arXiv:1207.6143v1 [math.CV] 25 Jul 2012

ON SCHWARZ-CHRISTOFFEL MAPPINGS

MARTIN CHUAQUI

ABSTRACT. Let f be a Schwarz-Christoffel mapping of the disk $\mathbb{D}$ onto a polygon. In this paper, we prove that in the formula $f''/f' = 2(B_1/B_2)/(1-zB_1/B_2)$ obtained recently in [3], the degrees d_1, d_2 of the finite Blaschke products B_1, B_2 are equal, respectively, to the number of convex vertices minus 1, and the number of concave vertices. A similar result is obtained for the corresponding representation formula for mappings onto the exterior of polygons.

1. INTRODUCTION AND RESULTS

The purpose of this note is to provide some results about Schwarz-Christoffel mappings that serve as complements to recent ones obtained in [2], [3]. See also [1] for related and interesting work regarding concave functions.

Let f be a Schwarz-Christoffel mapping of the unit disk $\mathbb{D}$ onto the interior of an $(n+1)$-gon. In [3], it is shown that the pre-Schwarzian of f has the form

$$\frac{f''}{f'} = \frac{2B_1/B_2}{1 - zB_1/B_2} \tag{1.1}$$

for some finite Blaschke products B_1, B_2 of degrees d_1, d_2, respectively, satisfying $d_1 + d_2 = n$. The polygon is convex iff $d_2 = 0$ (see also [2]). The representation for f''/f' is obtained from the well-known formula for such mappings,

$$\frac{f''}{f'} = -2\sum_{k=1}^{n+1} \frac{\beta_k}{z - z_k}\,, \tag{1.2}$$

where z_k are the pre-vertices and $2\pi\beta_k$ are the exterior angles, for which $-1 < \beta_k < 1$ and $\sum_{k=1}^{n+1} \beta_k = 1$. As a consequence, the pre-vertices were shown to be the roots of the equation

$$zB_1(z)/B_2(z) = 1\,. \tag{1.3}$$

It is interesting that (1.3) corresponds to a polynomial equation of degree $n+1$, all of its roots being simple and lying on $|z| = 1$. This is a particular feature of the pair of Blaschke products B_1, B_2 appearing from Schwarz-Christoffel mappings. Note that the topological degree of zB_1/B_2 on $\partial\mathbb{D}$ is $1 + d_1 - d_2$, so that zB_1/B_2

The author was partially supported by Fondecyt Grant #1110321.
Key words: Schwarz-Christoffel mappings, Blaschke product, convex, concave, vertex.
2000 AMS Subject Classification. Primary: 30C45, 30C20; Secondary: 30J10.

must be traversing in the negative sense at many of the pre-vertices. In fact, as the proof of Theorem 2 below shows, at a pre-vertex z_k, zB_1/B_2 is traversing $\partial\mathbb{D}$ in the positive or negative sense according to whether $f(z_k)$ is a convex or a concave vertex. In is also interesting to observe that when $d_2 = 0$, any solution of (1.1) will result in a univalent mapping because $1 + \mathrm{Re}\{zf''/f'\} \geq 0$. More generally, in [2] arbitrary convex mappings f are shown to correspond exactly to the solutions of

$$\frac{f''}{f'} = \frac{2h}{1-zh},$$

for some function h analytic in $\mathbb{D}$ bounded by 1. We can express h in terms of $p = f''/f'$ as

$$h = \frac{p}{2+zp}.$$

We draw the following result.

Theorem 1: *Let h be analytic in $\mathbb{D}$ with $|h(z)| \leq 1$ everywhere. Then there exists a sequence $\{B_n\}_{n\in\mathbb{N}}$ of finite Blaschke products converging to h locally uniformly in $\mathbb{D}$.*

Proof: Let f be the convex mapping corresponding to h as above, and let Ω_n be a sequence of convex polygons converging to $f(\mathbb{D})$ in the sense of Carathéodory. Properly normalized Schwarz-Christoffel mappings f_n of $\mathbb{D}$ onto Ω_n will converge locally uniformly to f. But each mapping f_n satisfies (1.1) for some finite Blaschke product $B_1 = B_{1,n}$ and $B_2 = 1$. The theorem now follows by expressing $B_{1,n}$ in terms of the pre-Schwarzian of f_n.

Next, we give an answer to an important issue left unresolved in [3], namely the connection between the degrees d_1, d_2 and the number of convex and concave vertices of the polygon.

Theorem 2: *Let f map $\mathbb{D}$ onto the interior of an $(n+1)$-gon, and let B_1, B_2 be the corresponding Blaschke products in the representation (1). Then d_2 is equal to the number of concave vertices, while $d_1 + 1$ is equal to the number of convex vertices.*

Proof: Let

$$\varphi(\theta) = \arg\left\{e^{i\theta}\frac{B_1}{B_2}(e^{i\theta})\right\},$$

with a well-defined branch of the argument whenever $e^{i\theta}$ lies between consecutive pre-vertices. In any case,

$$\varphi'(\theta) = 1 + e^{i\theta}\left(\frac{B_1'}{B_1}(e^{i\theta}) - \frac{B_2'}{B_2}(e^{i\theta})\right). \tag{1.4}$$

On the other hand, we see from (1.1) and (1.2) that

$$\frac{B_1/B_2}{zB_1/B_2 - 1} = \sum_{k=1}^{n+1} \frac{\beta_k}{z - z_k},,$$

hence

$$\beta_k = \lim_{z \to z_k} (z - z_k) \frac{B_1/B_2}{zB_1/B_2 - 1} = \frac{B_1/B_2}{(zB_1/B_2)'}(z_k) = \frac{1}{\varphi'(\theta_k)}\,,$$

where we have written $z_k = e^{i\theta_k}$. We say that z_k is convex or concave according to whether the polygon is convex or concave at $f(z_k)$. We conclude that $\varphi'(\theta_k)$ is positive at convex pre-vertices and negative at concave pre-vertices. It follows that

(1.5)

$$\int_{\theta_k}^{\theta_{k+1}} \varphi'(\theta)d\theta = \begin{cases} 2\pi \ , & \text{when } z_k, z_{k+1} \text{ are both convex} \\ 0 \ , & \text{when } z_k, z_{k+1} \text{ are one convex and one concave} \\ -2\pi \ , & \text{when } z_k, z_{k+1} \text{ are both concave}\,. \end{cases}$$

Let a be the number of consecutive convex pre-vertices, b the number of instances a vertex of one type is followed by one of the other type, and c the number of consecutive concave pre-vertices. Then $a + b + c = n + 1$, and we see by (1.4) and (1.5) that

$$\int_0^{2\pi} \varphi'(\theta)d\theta = 2\pi(1 + d_1 - d_2) = 2\pi(a - c)\,.$$

Hence

$$1 + d_1 - a = d = d_2 - d\,,$$

therefore

$$n + 1 = 1 + d_1 + d_2 = a + 2d + c\,.$$

We conclude that $2d = b$, which gives that

$$1 + d_1 = a + \frac{b}{2} \quad , \quad d_2 = c + \frac{b}{2}\,.$$

To obtain the theorem, we claim that $c + (b/2)$ is equal to the number of concave vertices (or pre-vertices). To see this, let $z_k, \dots, z_l$ be any maximal chain of consecutive concave pre-vertices. Hence z_{k-1} and z_{l+1} are convex pre-vertices. The collection $z_k, \dots, z_l$ of concave pre-vertices contributes with $l - k$ in the count of c, and with 2 in the count of b. Thus its contribution in the count of $c + (b/2)$ is exactly the number of points in the chain. This proves the claim, and completes the proof of the theorem.

Similar results hold for mappings f onto the exterior of an $(n+2)$-gon, having the important normalization $f(0)=\infty$. For such mappings we have that

$$\frac{f''}{f'} = 2\left(\sum_{k=1}^{n+2}\frac{\beta_k}{z-z_k} - \frac{1}{z}\right),$$

where, as before, z_k are the pre-vertices and $2\pi\beta_k$ are the exterior angles satisfying $-1<\beta_k<1$ and $\sum_{k=1}^{n+2}\beta_k = 1$. In [3], this was shown to lead to

$$\frac{f''}{f'} = \frac{2zB_2/B_2}{z^2B_1B_2-1}, \tag{1.6}$$

for Blaschke products B_1, B_2 of degree d_1, d_2 satisfying $d_1+d_2=n$. Again, the case $d_2=0$ corresponds exactly to when the polygon is convex. The pre-vertices appear as the solutions of the equation $z^2B_1=B_2$, yet no further information was provided in connection with the degrees of the Blaschke products. With a similar argument as in the proof of Theorem 1, one can show:

Theorem 3: *Let f map $\mathbb{D}$ onto the exterior of an $(n+2)$-gon, and let B_1, B_2 be the corresponding Blaschke products in the representation (1.6). Then d_2 is equal to the number of concave vertices, while d_1+2 is equal to the number of convex vertices.*

References

[1] B. Bhowmik, S. Ponnusamy, K.-J. Wirths, *Concave functions, Blaschke products, and polynomial mappings*, Siberian Math. J. 50 (2009), 609-615.

[2] M. Chuaqui, P. Duren, B. Osgood, *Schwarzian derivatives of convex mappings*, Ann. Acad. Scie. Fenn. Math. 36 (2011), 1-12.

[3] M. Chuaqui, P. Duren, B. Osgood, *Concave conformal mappings and pre-vertices of Schwarz-Christoffel mappings*, Proc. AMS 140 (2012), 3495-3505.

Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Casilla 306, Santiago 22, Chile, mchuaqui@mat.puc.cl