ADDENDUM TO A NOTE ON SCHLICHT FUNCTIONS

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In an earlier paper [**1**; Theorem 2] we proved that if

$$f(z) = z + \sum_2^\infty a_n z^n$$

is schlicht in $|z| < 1$ and maps $|z| < 1$ onto a bounded convex domain then $a_n = o(n^{-1})$ $(n \to \infty)$. We were unable to show whether or not this result is best possible but it was shown that at any rate the index of n is best possible, *i.e.* there is no absolute constant $c > 0$ such that

$$a_n = O(n^{-1-c}) \quad (n \to \infty).$$

Recently this point has been cleared up by Pommerenke, who has proved [**2**; Theorem 6] that to $f(z)$ corresponds a $\delta = \delta(f) > 0$ such that $a_n = O(n^{-1-\delta})$ $(n \to \infty)$. In this addendum we show that the conclusion of our earlier result is true if $f(z)$ maps $|z| < 1$ onto a domain bounded by a closed continuous rectifiable curve. It turns out that for this wider class of functions the conclusion is now best possible. Our results are given formally in the following two theorems.

Theorem 1. *If $f(z) = z + \sum_2^\infty a_n z^n$ is schlicht in $|z| < 1$ and maps $|z| < 1$ onto a domain whose boundary consists of a closed continuous rectifiable curve then $a_n = o(n^{-1})$ $(n \to \infty)$.*

Theorem 2. *If $\{\eta(n)\}$ is any sequence of positive numbers which converges to 0 as $n \to \infty$, then there is a function $f(z)$ of the kind considered in Theorem 1 such that $n|a_n| \geqslant \eta(n)$ for infinitely many n.*

In the proofs of the theorems we require the following two lemmas.

Lemma 1 [**3**; p. 203]. *Suppose that $f(z)$ is schlicht in $|z| < 1$ and maps $|z| < 1$ onto a domain D. A necessary and sufficient condition that the boundary of D is a closed continuous rectifiable curve is that for some constant K and all r $(0 \leqslant r < 1)$,*

$$\int_0^{2\pi} |f'(re^{i\theta})| \, r \, d\theta < K. \tag{1}$$

Lemma 2 [**4**; p. 285]. *Suppose that $g(z) = \sum_0^\infty b_n z^n$ is regular in $|z| < 1$. A necessary and sufficient condition that*

$$\int_0^{2\pi} |g(re^{i\theta})| \, d\theta$$

is bounded for $0 \leqslant r < 1$ is that the real and imaginary parts of $\sum_0^\infty b_n e^{in\theta}$ are Fourier series.

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If $f(z)$ satisfies the conditions of Theorem 1 then from Lemma 1 it follows that the inequality (1) holds. Hence, by Lemma 2, if $a_n = \alpha_n - i\beta_n$ then the real part of $z + \sum_2^\infty n a_n z^n$ $(z = e^{i\theta})$, *i.e.*

$$\cos\theta + \sum_2^\infty (n\alpha_n \cos n\theta + n\beta_n \sin n\theta)$$

is a Fourier series. By a well-known result it follows that $\alpha_n = o(n^{-1})$ $(n \to \infty)$ and $\beta_n = o(n^{-1})$ $(n \to \infty)$ so that $a_n = o(n^{-1}) (n \to \infty)$.

To prove Theorem 2 we make use of the result that if $\sum_2^\infty n|a_n| \leqslant 1$ then $f(z) = z + \sum_2^\infty a_n z^n$ is schlicht in $|z| < 1$. In fact $f(z)$ is starlike in $|z| < 1$ (see for example [**1**; Theorem 3]), but we do not require this result. Let $\{k_n\}$ be a sequence of increasing positive integers with $k_1 \geqslant 2$ such that $\sum_1^\infty \eta(k_n) \leqslant 1$. Define

$$a_n = \frac{\eta(k_\nu)}{k_\nu} \quad (n = k_\nu,\ \nu = 1, 2, \ldots),$$
$$= 0 \ \text{(otherwise)}.$$

Then $f(z) = z + \sum_2^\infty a_n z^n$ is schlicht in $|z| < 1$ and

$$\int_0^{2\pi} |f'(re^{i\theta})| r\, d\theta < 2\pi\left(1 + \sum_1^\infty \eta(k_n)\right) \quad (0 \leqslant r < 1).$$

Hence by Lemma 1, $f(z)$ maps $|z| < 1$ onto a domain bounded by a closed continuous rectifiable curve. Since $n|a_n| = \eta(n)$ $(n = k_\nu,\ \nu = 1, 2, \ldots)$ the proof of Theorem 2 is complete.

Finally, we remark that with reference to Theorem 1 "best possible" has been used in its usual sense. In this context "best possible" cannot be used in the strongest sense, *i.e.* if $\{\eta(n)\}$ is any sequence of positive numbers with limit 0 one cannot, in general, construct a function $f(z)$ of the kind considered in Theorem 1 such that for all large n we have $n|a_n| \geqslant \eta(n)$. This is because condition (1) implies that $\sum_2^\infty |a_n| < \infty$ and so one could not get the above result with $\eta(n) = n^{-\frac{1}{2}}$ for example.

References.

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