Commutative Algebra - Examples Sheet 1

All rings are assumed to be commutative with unit element.

(1) Let x be a nilpotent element of a ring A. Show that $1+x$ is a unit in A. Deduce that the sum of a nilpotent element and a unit is a unit.

(2) Let A be a ring and let $A[x]$ be the polynomial ring in one variable x with coefficients in A. Let $f = a_0 + a_1x + \ldots + a_nx^n \in A[x]$. Prove that
(a) f is a unit in $A[x] \iff a_0$ is a unit in A and $a_1, \ldots, a_n$ are nilpotent. (If $b_0 + \ldots + b_mx^m$ is the inverse of f, prove by induction on r that $a_n^{r+1}b_{m-r} = 0$. Hence show that a_n is nilpotent, and use exercise 1.)
(b) f is nilpotent $\iff a_0, \ldots, a_n$ are all nilpotent.
(c) The nilradical $\mathrm{Nil}(A[x])$ is the ideal generated by $\mathrm{Nil}(A)$ in $A[x]$, and the Jacobson radical $Jac(A[x])$ is equal to the nilradical $\mathrm{Nil}(A[x])$.

(3) Let A be a ring, x an indeterminate, and $A[[x]]$ the set of all *formal power series* $f = \sum_{n\in\mathbb{Z}_{\geq 0}} a_nx^n$ with coefficients $a_n \in A$. Addition of two formal power series is defined coefficientwise and multiplication is defined by

$$\left(\sum_{n\geq 0} a_nx^n\right) \cdot \left(\sum_{n\geq 0} b_nx^n\right) = \sum_{n\geq 0}\left(\sum_{i=0}^{n} a_ib_{n-i}\right) x^n$$

Show that:
(a) $f = \sum_n a_nx^n$ is a unit in $A[[x]] \iff a_0 \in A^*$.
(b) $f \in Jac(A[[x]]) \iff a_0 \in Jac(A)$. Deduce $Jac(A[[x]])$ is equal to the ideal generated by $Jac(A)$ and x.
(c) If A is a local ring with maximal ideal $\mathfrak{m}$, then $A[[x]]$ is a local ring with maximal ideal $(\mathfrak{m}, x)$.
(d) If $A = k$ is a field, then the non-zero ideals of $k[[x]]$ are exactly the principal ideals generated by x^n, $n \geq 0$.

(4) Let R be a commutative ring with unit. For $a, b \in R$ we say that a *divides* b (notation $a|b$) if $b = ac$ for some $c \in R$. Note that $a|1 \iff a \in R^*$. We say that a is *associated* to b iff $a = ub$ with $u \in R^*$ (notation $a \sim b$). If R is a domain, then $a \sim b \iff (a|b$ and $b|a)$. We call an element $a \in R - R^*$ *irreducible* if for any factorization $a = bc$ one of b, c is a unit in R. A non-zero non-unit a is called a *prime element* if a generates a prime ideal. An integral domain R is called a *unique factorization domain* (UFD) if the following two conditions are satisfied:

(i) every element $a \in R - \{0\}$, which is not a unit can be written as a product of (finitely many) irreducible elements;
(ii) if $a = x_1 \cdots x_r = y_1 \cdots y_s$ with all x_i, y_j irreducible, then $r = s$ and there is a permutation σ of $\{1, \ldots, r\}$ such that for all i: $x_i \sim y_{\sigma(i)}$.

(a) Show that in any domain R the prime elements are irreducible, and that in an UFD the irreducible elements are prime elements. Show further that a domain in which (i) holds and in which the irreducible elements are prime elements is an UFD.

(b) Show further that a PID is an UFD.

(5) An integral domain R is called an *euclidean domain* if there is a map $N : R - \{0\} \to \mathbb{Z}_{\geq 0}$ such that for all $a, b \in R$, $b \neq 0$, there are $d, r \in R$ with the property that

$$a = db + r\,,$$

with either $r = 0$ or $r \neq 0$ and $N(r) < N(b)$.

(a) Show that the ring of *Gaussian integers* $\mathbb{Z}[i] = \{a + ib \in \mathbb{C} \mid a, b \in \mathbb{Z}\}$ is an euclidean ring. (Hint: take $N(a + ib) = a^2 + b^2$ and use the graphic interpretation of elements of $\mathbb{Z}[i]$ as lattice points in $\mathbb{C}$.)
(b) Show that any euclidean domain is a PID. Deduce that $\mathbb{Z}[i]$ is an UFD.
(c) Let k be a field. Show that the ring $k[x]$ of polynomials in one variable over k is euclidean, hence a PID, and thus an UFD.

(6) Let A be an UFD. An element $f = a_0 + a_1x + \ldots + a_nx^n \in A[x]$ is said to be *primitive* if the coefficients $a_0, \ldots, a_n$ have no common divisor except units. Prove that if $f, g \in A[x]$, then fg is primitive iff f and g are both primitive.

(7) Let A be an UFD and K its field of fractions. Use the preceding exercise and the fact that $K[x]$ is a UFD to show that $A[x]$ is an UFD. Deduce that $A[x_1, \ldots, x_n]$ is an UFD.

(8) Show that $\mathbb{Z}[x]$ is not a PID, but that it is an UFD.

(9) Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 2x$, and consider the short exact sequence $0 \to \mathbb{Z} \xrightarrow{f} \mathbb{Z} \to \mathbb{Z}/(2) \to 0$, where the map in the right is the canonical projection. Apply the functor $\mathrm{Hom}_{\mathbb{Z}}(\,\cdot\,, \mathbb{Z}/(2))$ to this sequence and show that the resulting sequence is not exact. Do the same with $\mathrm{Hom}_{\mathbb{Z}}(\,\cdot\,, \mathbb{Z})$.

(10) Let $0 \to N' \xrightarrow{u} N \xrightarrow{v} N''$ be a sequence of A-modules and A-module homomorphisms. Show that this sequence is exact iff for all A-modules M the sequence $0 \to \mathrm{Hom}_A(M, N') \xrightarrow{u_*} \mathrm{Hom}_A(M, N) \xrightarrow{v_*} \mathrm{Hom}_A(M, N'')$ is exact.

(11) Let A be a ring, $\mathfrak{a}$ an ideal, M an A-module. Show that $A/\mathfrak{a} \otimes_A M$ is canonically isomorphic to $M/\mathfrak{a}M$. Deduce that if $\mathfrak{a}$, $\mathfrak{b}$ are ideals of A, then $A/\mathfrak{a} \otimes_A A/\mathfrak{b} \simeq A/(\mathfrak{a} + \mathfrak{b})$. What happens if we assume $\mathfrak{a}$ and $\mathfrak{b}$ coprime? Give an explicit 'formula' for $\mathbb{Z}/(m) \otimes_{\mathbb{Z}} \mathbb{Z}/(n)$. Compute $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/(n)$.

(12) Let $(M_i)_{i \in I}$ be a family of A-modules. Show that M_i is flat for all i iff their direct sum $\bigoplus_i M_i$ is flat. Deduce that the polynomial ring $A[x_1, \ldots, x_n]$ is flat over A.

(13) Let M be an A-module. Show that M is flat if and only if for any finitely generated ideal $\mathfrak{a} \subset A$ the sequence $0 \to \mathfrak{a} \otimes_A M \to M$ is exact, the second map being $a \otimes x \mapsto ax$.

(14) Let $\mathfrak{a}, \mathfrak{b}$ be ideals of the ring A. Show that if $A/\mathfrak{a}$ is isomorphic to $A/\mathfrak{b}$ as A-modules, then $\mathfrak{a} = \mathfrak{b}$.

(15) Suppose A is not the zero ring. Then, if $A^n \simeq A^m$ for non-negative integers m, n, then $n = m$. Therefore, if M is a finitely generated free A-module, then $M \simeq A^n$ for a unique $n \in \mathbb{Z}_{\geq 0}$ called the *rank* of M. (Hint: Tensor with $A/\mathfrak{m}$ for a maximal ideal $\mathfrak{m}$.)

(16) Let F be a free A-module and $f : M \to F$ be a surjective A-module homomorphism. Show that there is an A-module homomorphism $g : F \to M$ such that $f \circ g = \mathrm{id}_F$, i.e. $M = F \oplus \ker(f)$.

(17) Let A be a PID and n a non-negative integer. Show that every submodule U of A^n is free of rank $\leq n$. (By induction on n; consider the projection $p : A^n \to A$ onto the last component; then $p(U) = 0$ or $p(U) \simeq A$, now use the preceding exercise.)

(18) Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. Prove that if M' and M'' are finitely generated, then so is M.

(19) Suppose E is a flat A-module. Show that for every non-zero divisor $a \in A$ the map $E \to E, x \mapsto ax$, is injective. Suppose that A is an integral domain and has the property that every finitely generated ideal $\mathfrak{a} \subset A$ is principal (e.g. a PID). Then an A-module M is flat if and only if M is torsion free (i.e. for all $a \in A - \{0\}$ the map $M \to M, x \mapsto ax$, is injective).

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