ON STARLIKE AND CONVEX SCHLICHT FUNCTIONS

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1. The necessary and sufficient condition that $f(z) = z + \sum_{2}^{\infty} a_n z^n$ be schlicht and starlike in $|z| < 1$ is $\operatorname{Re} g(z) > 0$ in $|z| < 1$, where

$$g(z) = zf'(z)/f(z)$$

[2], p. 221. In dealing with the coefficients of $f(z)$ it has been customary to make use of the behaviour of $g(z)$. It is not difficult to see that we can write

$$g(z) = \frac{1+\omega(z)}{1-\omega(z)},$$

where $\omega(z)$ is regular and $|\omega(z)| < 1$ in $|z| < 1$, and $\omega(0) = 0$. In this paper we shall investigate the coefficients of starlike $f(z)$ which map $|z| < 1$ onto domains of finite area by relating these coefficients to $\omega(z)$ and not to $g(z)$. We first of all prove the following.

Theorem 1. *Suppose that $f(z) = z + \sum_{2}^{\infty} a_n z^n$ is starlike in $|z| < 1$ and maps $|z| < 1$ onto a domain of area Δ. Then*

$$|a_n| \leqslant \frac{2}{n-1} \cdot \left(\frac{\Delta}{\pi}\right)^{\frac{1}{2}} \quad (n \geqslant 2).$$

The question arises as to whether or not Theorem 1 is best possible. This can mean one of two different things. In the first place, if we put $A_n(\Delta) = \sup |a_n|$ where the sup is taken over all starlike $f(z)$ with $f'(0) = 1$ which map $|z| < 1$ onto a domain of area Δ, we see that

$$A_n(\Delta) \leqslant \frac{2}{n-1} \left(\frac{\Delta}{\pi}\right)^{\frac{1}{2}} \quad (n \geqslant 2).$$

We can now consider whether or not this result is best possible.

On the other hand if we fix our attention on a particular starlike $f(z)$ which maps $|z| < 1$ onto a domain of finite area, then Theorem 1 shows that $a_n = O(1/n)$ as $n \to \infty$. We can also consider whether this result is best possible.

In the first case we show that the result is nearly best possible in so far as to each $n \geqslant 2$ there corresponds a Δ such that

$$A_n(\Delta) \geqslant \frac{1}{2\sqrt{6}} \cdot \frac{2}{n-1} \cdot \left(\frac{\Delta}{\pi}\right)^{\frac{1}{2}}. \tag{1}$$

In the second case we produce an example which shows that the result is best possible.

The necessary and sufficient condition that $f(z) = z + \sum_{2}^{\infty} a_n z^n$ be schlicht and convex in $|z| < 1$ is $\operatorname{Re} g(z) > 0$, where $g(z) = 1 + zf''(z)/f'(z)$ ([2], p. 224).

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Expressing $g(z)$ in terms of $\omega(z)$ as before and then relating the coefficients of $f(z)$ to $\omega(z)$ we arrive at the next theorem.

THEOREM 2. *If $f(z)$ is schlicht and maps $|z| < 1$ onto a bounded convex domain then $a_n = o(1/n)$ as $n \to \infty$.*

(Note that a convex domain of finite area is necessarily bounded.)

We are unable to deal with $\sup |a_n|$ in this case as we could in Theorem 1. Nor can we prove that Theorem 2 is best possible. However, it is shown that the index of n in Theorem 2 is best possible, i.e. given $\alpha > 0$, however small, there is a schlicht function $f(z)$ mapping $|z| < 1$ onto a bounded convex domain and such that $|a_n| > C(\alpha) n^{-1-\alpha}$ for all n, where $C(\alpha)$ is a constant depending on α.

The next theorem is known [1], but we have included it because of the simplicity of our proof.

THEOREM 3. *If $\sum_2^\infty n |a_n| \leqslant 1$, then $f(z) = z + \sum_2^\infty a_n z^n$ is schlicht and star-like in $|z| < 1$.*

2. The proof of Theorem 1 is now given. Suppose that

$$\frac{zf'(z)}{f(z)} = \frac{1+\omega(z)}{1-\omega(z)},$$

where $\omega(z) = \sum_1^\infty \omega_n z^n$ and $|\omega(z)| < 1$ in $|z| < 1$. Then we see that

$$\{zf'(z) + f(z)\} \, \omega(z) = zf'(z) - f(z),$$

or

$$\left\{2z + \sum_2^\infty (k+1) a_k z^k\right\} . \sum_1^\infty \omega_k z^k = \sum_2^\infty (k-1) a_k z^k. \tag{2}$$

Equating corresponding coefficients on both sides of (2) we find that

$$2\omega_{n-1} + 3a_2 \omega_{n-2} + \ldots + n a_{n-1} \omega_1 = (n-1) a_n \quad (n \geqslant 2).$$

This means that the coefficient a_n on the right of (2) depends only on $a_2, \ldots, a_{n-1}$ on the left of (2). Hence for $n \geqslant 2$,

$$\left\{2z + \sum_2^{n-1} (k+1) a_k z^k\right\} \omega(z) = \sum_2^n (k-1) a_k z^k + \sum_{n+1}^\infty b_k z^k, \tag{3}$$

say. Squaring the moduli of both sides of (3) and integrating round $|z| = r < 1$ we get, using the fact that $|\omega(z)| < 1$ in $|z| < 1$,

$$\sum_2^n (k-1)^2 |a_k|^2 r^{2k} + \sum_{n+1}^\infty |b_k|^2 r^{2k} < 4 + \sum_2^{n-1} (k+1)^2 |a_k|^2.$$

Let $r \to 1$ and we find that

$$\sum_2^n (k-1)^2 |a_k|^2 \leqslant 4 + \sum_2^{n-1} (k+1)^2 |a_k|^2,$$

or

$$(n-1)^2 |a_n|^2 \leqslant 4\left(+ \sum_2^{n-1} k |a_k|^2\right) \quad (n \geqslant 2). \tag{4}$$

Now Δ, the area of the image of $|z|<1$ by $f(z)$, is given by

$$\Delta = \int_0^1 \int_0^{2\pi} |f'(re^{i\theta})|^2 r\,dr\,d\theta$$

$$= 2\pi \int_0^1 \left(1+\sum_2^\infty k^2 |a_k|^2 r^{2k-2}\right) r\,dr$$

$$= \pi\left(1+\sum_2^\infty k|a_k|^2\right). \tag{5}$$

Using (5), we find from (4) that

$$(n-1)^2 |a_n|^2 \leqslant 4\frac{\Delta}{\pi} \quad (n \geqslant 2),$$

or

$$|a_n| \leqslant \frac{2}{n-1}\left(\frac{\Delta}{\pi}\right)^{\frac{1}{2}} \quad (n \geqslant 2).$$

This completes the proof of Theorem 1.

Consider now

$$f(z) = z + z^n/n \quad (n \geqslant 2).$$

We have

$$z\frac{f'(z)}{f(z)} = \frac{z+z^n}{z+z^n/n} = 1+\frac{(n-1)z^n}{nz+z^n},$$

and since

$$\left|\frac{(n-1)z^n}{nz+z^n}\right| \leqslant \frac{n-1}{n-1} = 1 \quad (|z|<1),$$

it follows that $f(z)$ is schlicht and starlike in $|z|<1$. Also, using (5), we see that $f(z)$ maps $|z|<1$ onto a domain of area $\Delta = \pi(1+1/n)$. For this $f(z)$ we have

$$|a_n| = \frac{1}{n} = \left\{\frac{n-1}{2n}\cdot\left(\frac{n}{n+1}\right)^{\frac{1}{2}}\right\}\cdot\frac{2}{n-1}\left(\frac{\Delta}{\pi}\right)^{\frac{1}{2}},$$

and since the minimum value of the factor multiplying $\frac{2}{n-1}\left(\frac{\Delta}{\pi}\right)^{\frac{1}{2}}$ is $\frac{1}{2\sqrt{6}}$, attained at $n=2$, we obtain (1).

We now construct a starlike $f(z) = z+\sum_2^\infty a_n z^n$ which is bounded and such that $a_n > 1/n$ for infinitely many n. Consider the formal expansion

$$\prod_0^\infty (1+\cos 4^n\theta) = 1+\sum_1^\infty b_n \cos n\theta.$$

It is known that $0 \leqslant b_n \leqslant 1$ $(n \geqslant 1)$; that $b_n = 1$ when $n = 4^m$ $(m \geqslant 0)$; and that $\sum_1^\infty b_n/n < \infty$ ([3], p. 442). Now define, for $|z|<1$,

$$h(z) = 1+\sum_1^\infty b_n z^n$$

and put

$$z\frac{f'(z)}{f(z)} = h(z),$$

so that $$f(z)=z\exp\left\{\int_0^z \frac{h(\zeta)-1}{\zeta}\,d\zeta\right\}=z\exp\left(\sum_1^\infty \frac{b_n}{n}z^n\right).$$
It is not difficult to show that $\operatorname{Re}\{h(z)\}>0$ in $|z|<1$, so that $f(z)$ is schlicht and starlike in $|z|<1$. Since $\sum_1^\infty b_n/n<\infty$ it is also clear that $f(z)$ is bounded in $|z|<1$. Using the fact that $0\leqslant b_n\leqslant 1$ we see that if $f(z)=z+\sum_2^\infty a_n z^n$, then $a_n\geqslant b_{n-1}/(n-1)$ $(n\geqslant 2)$. Remembering that $b_n=1$ when n is of the form 4^m we arrive at the required result.

3. We now give the proof of Theorem 2. Let
$$1+z\frac{f''(z)}{f'(z)}=\frac{1+\omega(z)}{1-\omega(z)},$$
where $\omega(z)=\sum_1^\infty \omega_n z^n$ and $|\omega(z)|<1$ in $|z|<1$. Then it follows that
$$\{2f'(z)+zf''(z)\}\,\omega(z)=zf''(z),$$
or
$$\left\{2+\sum_2^\infty k(k+1)\,a_k z^{k-1}\right\}\sum_1^\infty \omega_k z^k=\sum_2^\infty k(k-1)\,a_k z^{k-1}. \tag{7}$$
Equating corresponding coefficients on both sides of (7), we obtain
$$2\omega_{n-1}+2\,.\,3a_2\,\omega_{n-2}+\ldots+(n-1)\,na_{n-1}\,\omega_1=n(n-1)\,a_n \quad (n\geqslant 2).$$
Hence for $n\geqslant 2$, as before,
$$\left\{2+\sum_{k=2}^{n-1} k(k+1)\,a_k z^{k-1}\right\}\omega(z)=\sum_2^n k(k-1)\,a_k z^{k-1}+\sum_n^\infty d_k z^k,$$
say, and this leads to
$$\sum_2^n\{k(k-1)\}^2|a_k|^2\leqslant 4+\sum_2^{n-1}\{k(k+1)\}^2|a_k|^2$$
or
$$\{n(n-1)\}^2|a_n|^2\leqslant 4\left(1+\sum_2^{n-1}k^3|a_k|^2\right) \quad (n\geqslant 2). \tag{8}$$

If $f(z)$ is bounded it follows from (5) that $\sum_2^\infty k|a_k|^2<\infty$ and so $r_n=\sum_n^\infty k|a_k|^2\to 0$ as $n\to\infty$. We have
$$\begin{aligned}\sum_2^{n-1}k^3|a_k|^2 &= \sum_2^{n-1}k^2(r_k-r_{k+1})\\ &= 4r_2+\sum_{k=3}^{n-1}(2k-1)\,r_k-(n-1)^2 r_n\\ &= o(n^2)\end{aligned} \tag{9}$$
as $n\to\infty$. From (8) and (9) we get $a_n=o(1/n)$ as $n\to\infty$, the result given in Theorem 2.

To show that the power of n in Theorem 2 is best possible consider $f_\alpha(z) = \alpha^{-1}[1-(1-z)^\alpha]$ $(0 < \alpha < 1)$. Since $f_\alpha'(z) = (1-z)^{\alpha-1}$ it follows that

$$1+z\frac{f_\alpha''(z)}{f_\alpha'(z)} = 1+\frac{(1-\alpha)z}{1-z} = \alpha+\frac{1-\alpha}{1-z}.$$

The function on the right-hand side above has its real part positive in $|z| < 1$ and so $f_\alpha(z)$ is schlicht and convex in $|z| < 1$. It is obvious that $f_\alpha(z)$ is bounded in $|z| < 1$. From known results it is easily shown that if $f_\alpha(z) = z+\sum_2^\infty a_n z^n$, then

$$a_n \sim K(\alpha) . n^{-1-\alpha}$$

as $n \to \infty$, where $K(\alpha)$ is a non-zero constant depending on α. As α can be any number satisfying $0 < \alpha < 1$ we have obtained the desired result.

4. The proof of Theorem 3 follows. Suppose that $\sum_2^\infty n|a_n| \leqslant 1$ and that $f(z) = z+\sum_2^\infty a_n z^n$. Then in $|z| < 1$,

$$\begin{aligned} |zf'(z)-f(z)| - |f(z)| &= \left|\sum_2^\infty (n-1)\, a_n z^n\right| - \left|z+\sum_2^\infty a_n z^n\right| \\ &< \sum_2^\infty (n-1)|a_n| - \left(1-\sum_2^\infty |a_n|\right) \\ &= \sum_2^\infty n|a_n| - 1 \\ &\leqslant 0. \end{aligned}$$

Hence it follows that in $|z| < 1$,

$$\left|z\frac{f'(z)}{f(z)} - 1\right| < 1.$$

This shows that $\text{Re}\,\{zf'(z)/f(z)\} > 0$, and so $f(z)$ is schlicht and starlike in $|z| < 1$.

References.

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