

# THE ORDER OF A LINEARLY INVARIANT FAMILY IN $\mathbb{C}^n$

MARTIN CHUAQUI AND RODRIGO HERNÁNDEZ

ABSTRACT. We study the (trace) norm of a linearly invariant family in the ball in  $\mathbb{C}^n$ . By adapting an approach that in one variable yields optimal results, we are able to derive an upper bound for the norm of the family in terms of the Schwarzian norm and the dimension  $n$ .

## 1. INTRODUCTION

The purpose of this paper is to obtain an upper bound for the trace order of a certain linearly invariant family of locally biholomorphic mappings defined in the unit ball  $\mathbb{B}^n$  in  $\mathbb{C}^n$ . The family is defined in terms of the Schwarzian derivative  $SF$ , which inherits from the Bergman metric in  $\mathbb{B}^n$  a natural norm  $\|SF\|$  that is invariant under the automorphism group [2]. Disregarding certain normalizations, the families  $\mathcal{F}_\alpha$  considered in this paper are defined by the condition  $\|SF\| \leq \alpha$ . Linearly invariant families of holomorphic mappings were introduced in one complex variable by Pommerenke in two seminal papers that offered a systematic treatment of such families [10]. He showed that relevant aspects of the family  $\mathcal{F}$ , such as growth and covering, are determined by its order  $\sup_{f \in \mathcal{F}} |a_2(f)|$ . If  $Sf$  is the usual Schwarzian derivative and  $\|Sf\| = \sup_{|z| < 1} (1 - |z|^2)^2 |Sf(z)|$ , then the family of properly normalized locally univalent mappings in the disc  $\mathbb{D}$  for which  $\|Sf\| \leq \alpha$  is linearly invariant. By means of a variational method, Pommerenke determined the sharp value  $\sqrt{1 + \frac{1}{2}\alpha}$  for its order. In several variables, the concept of order of a linearly invariant family appears in the form of the (trace) order and the (norm) order [1]. In this work, we mimic the variational approach in several variables to estimate the order of  $\mathcal{F}_\alpha$  in terms of  $\alpha$  and the dimension  $n$ . Much like in the analysis found in [10], we are lead to a characteristic equation involving derivatives of order up to three that must be satisfied by any mapping extremal for the trace order. Finally, the estimate on the trace order is used to obtain a similar estimate for the norm order of the family  $\mathcal{F}_\alpha$ .

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## 2. PRELIMINARIES

In [9] T.Oda generalizes the concept of Schwarzian derivative to the case of locally biholomorphic mappings in several variables. For such a mapping  $F = F = (f_1, \dots, f_n) : \Omega \subset \mathbb{C}^n \rightarrow \mathbb{C}^n$  he introduces a family of Schwarzian derivatives by

$$(2.1) \quad S_{ij}^k F = \sum_{l=1}^n \frac{\partial^2 f_l}{\partial z_i \partial z_j} \frac{\partial z_k}{\partial f_l} - \frac{1}{n+1} \left( \delta_i^k \frac{\partial}{\partial z_j} + \delta_j^k \frac{\partial}{\partial z_i} \right) \log JF,$$

where  $i, j, k = 1, 2, \dots, n$ ,  $JF = \det(DF)$  is the jacobian determinant of the differential  $DF$  and  $\delta_i^k$  are the Kronecker symbols. Two important aspects of the one dimensional Schwarzian are also present in this context. First,

$$(2.2) \quad S_{ij}^k F = 0 \quad \text{for all } i, j, k = 1, 2, \dots, n \quad \text{iff } F(z) = M(z),$$

for some Möbius transformation

$$M(z) = \left( \frac{l_1(z)}{l_0(z)}, \dots, \frac{l_n(z)}{l_0(z)} \right),$$

where  $l_i(z) = a_{i0} + a_{i1}z_1 + \dots + a_{in}z_n$  with  $\det(a_{ij}) \neq 0$ . Next, under composition we have the chain rule

$$(2.3) \quad S_{ij}^k(G \circ F)(z) = S_{ij}^k F(z) + \sum_{l,m,r=1}^n \mathbb{S}_{lm}^r G(w) \frac{\partial w_l}{\partial z_i} \frac{\partial w_m}{\partial z_j} \frac{\partial z_k}{\partial w_r}, \quad w = F(z).$$

Thus, if  $G$  is a Möbius transformation then  $S_{ij}^k(G \circ F) = S_{ij}^k F$ . The  $S_{ij}^0 F$  coefficients are given by

$$S_{ij}^0 F(z) = (JF)^{-\frac{1}{n+1}} \left( \frac{\partial^2}{\partial z_i \partial z_j} (JF)^{-\frac{1}{n+1}} - \sum_{k=1}^n \frac{\partial}{\partial z_k} (JF)^{-\frac{1}{n+1}} S_{ij}^k F(z) \right).$$

One can find in the literature other equivalent formulations of the Schwarzian in several variables, which also come in the form of differential operators of orders two and three (see, *e.g.*, [5], [6], [8]). In order to recover a mapping from its Schwarzian derivatives we can consider the following overdetermined system of partial differential equations,

$$(2.4) \quad \frac{\partial^2 u}{\partial z_i \partial z_j} = \sum_{k=1}^n P_{ij}^k(z) \frac{\partial u}{\partial z_k} + P_{ij}^0(z) u, \quad i, j = 1, 2, \dots, n,$$

where  $z = (z_1, z_2, \dots, z_n) \in \Omega$  and  $P_{ij}^k(z)$  are holomorphic functions in  $\Omega$ , for  $i, j, k = 0, \dots, n$ . The system (2.4) is called *completely integrable* if there are  $n+1$  (maximun) linearly independent solutions. The system is said to be in *canonical*

form (see [11]) if the coefficients satisfy

$$\sum_{j=1}^n P_{ij}^j(z) = 0, \quad i = 1, 2, \dots, n.$$

An important result established by Oda is that (2.4) is completely integrable and in canonical form if and only if  $P_{ij}^k = S_{ij}^k F$  for a locally biholomorphic mapping  $F = (f_1, \dots, f_n)$ , where  $f_i = u_i/u_0$  for  $1 \leq i \leq n$  and  $u_0, u_1, \dots, u_n$  is a set of linearly independent solutions of the system. It was also observed by the author that  $u_0 = (JF)^{-\frac{1}{n+1}}$  is always a solution of (2.4) with  $P_{ij}^k = S_{ij}^k F$ . The following result not stated in the work of Oda will be important in the rest of the paper.

The individual components  $S_{ij}^k F$  can be gathered to conform an operator in the following form (see[2]).

**Definition 2.1.** For  $k = 1, \dots, n$  let  $\mathbb{S}^k F$  be the matrix

$$\mathbb{S}^k F = (S_{ij}^k F), \quad i, j = 1, \dots, n.$$

**Definition 2.2.** We define the *Schwarzian derivative operator* as the mapping  $\mathcal{S}F(z) : T_z\Omega \rightarrow T_{F(z)}F(\Omega)$  given by

$$\mathcal{S}F(z)(\vec{v}) = (\vec{v}^t \mathbb{S}^1 F(z) \vec{v}, \vec{v}^t \mathbb{S}^2 F(z) \vec{v}, \dots, \vec{v}^t \mathbb{S}^n F(z) \vec{v}),$$

where  $\vec{v} \in T_z\Omega$ .

As an operator  $\mathcal{S}F(z)$  inherits a norm from the metric in the domain:

$$(2.5) \quad \|\mathcal{S}F(z)\| = \sup_{\|\vec{v}\|=1} \|\mathcal{S}F(z)(\vec{v})\|,$$

and finally, we let

$$(2.6) \quad \|\mathcal{S}F\| = \sup_{z \in \Omega} \|\mathcal{S}F(z)\|.$$

Our interest is to study certain classes of locally biholomorphic mappings  $F$  defined in the unit ball  $\mathbb{B}^n$ . The Bergman metric  $g_B$  on  $\mathbb{B}^n$  is the hermitian product defined by

$$(2.7) \quad g_{ij}(z) = \frac{n+1}{(1-|z|^2)^2} [(1-|z|^2)\delta_{ij} + \bar{z}_i z_j].$$

The automorphisms of  $\mathbb{B}^n$  act as isometries of the Bergman metric, and are given by

$$\sigma(z) = \frac{Az + B}{Cz + D},$$

where  $A$  is  $n \times n$ ,  $B$  is  $n \times 1$ ,  $C$  is  $1 \times n$  and  $D$  is  $1 \times 1$  with

$$\begin{aligned} A^t \overline{A} - C^t \overline{C} &= \text{Id}, \\ |D|^2 - B^t \overline{B} &= 1, \\ A^t \overline{B} - C^t \overline{D} &= 0, \end{aligned}$$

(see, e.g., [4]).

By appealing to the chain rule (2.3), it was shown in [2] that

$$\|\mathcal{S}(F \circ \sigma)(z)\| = \|\mathcal{S}F(\sigma(z))\|,$$

from which

$$(2.8) \quad \|\mathcal{S}F\| = \|\mathcal{S}(F \circ \sigma)\|.$$

In this paper we will consider the family  $\mathcal{F}_\alpha$  defined by

$$\mathcal{F}_\alpha = \{ F : \mathbb{B}^n \rightarrow \mathbb{C}^n \mid F \text{ locally biholomorphic, } F(0) = 0, DF(0) = \text{Id}, \|\mathcal{S}F\| \leq \alpha \}.$$

The family  $\mathcal{F}_\alpha$  is linearly invariant and also compact [2]. We are interested in studying its (trace) order [1], given by

$$(2.9) \quad \text{ord } \mathcal{F}_\alpha = \sup_{F \in \mathcal{F}_\alpha} \sup_{|w|=1} \frac{1}{2} \left| \sum_{i,j=1}^n \frac{\partial^2 f_j}{\partial z_i \partial z_j}(0) w_i \right|.$$

Because the family is compact, the order is finite. An equivalent form of the order is given by

$$\mathcal{A}_\alpha = \sup_{f \in \mathcal{F}_\alpha} |\nabla(JF)(0)|,$$

which is shown in [2] to satisfy

$$\mathcal{A}_\alpha = 2 \text{ord } \mathcal{F}_\alpha.$$

A second measure of the size of a linearly invariant family  $\mathcal{F}$  is given by the norm order, defined by

$$\|\text{ord}\|\mathcal{F} = \sup_{f \in \mathcal{F}} \frac{1}{2} \|D^2 F(0)\|,$$

where

$$F(z) = z + \frac{1}{2} D^2 F(0)(z, z) + \dots$$

In general,  $\text{ord } \mathcal{F} \leq n \|\text{ord}\|\mathcal{F}$ . For the family  $\mathcal{F}_\alpha$  in particular, it was shown in [2] that

$$(2.10) \quad 1 + \frac{\sqrt{3}}{2} \alpha \leq \|\text{ord}\|\mathcal{F}_\alpha \leq \frac{2}{n+1} \text{ord } \mathcal{F}_\alpha + \frac{\sqrt{n+1}}{2} \alpha.$$

## 3. VARIATIONS AND EXTREMAL MAPPINGS

Let  $F_0 \in \mathcal{F}_\alpha$  be a mapping for which  $\mathcal{A}_\alpha$  is maximal, with  $\nabla(JF_0)(0) = \Lambda = (\lambda_1, \dots, \lambda_n)$ . Let  $\sigma$  be an automorphism of  $\mathbb{B}^n$  with  $\sigma(0) = \zeta$ , and consider the Koebe transform

$$G(z) = D\sigma(0)^{-1}DF_0(\zeta)^{-1}[F_0(\sigma(z)) - F_0(\zeta)] .$$

The mapping  $G \in \mathcal{F}_\alpha$  represents a variation of the extremal mapping  $F_0$  when  $|\zeta|$  is small. With this in mind, we need to compute  $\nabla(JG)(0)$ . We have that

$$DG(z) = D\sigma(0)^{-1}DF_0(\zeta)^{-1}DF_0(\sigma(z))D\sigma(z) ,$$

hence

$$JG(z) = J\sigma(0)^{-1}JF_0(\zeta)^{-1}JF_0(\sigma(z))J\sigma(z) ,$$

so that

$$(3.1) \quad \nabla(JG)(0) = \frac{\nabla(JG)}{JG}(0) = \frac{\nabla(JF_0)}{JF_0}(\zeta)D\sigma(0) + \frac{\nabla(J\sigma)}{J\sigma}(0) .$$

In order to proceed with the analysis, we need the expansion of  $\nabla(JG)(0)$  in powers of  $\zeta$ .

**Lemma 3.1.** *Let*

$$B_{ij} = \sum_{k=1}^n S_{ij}^k F_0(0) \lambda_k \quad , \quad B_{ij}^0 = S_{ij}^0 F_0(0) .$$

Then

$$\frac{\nabla(JF_0)}{JF_0}(\zeta) = \Lambda + A \cdot \zeta + O(|\zeta|^2) \quad , \quad |\zeta| \rightarrow 0 ,$$

where  $A = (A_{ij})$  is the matrix given by

$$(3.2) \quad A_{ij} = B_{ij} - (n+1)B_{ij}^0 + \frac{\lambda_i \lambda_j}{n+1} .$$

*Proof.* Let  $u_0 = (JF_0)^{-\frac{1}{n+1}}$  and  $\phi(\zeta) = \frac{\nabla(JF_0)}{JF_0}(\zeta)$ . Then  $\phi(0) = \Lambda$  because  $JF_0(0) = 1$ . We have that  $\phi = (\phi_1, \dots, \phi_n)$ , where

$$\phi_i(\zeta) = \partial_i \log(JF_0)(\zeta) = -(n+1)\partial_i(\log u_0)(\zeta) \quad , \quad \partial_i = \partial/\partial z_i .$$

Since  $u_0$  is a solution of (2.4) with  $u_0(0) = 1$  and  $\nabla u_0(0) = -\frac{1}{n+1}\nabla(JF_0)(0)$ , we see that

$$\begin{aligned} \partial_j \phi_i(0) &= -(n+1)\partial_j \left[ \frac{\partial_i u_0}{u_0} \right] (0) = -(n+1) \left[ \partial_{ij}^2 u_0(0) - \partial_i u_0(0)\partial_j u_0(0) \right] , \\ &= -(n+1) \left[ -\frac{B_{ij}}{n+1} + B_{ij}^0 - \frac{\lambda_i \lambda_j}{(n+1)^2} \right] , \end{aligned}$$

which gives that the differential  $D\phi(0)$  is given by the matrix  $A = (A_{ij})$ . This proves the lemma.  $\square$

**Lemma 3.2.** *With the notation as before, one can choose  $\sigma$  so that*

$$\begin{aligned} D\sigma(0) &= \text{Id} + O(|\zeta|^2), \\ \frac{\nabla(J\sigma)}{J\sigma}(0) &= -(n+1)\bar{\zeta}. \end{aligned}$$

*Proof.* Assume first that  $\sigma(0) = \zeta = (\zeta_1, 0, \dots, 0)$ . Then we may take

$$\sigma(z) = \left( \frac{z_1 + \zeta_1}{1 + \bar{\zeta}_1 z_1}, \frac{\sqrt{1 - |\zeta_1|^2} z_2}{1 + \bar{\zeta}_1 z_1}, \dots, \frac{\sqrt{1 - |\zeta_1|^2} z_n}{1 + \bar{\zeta}_1 z_1} \right),$$

and one finds that

$$J\sigma(z) = \frac{(1 - |\zeta_1|^2)^{\frac{n+1}{2}}}{(1 + \bar{\zeta}_1 z_1)^{n+1}},$$

together with

$$D\sigma(0) = \begin{pmatrix} (1 - |\zeta_1|^2) & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{1 - |\zeta_1|^2} & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{1 - |\zeta_1|^2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \sqrt{1 - |\zeta_1|^2} \end{pmatrix},$$

from which the lemma follows for  $\zeta$  of the form  $(\zeta_1, 0, \dots, 0)$ . The general case obtains after considering a rotation of the ball.  $\square$

In light of Lemmas 3.1 and 3.2, we can rewrite equation (3.1) as

$$(3.3) \quad \nabla(JG)(0) = \Lambda + A \cdot \zeta - (n+1)\bar{\zeta} + O(|\zeta|^2).$$

**Theorem 3.3.** *Let  $F_0 \in \mathcal{F}_\alpha$  be extremal for the order, with  $\nabla(JF_0)(0) = \Lambda$ . Then*

$$(3.4) \quad A \cdot \bar{\Lambda} = (n+1)\bar{\Lambda}.$$

*Proof.* The proof is based on the observation that, in reference to equation (3.3), we must have

$$|\nabla(JG)(0)| \leq |\Lambda|, \quad |\zeta| \rightarrow 0.$$

Let  $\langle v, w \rangle = v_1 \bar{w}_1 + \dots + v_n \bar{w}_n$ . Then

$$\begin{aligned} |\nabla(JG)(0)|^2 &= |\Lambda|^2 + \text{Re}\langle \Lambda, A \cdot \zeta \rangle - 2(n+1)\text{Re}\langle \Lambda, \zeta \rangle + O(|\zeta|^2) \\ &= |\Lambda|^2 + 2\text{Re}\langle \bar{A}^t \cdot \Lambda - (n+1)\Lambda, \zeta \rangle + O(|\zeta|^2). \end{aligned}$$

Since  $\zeta = (\zeta_1, \dots, \zeta_n)$  can be chosen small but otherwise arbitrary, we conclude that

$$\bar{A}^t \cdot \Lambda - (n+1)\Lambda = 0,$$

which proves the theorem because  $A^t = A$ .  $\square$

In order to facilitate the use of equation (3.4) to estimate the order of  $\mathcal{F}_\alpha$ , we use linear invariance to assume that  $\Lambda = (\lambda, 0, \dots, 0)$  with  $\lambda > 0$ . This normalization has a decoupling effect on (3.4), with the matrix  $A$  now given by

$$(3.5) \quad A_{ij} = S_{ij}^1 \lambda - (n+1)S_{ij}^0 + \frac{\delta_i^1 \delta_j^1}{n+1} \lambda^2,$$

where  $S_{ij}^k = S_{ij}^k F_0(0)$ . By equating the first components of (3.4) we obtain

$$(3.6) \quad \lambda^2 + (n+1)S_{11}^1 \lambda - (n+1)^2 S_{11}^0 - (n+1)^2 = 0,$$

while the remaining components give

$$(3.7) \quad S_{1j}^1 \lambda - (n+1)S_{1j}^0 = 0, \quad j = 2, \dots, n.$$

We are now in position to estimate the order of the family  $\mathcal{F}_\alpha$ .

**Theorem 3.4.** *The order of  $\mathcal{F}_\alpha$  satisfies*

$$(3.8) \quad \text{ord } \mathcal{F}_\alpha \leq \frac{1}{2}(n+1) \left[ \frac{1}{2} \sqrt{n+1} \alpha + \sqrt{1 + \frac{1}{4}(n+1)\alpha^2 + C(n, \alpha)} \right],$$

where

$$C(n, \alpha) \leq 6n^2 \alpha^2 + 16\sqrt{n} \alpha.$$

*Proof.* From (3.7), we see that

$$\left( \lambda + \frac{1}{2}(n+1)S_{11}^1 \right)^2 = (n+1)^2 \left( 1 + \frac{1}{4}(S_{11}^1)^2 + S_{11}^0 \right).$$

In [2], the following bounds were established for the quantities  $S_{11}^1, S_{11}^0$ :

$$|S_{11}^1| \leq \sqrt{n+1} \alpha, \quad |S_{11}^0| \leq C(n, \alpha),$$

where

$$C(n, \alpha) = \left( 4n^2 + 2n - 2 + \frac{n+1}{n-1} \right) \alpha^2 + \left( 4\sqrt{n+1} + 8\frac{\sqrt{n+1}}{n-1} \right) \alpha.$$

The inequality (3.8) follows at once from the estimates on  $S_{11}^1, S_{11}^0$ . Finally, it is not difficult to see that

$$C(n, \alpha) \leq 6n^2 \alpha^2 + 16\sqrt{n} \alpha.$$

$\square$

The following corollary is obtained at once from (2.10).

**Corollary 3.5.** *For the family  $\mathcal{F}_\alpha$  we have*

$$\|\text{ord}\| \mathcal{F}_\alpha \leq (n+1)\alpha + \sqrt{1 + \frac{1}{4}(n+1)\alpha^2 + C(n, \alpha)}.$$

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Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Casilla 306, Santiago 22, Chile, mchuaqui@mat.puc.cl

Facultad de Ciencias y Tecnología, Universidad Adolfo Ibáñez, Av. Diagonal las Torres 2640, Peñalolen, Chile, rodrigo.hernandez@uai.cl