(i) The generalized third derivative and its application to the theory of trigonometric series; (ii) On summable trigonometric series : S. Verblunsky.

On a class of projectively flat affine connections: T N. C. Whitehead.

On hyperlemniscate functions, a family of automorphic functions : E. T. Whittaker.

The uniformization of algebraic curves : J. M. Whittaker.

A class of everywhere divergent trigonometrical series: J. R. Wilton.

On certain applications of prolate spheroidal harmonics : D. M. Wrinch.

On quantitative substitutional analysis, (IV) and (V) : A. Young.

ON MEROMORPHIC AND INTEGRAL FUNCTIONS

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1. Let $f(z)$ be a meromorphic function, and, in the notation that is now conventional, write $n(r, x) \equiv n(r, f-x)$ for the number of zeros of $f(z)-x$ in the circle $|z| \leqslant r$, x being any complex number and multiple zeros being credited with their order of multiplicity;

$$N(r, f-x) = \int_0^r \frac{n(y, x)-n(0, x)}{y} dy + n(0, x) \log r;$$

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta;$$

and
$$T(r, f) = m(r, f) + N\left(r, \frac{1}{f}\right).$$

The function $T(r, f)$, introduced into the theory by R. Nevanlinna†, characterizes the growth of $f(z)$. If $f(z)$ is an integral function,

$$T(r, f) = m(r, f).$$

The researches of various writers using the methods introduced by Nevanlinna have shown that, for large values of r, the behaviour of

* Received and read 16 May, 1929.

† R. Nevanlinna, "Zur Theorie der meromorphen Funktionen", *Acta Math.*, 46 (1925), 1-99, where further references will be found.

$N(r, f-x)$ approximates to that of $T(r, f)$, except for a certain set of values of x which may properly be regarded as exceptional.

I write

$$D(r, x) \equiv D(r, f-x) = T(r, f) + K(f, x) - N(r, f-x),$$

where $$K(f, x) = \log^+ |x| + \log 2 - C(f-x),$$

$C(f-x)$ being the logarithm of the modulus of the first non-vanishing coefficient in the Laurent expansion of $f(z)-x$ about the origin. It follows at once from Jensen's theorem that

$$0 \leqslant D(r, x)$$

for all values of x and r. Plainly no significant complementary inequality holds for all values of x and r; in stating such an inequality it is always necessary to exclude a certain exceptional set of values of x. For instance, Nevanlinna proves that

$$\overline{\lim_{r=\infty}} \frac{N(r, x)}{T(r, f)} > \tfrac{1}{3} \quad \text{or} \quad \underline{\lim_{r=\infty}} \frac{D(r, x)}{T(r, f)} < \tfrac{2}{3}$$

for all values of x, with two possible exceptions. The sharpest result of this kind hitherto known is due to Valiron*, who proved that

$$N(r, x) \sim T(r, f) \quad \text{or} \quad D(r, x) = o\{T(r, f)\}$$

for all values of x, with the possible exception of a set of linear measure zero, and showed by examples that the exceptional set may actually have the cardinal number of the continuum. Though he does not state the result explicitly, his argument proves that

$$D(r, x) = O\{T(r, f)\}^{\frac{1}{2}}$$

outside the exceptional set of values of x.

It is natural to search for the sharpest possible *significant* inequality for $D(r, x)$, that is to say, the sharpest inequality that holds outside a set E of insignificant measure. Approaching this problem from a point of view quite different from that of Valiron I have obtained, for the limited class of integral functions of zero or finite order, a result considerably sharper than his. It is†

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* G. Valiron, "Distribution des valeurs des fonctions méromorphes", *Acta Math.*, 47 (1926), 117–142.

† Since this paper was read before the Society a quite different proof of Theorem I for *meromorphic* functions of finite order, together with an analogue for functions of infinite order, has been published by R. Nevanlinna in his book, *La théorie des fonctions méromorphes* (Gauthier-Villars, 1929), where an admirably clear account of the theory will be found.

THEOREM I. *If $f(z)$ is an integral function of finite or zero order, then*

(1) $$D(r,\ x) = O(\log r)$$

for all values of x outside a possible exceptional set E of linear measure zero.

We know that $\log r = o\{m(r,\ f)\}^{a}$ for any $a > 0$.

Therefore, if $f(z)$ satisfies a condition of growth such as

$$\varliminf_{r=\infty} \frac{\log m(r, f)}{\log r} \geqslant \epsilon > 0,$$

we may replace the term $O(\log r)$ in (1) by $O\{\log m(r,\ f)\}$. The question whether this is true in the general case I am unable to answer.

The proof of Theorem I is somewhat elaborate. It depends upon a general proposition of a rather unusual type about meromorphic functions which I state in the next paragraph.

2. When $f(z)$ is a meromorphic function, the trivial inequality

$$m\left(r, \frac{1}{f-x}\right) < \max\left(\log^{+}\left|\frac{1}{f(re^{i\theta})-x}\right|\right)$$

shows that, if $m\{r,\ 1/(f-x)\} > 0$, there is at least one arc of the circumference $|z| = r$ on which

$$|f(z)-x| \leqslant \exp\left\{-m\left(r, \frac{1}{f-x}\right)\right\} = \mu(r,\ x).$$

Any such arc belongs to a connected domain $\Delta_{\mu(r,\ x)}(x)$ in which this inequality holds. If a domain $\Delta_{\mu(r,\ x)}(x)$ is bounded and contains no zero of $f'(z)$, I say that it is of class (a); if it is unbounded or, though bounded, contains a zero of $f'(z)$, I say that it is of class (β). I now define the number $\sigma_f = \sigma_f(r,\ x)$ as follows for all values of r and x :—

(i) If $m\{r,\ 1/(f-x)\} = 0$, then $\sigma_f = 1$.

(ii) If $m\{r,\ 1/(f-x)\} > 0$, and at least one of the domains $\Delta_{\mu(r,\ x)}(x)$ intercepted by $|z| = r$ is of class (β), then

$$\sigma_f = \exp\left\{-m\left(r, \frac{1}{f-x}\right)\right\}.$$

(iii) If $m\{r,\ 1/(f-x)\} > 0$ and every $\Delta_{\mu(r,\ x)}(x)$ intercepted by $|z| = r$ is of class (a), each of them is contained in a similar domain

$\Delta_\lambda(x)$ $[\lambda > \mu(r, x)]$ in which $|f(z)-x| \leqslant \lambda$. Then σ_f is the greatest value of λ for which every $\Delta_\lambda(x)$ intercepted by $|z| = r$ is of class (a).

With this definition of σ_f we have

THEOREM II. *If $f(z)$ is any meromorphic function and x any complex number, then*

$$\log \sigma_f(r, x) < -m\left(r, \frac{1}{f-x}\right) + \log^+ 8r + m\left(r, \frac{f'}{f-x}\right) \tag{2}$$

for any $r > 0$.

The proof requires the definition of σ_f to be stated in terms of the inverse function $z = \phi(w)$ of $w = f(z)$, and then proceeds from a simple application of Koebe's Verzerrungssatz to the schlicht elements of certain branches of $\phi(w)$ in the neighbourhood of $w = x$. Here I adopt the direct definition of σ_f in order to avoid the somewhat elaborate preliminaries about $\phi(w)$ required by the inverse definition.

In an earlier note* I proved, by a similar argument, a proposition about integral functions, which is, in fact, a corollary of Theorem II.

3. To apply Theorem II observe that, since

$$D(r, x) < m\left(r, \frac{1}{f-x}\right) + 2(\log^+ |x| + \log 2),$$

(2) gives

$$\log \sigma_f(r, x) < -D(r, x) + \log^+ 8r + m\left(r, \frac{f'}{f-x}\right) + 2(\log^+ |x| + \log 2). \tag{3}$$

Nevanlinna has shown that, if $f(z)$ is of finite order $\rho \geqslant 0$, there is a constant $K(\rho)$ such that, for $r > r_x$,

$$m\left(r, \frac{f'}{f-x}\right) < K(\rho) \log r;$$

so that it follows from (3) that the set E of x, for which (1) does not hold, is included in the set $F(J)$ of values of x for which

$$\varlimsup_{r=\infty} \frac{-\log \sigma_f(r, x)}{\log r} > J - K(\rho),$$

where $J > K(\rho)$ but is otherwise arbitrary.

In the case of integral functions of finite order $\rho \geqslant 0$, I am able to show that we can choose J such that $F(J)$ is of linear measure zero.

The proofs of these results will be published elsewhere.

* "Sur les valeurs exceptionnelles des fonctions entières d'ordre fini", *Comptes rendus*, 179 (1924), 1125–1127.