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On some log-cosine integrals related to $\zeta(3)$, $\zeta(4)$, and $\zeta(6)$

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Abstract

We evaluate a variety of log-cosine integrals, one of which has value a rational multiple of $\pi\zeta(4)$, while others evaluate to the form $\pi[a\zeta(6)+b(3/4)\zeta^2(3)]$, where $a>0$ is rational and b is 1 or 2, and ζ is the Riemann zeta function. We show that these results may be obtained starting from a known tabulated result. The derivation yields results for certain classes of logarithm-trigonometric integrals, and indicates how further analytic results on digamma series may be obtained.

MSC: 33B; 33E; 11M

Keywords: Gamma function; Digamma function; Polygamma function; Riemann zeta function; Dilogarithm function; Trilogarithm function; Log-cosine integrals

Previously, a certain log-cosine integral has been considered in connection with certain digamma series [3,6]. This integral has value a rational multiple of $\pi\zeta(4)$, where ζ is the Riemann zeta function [7,10,15,12]. We show that this integral may be alternatively evaluated starting from a known tabulated result [9]. The derivation yields results suitable for certain classes of logarithmic-trigonometric integrals. We then describe in Appendix A how some integrals related to $\zeta^2(3)$ may be further used in the study of digamma series of higher degree. Appendix B additionally discusses the nonlinear Euler sum introduced in Appendix A. Besides demonstrating an alternative approach, our logarithmic integrals have potential application in the evaluation of classical, semiclassical, and quantum entropies of position and momentum [13,5]. Other quantum mechanical applications could include the evaluation of matrix elements of powers of the logarithm function. In the classical case, these integrals would be of interest when the position distribution function of the motion is written in terms of trigonometric functions. In the quantum case, trigonometric wavefunctions lead to similar integrals.

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As a result of investigating other integrals containing parameters, we first verify the evaluation

$$\frac{1}{\pi}\int_0^{\pi} \theta^2 \ln^2(2\cos\theta/2)\,d\theta = \frac{11\pi^4}{180} = \frac{11}{2}\zeta(4). \tag{1}$$

As corollaries, results for some digamma series follow [3]. With an elementary change of variable, the integral of Eq. (1) is equivalent to

$$\frac{8}{\pi}\int_0^{\pi/2} u^2[\ln 2 + \ln(\cos u)]^2\,du. \tag{2}$$

This expression gives a sum of three integrals, one of which is completely elementary. Before evaluating the other two logarithmic integrals, we develop some analytic results.

The evalution of the integral

$$I(a,p) = \int_0^{\pi/2} x\cos^{p-1}x\sin ax\,dx, \quad \mathrm{Re}\, p > 0,\ |a| < |p+1| \tag{3}$$

is known in terms of the gamma Γ and digamma ψ functions [9, p. 453]. We may note in passing that integrating this result with respect to the parameter a gives an integral closely related to that considered by other authors [3],

$$\int_y^1 I(a,p)\,da = \int_0^{\pi/2} \cos^{p-1}x\cos yx\,dx - \frac{\sqrt{\pi}}{2}\frac{\Gamma[(p+1)/2]}{\Gamma(p/2+1)}, \tag{4}$$

where the last term on the right-hand side follows from the beta function B. From the tabulation [9],

$$I(a,p) = \pi 2^{-(p+1)}\Gamma(p)\frac{[\psi(p_1)-\psi(p_2)]}{\Gamma(p_1)\Gamma(p_2)}, \quad p_1 = (p+a+1)/2, \quad p_2 = (p-a+1)/2, \tag{5}$$

it follows that

$$\frac{\partial I(a,p)}{\partial a} = \pi\frac{2^{-(p+2)}\Gamma(p)}{\Gamma(p_1)\Gamma(p_2)}\{-[\psi(p_1)-\psi(p_2)]^2 + \psi'(p_1) + \psi'(p_2)\}, \tag{6}$$

where ψ' is the derivative of the digamma function. By putting $a=0$, we have

$$J_p \equiv \left.\frac{\partial I(a,p)}{\partial a}\right|_{a=0} = \int_0^{\pi/2} x^2\cos^{p-1}x\,dx = \pi\frac{2^{-(p+1)}\Gamma(p)}{\Gamma^2[(p+1)/2]}\psi'[(p+1)/2]. \tag{7}$$

Then, differentiating with respect to the exponent p, we have

$$\begin{aligned}\frac{1}{\pi}\frac{dJ_p}{dp} &= \frac{1}{\pi}\int_0^{\pi/2} x^2\cos^{p-1}x\ln(\cos x)\,dx \\ &= \frac{2^{-(p+1)}\Gamma(p)}{\Gamma^2[(p+1)/2]}\psi'[(p+1)/2]\left\{-\ln 2 + \psi(p) - \psi[(p+1)/2] + \frac{\psi''[(p+1)/2]}{2\psi'[(p+1)/2]}\right\}. \end{aligned}\tag{8}$$

In particular, we have for the second term in expression (2),

$$\frac{1}{\pi}\left.\frac{dJ_p}{dp}\right|_{p=1} = \frac{1}{\pi}\int_0^{\pi/2} x^2 \ln(\cos x)\,dx = \frac{\pi^2}{24}\left[-\ln 2 + \frac{\psi''(1)}{2\psi'(1)}\right], \tag{9}$$

where $\gamma = -\psi(1) \simeq 0.57721566490$ is the Euler constant, $\psi'(1) = \pi^2/6$, and $\psi''(1) = -2\zeta(3)$.

Similarly, taking the second derivative of J_p gives

$$\begin{aligned}\frac{1}{\pi}\frac{d^2J_p}{dp^2} &= \frac{1}{\pi}\int_0^{\pi/2} x^2 \cos^{p-1} x \ln^2(\cos x)\,dx \\ &= \frac{2^{-(p+1)}\Gamma(p)}{\Gamma^2[(p+1)/2]}\psi'[(p+1)/2]\left\{\left[-\ln 2 + \psi(p) - \psi[(p+1)/2] + \frac{\psi''[(p+1)/2]}{2\psi'[(p+1)/2]}\right]^2\right. \\ &\quad \left. + \psi'(p) - \frac{1}{2}\psi'[(p+1)/2] + \frac{1}{2}\frac{d}{dp}\frac{\psi''[(p+1)/2]}{\psi'[(p+1)/2]}\right\}.\end{aligned} \tag{10}$$

In particular, since $\psi'''(1) = \pi^4/15$, we have

$$\left.\frac{d^2J_p}{dp^2}\right|_{p=1} = \int_0^{\pi/2} x^2 \ln^2(\cos x)\,dx = \frac{\pi}{1440}[11\pi^4 + 60\pi^2 \ln^2 2 + 720 \ln 2\zeta(3)], \tag{11}$$

giving the third term in expression (2). By combining terms, including Eqs. (9) and (11), we obtain Eq. (1). In the final result (1), both the $\ln^2 2$ and $\zeta(3)$ contributions cancel out.

Similarly, starting with successive derivatives with respect to the parameter a in Eq. (6), one may obtain logarithmic integrals with higher powers of x in the integrand. On the other hand, using higher derivatives of J_p than in Eq. (10) gives integrands with log-cosine to degrees greater than quadratic. These types of integrals have potential application in classical and quantum mechanics in the calculation of position and momentum entropies [13,5] and matrix elements of the logarithm function.

In addition, the asymptotic relation between classical and quantum entropies involves log-trigonometric integrals [13]. These integrals arise because the WKB approximation [14] to the wave-function has a cosine phase function factor.

Next, we remark on the classes of integrals which may be obtained by first differentiating the integral I of Eq. (3) with respect to the exponent p,

$$\begin{aligned}\frac{\partial I(a,p)}{\partial p} &= \pi\frac{2^{-(p+1)}\Gamma(p)}{\Gamma(p_1)\Gamma(p_2)}[\psi(p_1) - \psi(p_2)] \\ &\quad \times\left\{-\ln 2 + \psi(p) + \frac{1}{2}[\psi(p_1) + \psi(p_2)] + \frac{1}{2}\left[\frac{\psi'(p_1) - \psi'(p_2)}{\psi(p_1) - \psi(p_2)}\right]\right\}.\end{aligned} \tag{12}$$

The value at $p=1$ is

$$K_a \equiv \left.\frac{\partial I(a,p)}{\partial p}\right|_{p=1}$$
$$=\frac{\pi}{4}\frac{[\psi(a/2+1)-\psi(1-a/2)]}{\Gamma(a/2+1)\Gamma(1-a/2)}\left\{-\ln 2-\gamma+\frac{1}{2}[\psi(a/2+1)+\psi(1-a/2)]\right.$$
$$\left.+\ \frac{1}{2}\left[\frac{\psi'(a/2+1)-\psi'(1-a/2)}{\psi(a/2+1)-\psi(1-a/2)}\right]\right\}, \qquad (13)$$

where $1/\Gamma(a/2+1)\Gamma(1-a/2)=2\sin(\pi a/2)/\pi a$. In turn, we have such integrals as

$$\frac{dK_a}{da}=\int_0^{\pi/2} x^2\cos ax\ln(\cos x)\,dx. \qquad (14)$$

We describe in Appendix A how the logarithmic-trigonometric integrals mentioned above may be used to obtain higher degree digamma series. In particular, the integrals of the form $\int_0^{\pi/2} u^\alpha \ln^\beta(2\cos u)\,du$, where α or β is, respectively, either 2 or 4, given in Eqs. (A.10) and (A.11), evaluate to a very simple sum of the form $\pi[a\zeta(6)+b(3/4)\zeta^2(3)]$, where $a>0$ is rational and b is 1 or 2.

Finally, in Appendix B we further discuss the nonlinear Euler series $\sum_{n=1}^\infty H_n^4/(n+1)^2$ introduced in Appendix A, where H_n are harmonic numbers. While Appendix A follows a Fourier series-based approach, Appendix B uses results from the calculus of residues.

Appendix A. Fourier series approach

We consider how the series $\sum_{n=1}^\infty H_n^4/(n+1)^2$ may be obtained via Parseval's theorem for Fourier series, where the harmonic sum $H_n \equiv \sum_{k=1}^n 1/k=\psi(n+1)-\psi(1)$. Previously, the much simpler sum $\sum_{n=1}^\infty H_n^2/(n+1)^2=(11/4)\zeta(4)$ has been obtained [3]. A sum of a good many log-cosine integrals discussed in the text is necessary for the evaluation, together with additional parametric integrations. Accordingly, we are content here to partially evaluate the necessary L_2 function norm of Eq. (A.9) below.

From the generating function

$$\sum_{n=1}^\infty H_n^2 z^n=\frac{\ln^2(1-z)}{1-z}+\frac{Li_2(z)}{1-z}, \quad |z|<1, \qquad (A.1)$$

it follows that

$$\sum_{n=1}^\infty \frac{H_n^2 z^{n+1}}{n+1}=-\frac{1}{3}\ln^3(1-z)+\int_0^z \frac{Li_2(t)}{(1-t)}\,dt, \quad |z|<1. \qquad (A.2)$$

In Eq. (A.1), the dilogarithm function is defined as $Li_2(z) = \sum_{j=1}^{\infty} z^j/j^2$ [11]. We have the expression

$$\int_0^z \frac{Li_2(t)}{(1-t)}\,\mathrm{d}t = -2Li_2(z)\ln(1-z) - \frac{\pi^2}{12}\ln(1-z) + \sum_{k=1}^{\infty}\sum_{\ell=1}^{k}\frac{z^{k+1-\ell}}{k+1-\ell}, \quad |z|<1 \tag{A.3}$$

obtained from multiple integration by parts [9] and the alternative form

$$\int_0^z \frac{Li_2(t)}{(1-t)}\,\mathrm{d}t = -\ln(1-z)[Li_2(1-z)+\zeta(2)] + 2[Li_3(1-z)-\zeta(3)], \quad |z|<1, \tag{A.4}$$

where $Li_3(z) = \sum_{j=1}^{\infty} z^j/j^3$, based upon the property

$$Li_n(z) = \int_0^z \frac{Li_{n-1}(t)}{t}\,\mathrm{d}t \tag{A.5}$$

and the fact that $Li_3(1) = \zeta(3)$ [11].

We obtain a Fourier sine series from Eq. (A.2) by putting $z = r\exp(it)$, $0<r<1$, $0<t\leqslant\pi$, letting $r\to 1-$ and taking the imaginary part. These operations, justified on the basis of Abel's theorem, give

$$\sum_{n=1}^{\infty}\frac{H_n^2}{(n+1)}\sin(n+1)t = g(t)+h(t), \tag{A.6}$$

where

$$g(t) \equiv -\frac{1}{6}(t-\pi)\left[3\ln^2(2\sin t/2) - \frac{1}{4}(t-\pi)^2\right] \tag{A.7}$$

and

$$h(t) \equiv \mathrm{Im}\left(\lim_{r\to 1-}\int_0^z \frac{Li_2(t)}{(1-t)}\,\mathrm{d}t,\right), \quad |z|<1. \tag{A.8}$$

It then follows from Parseval's theorem that

$$\sum_{n=1}^{\infty}\frac{H_n^4}{(n+1)^2} = \frac{2}{\pi}\int_0^{\pi}[g(t)+h(t)]^2\,\mathrm{d}t. \tag{A.9}$$

Fig. 1 illustrates various partial sums of the sine series of Eq. (A.6).

We have determined, using the methods presented in the text, that

$$\int_0^{\pi/2} u^4\ln^2(2\cos u)\,\mathrm{d}u = \frac{5\pi^7}{8064} + \frac{3\pi}{4}\zeta^2(3) \tag{A.10}$$

and

$$\int_0^{\pi/2} u^2\ln^4(2\cos u)\,\mathrm{d}u = \frac{33\pi^7}{4480} + \frac{3\pi}{2}\zeta^2(3), \tag{A.11}$$

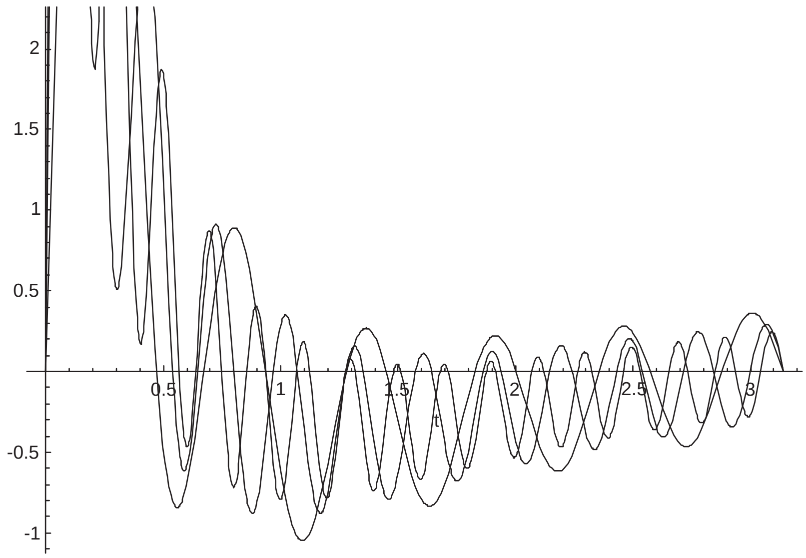

Fig. 1. The partial sums of the Fourier sine series of Eq. (A.6) are plotted for 10, 20, and 30 terms.

giving

$$\frac{2}{\pi}\int_0^{\pi} g^2(t)\,dt = \frac{107\pi^6}{3780} + 4\zeta^2(3). \tag{A.12}$$

From Eqs. (A.4) and (A.8), we find that

$$h(t) = \tfrac{1}{2}(\pi - t)[\mathrm{Re}\,Li_2(1-z)|_{r\to 1-} + \zeta(2)] - \ln(2\sin t/2)\,\mathrm{Im}[Li_2(1-z)]_{r\to 1-} + 2\,\mathrm{Im}[Li_3(1-z)]_{r\to 1-}, \tag{A.13}$$

where

$$\mathrm{Im}[Li_3(1-z)]_{r\to 1-} = -\sum_{n=1}^{\infty}\frac{(-2)^n}{n^3}[\cos(n\pi/2)\sin(nt/2) + \sin(n\pi/2)\cos(nt/2)]\sin^n(t/2), \tag{A.14a}$$

$$\mathrm{Re}[Li_2(1-z)]_{r\to 1-} = -\sum_{n=1}^{\infty}\frac{(-2)^n}{n^2}[\cos(n\pi/2)\cos(nt/2) - \sin(n\pi/2)\sin(nt/2)]\sin^n(t/2) \tag{A.14b}$$

and

$$\mathrm{Im}[Li_2(1-z)]_{r\to 1-} = -\sum_{n=1}^{\infty}\frac{(-2)^n}{n^2}[\cos(n\pi/2)\sin(nt/2) + \sin(n\pi/2)\cos(nt/2)]\sin^n(t/2). \tag{A.14c}$$

Since $\cos(n\pi/2) = (-1)^{n/2}$ for n even and is zero otherwise, and $\sin(n\pi/2) = (-1)^{(n-1)/2}$ for n even and is zero otherwise, it is obvious that Eqs. (A.14) can be written as separate sums over even and odd indices, and then reindexed if desired.

The evaluation of the term $\int_0^\pi g(t)h(t)\,dt$ in Eq. (A.9) requires integrals of the form

$$-\left.\frac{\partial^3 I(a,p)}{\partial p \partial a^2}\right|_{a=p-1} = \int_0^{\pi/2} x^3 \ln(\cos x)\sin(p-1)x \cos^{p-1} x\,dx, \tag{A.15}$$

$$-\left.\frac{\partial^4 I(a,p)}{\partial p \partial a^3}\right|_{a=p-1} = \int_0^{\pi/2} x^4 \ln(\cos x)\cos(p-1)x \cos^{p-1} x\,dx \tag{A.16}$$

and

$$-\left.\frac{\partial^4 I(a,p)}{\partial p^2 \partial a^2}\right|_{a=p-1} = \int_0^{\pi/2} x^3 \ln^2(\cos x)\sin(p-1)x \cos^{p-1} x\,dx, \tag{A.17}$$

where I is given in Eq. (3). In addition, we introduce the parametric integral

$$I_2(a,p) \equiv \int_0^{\pi/2} \cos^{p-1} x \sin ax\,dx, \quad \mathrm{Re}\, p > 0, \tag{A.18}$$

such that

$$-\left.\frac{\partial^4 I_2(a,p)}{\partial p \partial a^3}\right|_{a=p-1} = \int_0^{\pi/2} x^3 \ln(\cos x)\cos(p-1)x \cos^{p-1} x\,dx, \tag{A.19}$$

$$\left.\frac{\partial^5 I_2(a,p)}{\partial p^2 \partial a^3}\right|_{a=p-1} = \int_0^{\pi/2} x^4 \ln(\cos x)\sin(p-1)x \cos^{p-1} x\,dx \tag{A.20}$$

and

$$-\left.\frac{\partial^5 I_2(a,p)}{\partial p^2 \partial a^3}\right|_{a=p-1} = \int_0^{\pi/2} x^3 \ln^2(\cos x)\cos(p-1)x \cos^{p-1} x\,dx. \tag{A.21}$$

The value of I_2 can be written with the use of the generalized hypergeometric function ${}_3F_2$ at unit argument:

$$I_2(a,p) = \frac{a}{p}\,{}_3F_2[1,(a+1)/2,(1-a)/2;1+p/2,3/2;1], \quad \mathrm{Re}\, p > 0. \tag{A.22}$$

This equation can be derived by writing $\sin ax = a \sin x\, {}_2F_1[(a+1)/2,(1-a)/2;3/2;\sin^2 x]$ in terms of the hypergeometric function ${}_2F_1$, performing a change of variable in Eq. (A.18), and then applying a standard result [9] for ${}_pF_q$ as an integral over ${}_{p-1}F_{q-1}$. The drastic simplifications in the special cases $I(p-1,p) = \pi 2^{-(p+1)} H_{p-1}$ and $I_2(p-1,p) = 2^{-p}\sum_{j=1}^{p-1} 2^j/j$ may be noted. These cases are in agreement with previous evaluations [2].

We have also derived the following expression for I_2, containing ${}_2F_1$ at minus unit argument:

$$I_2(a,p) = 2^{-p}\left[\frac{2}{(a-p+1)}\,{}_2F_1[1-p,(a-p+1)/2;(a-p+1)/2+1;-1] + \sin[(a-p)\pi/2]\frac{\Gamma(p)\Gamma[(a-p+1)/2]}{\Gamma[(a+p-1)/2+1]}\right]. \tag{A.23}$$

In light of Eq. (A.22), it is possible that this result represents a new reduction of the particular ${}_3F_2$ function at unit argument. We have obtained Eq. (A.23) by taking the real part of the integral $\int_{-1}^{1}(1+w)^x w^\eta\,dw$ considered in Ref. [3], employing [9]

$$\int_0^1 (1+u)^x u^\eta\,du = \frac{1}{\eta+1}\,{}_2F_1(-x,\eta+1;\eta+2;-1), \quad \mathrm{Re}\,\eta > -1 \tag{A.24}$$

and performing some manipulations.

The expressions for the integrals of Eqs. (A.15)–(A.17) and (A.19)–(A.21) are lengthy. Therefore, we present only that of Eq. (A.15) as an example:

$$\begin{aligned}
&\int_0^{\pi/2} x^3 \ln(\cos x)\sin(p-1)x\cos^{p-1}x\,dx \\
&\quad = -\frac{\pi}{15}2^{-(p+6)}[60\gamma^4 - 60\gamma^2\pi^2 + \pi^4 + 60\gamma(\pi^2-2\gamma^2)\ln 2 \\
&\qquad + 60\psi'''(p) + 60(-(\pi^2+6\gamma(-\gamma+\ln 2))\psi^2(p) \\
&\qquad + (4\gamma - 2\ln 2)\psi^3(p) + \psi^4(p) + 6\gamma\ln 2\psi'(p) \\
&\qquad - 3[\psi'(p)]^2 - 2(\gamma+\ln 2)\psi''(p) + (8\gamma - 4\ln 2)\zeta(3) \\
&\qquad + \psi(p)(4\gamma^3 - 2\gamma\pi^2 + (\pi^2-6\gamma^2)\ln 2 \\
&\qquad + 6\ln 2\psi'(p) - 2\psi''(p) + 8\zeta(3)))].
\end{aligned} \tag{A.25}$$

Taking derivatives of the ${}_3F_2$ or ${}_2F_1$ function in the expression for I_2, with respect to either numerator or denominator parameters, is possible by knowing how to differentiate the Pochhammer symbol. Details of such procedures have been recorded elsewhere [4].

In principle, Eqs. (A.15)–(A.23), together with a few elementary integrals, permit the evaluation of the term $\int_0^\pi g(t)h(t)\,dt$ in Eq. (A.9), while evaluation of the term $\int_0^\pi h^2(t)\,dt$ requires some more extensions.

Appendix B. Contour integral approach

Recently Flajolet and Salvy have presented a method for evaluating Euler sums via a contour integral representation [8]. In this appendix, we describe how this approach applies to the sum $\sum_{n=1}^\infty H_n^4/(n+1)^2$ of particular interest in this paper. For this purpose, we introduce a generalization of harmonic numbers, $H_n^{(j)} \equiv \sum_{k=1}^n 1/k^j$. By Theorem 5.3 of Ref. [8], we know that the Euler sum $\sum_{n=1}^\infty H_n^4/n^2$ may be reduced to a combination of sums of lower order. This is because this Euler sum of weight 6, degree 4, and order $r=4$ has an order and weight of the same parity. In turn, this theorem implies that (at least in principle) our closely related Euler sum can be written as a sum of zeta values.

What is required here is to evaluate Euler sums and corresponding contour integrals of higher degree than explicitly considered in Ref. [8]. In particular, we need to evaluate the contour integral

$$\frac{1}{2\pi i}\int_O r(s)[\psi(-s)+\gamma]^5\,ds, \tag{B.1}$$

where the contour may be taken as a circle of arbitrarily large radius centered upon the origin, and once again ψ is the digamma function and γ the Euler constant. In Eq. (B.1), the rational function $r(s)$ must decay sufficiently rapidly at infinity in order to ensure convergence. Our particular case of interest, $r(s) = 1/s^2$, does so.

In order to develop the summation formula appropriate for Eq. (B.1), we first note the local expansion about the origin,

$$\psi(-s) + \gamma = \frac{1}{s} - \zeta(2)s - \zeta(3)s^2 - \zeta(4)s^3 - \zeta(5)s^4 - \zeta(6)s^5 - \zeta(7)s^6 - \zeta(8)s^7 + O(s^8) \tag{B.2}$$

and about the positive integers $n \geqslant 0$,

$$\psi(-s) + \gamma \underset{s\to n}{=} \frac{1}{s-n} + H_n + \sum_{k=1}^{\infty} [(-1)^k H_n^{(k+1)} - \zeta(k+1)](s-n)^k. \tag{B.3}$$

We then require the notion of the special residue sum [8] $\mathscr{R}$, which is the finite sum of residues of the integrand of Eq. (B.1) over the poles of the factor $r(s)$, including the origin. We may now present the summation result

$$\begin{aligned}
&-\mathscr{R}[r(s)(\psi(-s)+\gamma)^5] \\
&\quad = 5\sum_n r(n)H_n^4 + 10\sum_n r'(n)H_n^3 + 5\sum_n r''(n)H_n^2 + \frac{5}{6}\sum_n r'''(n)H_n + \sum_n r^{(iv)}(n)/24 \\
&\qquad -30\sum_n r(n)H_n^{(2)}H_n^2 - 20\sum_n r'(n)H_nH_n^{(2)} - \frac{5}{2}\sum_n r''(n)H_n^{(2)} + 10\sum_n r(n)[H_n^{(2)}]^2 \\
&\qquad +20\sum_n r(n)H_nH_n^{(3)} + 5\sum_n r'(n)H_n^{(3)} - 5\sum_n r(n)H_n^{(4)} - 30\zeta(2)\sum_n r(n)H_n^2 \\
&\qquad -20\zeta(2)\sum_n r'(n)H_n - \frac{5}{2}\zeta(2)\sum_n r''(n) + 20\zeta(2)\sum_n r(n)H_n^{(2)} + 10\zeta^2(2)\sum_n r(n) \\
&\qquad -20\zeta(3)\sum_n r(n)H_n - 5\zeta(3)\sum_n r'(n) - 5\zeta(4)\sum_n r(n).
\end{aligned} \tag{B.4}$$

In particular, Eq. (B.4) is an extension of Ref. [8] to the case of quartic Euler sums.

When $r(s) = 1/s^2$, the left-hand side of this equation is given by

$$\mathscr{R} = \mathrm{Res}[\psi(-s)+\gamma]^5/s^2|_{s=0} = 5[-2\zeta^3(2) + 2\zeta^2(3) + 4\zeta(2)\zeta(4) - \zeta(6)]. \tag{B.5}$$

Then, accounting for explicit zeta values in $\mathscr{R}$, Eq. (B.4) becomes

$$\begin{aligned}
&-20\zeta^2(3) \\
&\quad = 5\sum_n \frac{H_n^4}{n^2} - 20\sum_n \frac{H_n^3}{n^3} + 30\sum_n \frac{H_n^2}{n^4} - 20\sum_n \frac{H_n}{n^5} \\
&\qquad -30\sum_n \frac{H_n^{(2)}H_n^2}{n^2} + 40\sum_n \frac{H_nH_n^{(2)}}{n^3} - 15\sum_n \frac{H_n^{(2)}}{n^4} + 10\sum_n \frac{[H_n^{(2)}]^2}{n^2}
\end{aligned}$$

$$+20\sum_n \frac{H_n H_n^{(3)}}{n^2} - 10\sum_n \frac{H_n^{(3)}}{n^3} - 5\sum_n \frac{H_n^{(4)}}{n^2} - 30\zeta(2)\sum_n \frac{H_n^2}{n^2}$$

$$+40\zeta(2)\sum_n \frac{H_n}{n^3} + 20\zeta(2)\sum_n \frac{H_n^{(2)}}{n^2} - 20\zeta(3)\sum_n \frac{H_n}{n^2}. \tag{B.6}$$

The linear Euler sums appearing in this equation are known [3,8,1]. These include

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3), \quad \sum_{n=1}^{\infty} \frac{H_n}{n^3} = \frac{1}{2}[5\zeta(4) - \zeta^2(2)] \tag{B.7a}$$

and

$$\sum_{n=1}^{\infty} \frac{H_n}{n^5} = \frac{1}{2}[7\zeta(6) - 2\zeta(2)\zeta(4) - \zeta^2(3)]. \tag{B.7b}$$

Known nonlinear Euler sums appearing in Eq. (B.6) are [8]

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^2} = \frac{17}{4}\zeta(4), \quad \sum_{n=1}^{\infty} \frac{H_n^2}{n^4} = \frac{97}{24}\zeta(6) - 2\zeta^2(3) \tag{B.8a}$$

and

$$\sum_{n=1}^{\infty} \frac{H_n^3}{n^3} = 3\zeta(2)\zeta(4) - \frac{5}{2}\zeta^2(3) + \frac{9}{16}\zeta(6). \tag{B.8b}$$

Furthermore, we have the linear sums [8]

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^4} = \zeta^2(3) - \frac{1}{3}\zeta(6), \quad \sum_{n=1}^{\infty} \frac{H_n^{(3)}}{n^3} = \frac{1}{2}\zeta^2(3) + \frac{1}{2}\zeta(6) \tag{B.9a}$$

and

$$\sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^2} = \frac{7}{4}\zeta(4). \tag{B.9b}$$

In addition to the linear Euler sum $\sum_{n=1}^{\infty} H_n^{(4)}/n^2$, there are four remaining quadratic Euler sums in Eq. (B.6) to be determined before the term $\sum_{n=1}^{\infty} H_n^4/n^2$ is known. Once it is, the sum of interest $\sum_{n=1}^{\infty} H_n^4/(n+1)^2$ can be easily found by using $H_{n-1}^4 = (H_n - 1/n)^4$, a shift of index, expanding terms and evaluating successive lower order Euler sums. Another means to obtaining the quartic Euler sum of interest would be to evaluate a contour integral of the form

$$\frac{\pi}{2\pi i}\int_O r(s)[\psi(-s)+\gamma]^4 \cot \pi s \, \mathrm{d}s. \tag{B.10}$$

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