_{j=k}^{\infty} E_j$ satisfy $E_k \subset E_k^*$, $E_{k+1}^* \subset E_k$, and $|E_k^*| < 2^{-k-1}$. Therefore, there are closed sets B_k with $B_k \subset \partial\mathbb{D}$ with

$$B_k \subset \partial\mathbb{D} \setminus E_k^* \subset \partial\mathbb{D} \setminus E_k, \qquad B_k \subset B_{k+1}, \qquad |B_k| > 2\pi - 2^{-k}, \tag{5.6}$$

and such that the angular limits of $\varphi_n(\zeta)$ exist for all n and all $\zeta \in B_k$. Now we can define the sets $A_k := \cup_{\zeta \in B_k} \overline{\Omega}_{\rho_k}(\zeta)$. It is clear that A_k is a closed subset of $\overline{\mathbb{D}}$ and that $\varphi_n(z)$ exists for all $z \in A_k$. Let us see that this family of sets satisfies the theorem.

Let $z \in A_k$ and $n \ge m_k$. There is $\zeta \in B_k$ such that $z \in \overline{\Omega}_{\rho_k}(\zeta)$. Thus $\zeta \in \partial\mathbb{D} \setminus E_k$ and we have that $|\varphi_n(z) - 1| \le 2^{-k}$, getting (i). Since $\partial\mathbb{D} \cap A_k = B_k$, we obtain that $|\partial\mathbb{D} \cap A_k| > 2\pi - 2^{-k}$ and (ii) holds. Furthermore (iii) follows from the fact that $B_k \subset B_{k+1}$ and $\overline{\Omega}_{\rho_k}(\zeta) \subset \overline{\Omega}_{\rho_{k+1}}(\zeta)$. Finally, (iv) and (v) follow from $\overline{\Omega}_{\rho_k}(\zeta) \subset A_k$.

Finally, if $z \in A_k$ and $j \ge k$, from (iii) we know that $z \in A_j$ and, by (i), we deduce that

$$|\varphi_n(z) - 1| \le 2^{-j} \quad \text{for } n \ge m_j. \quad \square$$

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