AHLFORS-WEILL EXTENSIONS FOR A CLASS OF MINIMAL SURFACES

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ABSTRACT. The Ahlfors-Weill extension of a conformal mapping of the disk is generalized to the lift of a harmonic mapping of the disk to a minimal surface, producing homeomorphic and quasiconformal extensions. The extension is obtained by a reflection across the boundary of the surface using a family of Euclidean circles orthogonal to the surface. This gives a geometric generalization of the Ahlfors-Weill formula and extends the minimal surface. Thus one obtains a homeomorphism of $\overline{\mathbb{C}}$ onto a toplological sphere in $\overline{\mathbb{R}^3} = \mathbb{R}^3 \cup \{\infty\}$ that is real-analytic off the boundary. The hypotheses involve bounds on a generalized Schwarzian derivative for harmonic mappings in term of the hyperbolic metric of the disk and the Gaussian curvature of the minimal surface. Hyperbolic convexity plays a crucial role.

1. Introduction

If f is an analytic, locally injective function its Schwarzian derivative is

$$Sf = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

We owe to Nehari [10] the discovery that the size of the Schwarzian derivative of an analytic function is related to its injectivity, and to Ahlfors and Weill [2] the discovery of an allied, stronger phenomenon of quasiconformal extension of the function. We state the combined results as follows:

Theorem 1. *Let f be analytic and locally injective in the unit disk, $\mathbb{D}$.*

(a) If

$$|Sf(z)| \le \frac{2}{(1-|z|^2)^2}, \quad z \in \mathbb{D}, \tag{1}$$

then f is injective in $\mathbb{D}$.

(b) If for some $t < 1$

$$|Sf(z)| \le \frac{2t}{(1-|z|^2)^2}, \quad z \in \mathbb{D}, \tag{2}$$

then f has a $\frac{1+t}{1-t}$-quasiconformal extension to $\overline{\mathbb{C}}$.

A remarkable aspect of Ahlfors and Weill's theorem is the explicit formula they give for the quasiconformal extension. They need the stronger inequality (2) to show, first of all, that the extended mapping has a positive Jacobian and is hence a local homeomorphism. Global injectivity then follows from the monodromy theorem and quasiconformality from a calculation of the dilatation. The topological argument cannot get started without (2), but a different approach in [5] shows that the same formula still provides a homeomorphic extension even when f satisfies the weaker inequality (1) and $f(\mathbb{D})$ is a Jordan domain. As to the latter requirement, if f satisfies (1) then $f(\mathbb{D})$ fails to be a Jordan domain only when $f(\mathbb{D})$ is a parallel strip or the image of a parallel strip under a Möbius transformation, as shown by Gehring and Pommerenke [8].

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In earlier work we introduced a Schwarzian derivative for plane harmonic mappings and we established an injectivity criterion analogous to (1) for the Weierstrass-Enneper lift of a harmonic mapping of $\mathbb{D}$ to a minimal surface. In this paper we show that injective and quasiconformal extensions also obtain in this more general setting under conditions analogous to (1) and (2), respectively. The construction is a geometric generalization of the Ahlfors-Weill formula and extends the minimal surface. Thus one obtains a homeomorphism of $\overline{\mathbb{C}}$ onto a toplological sphere in $\overline{\mathbb{R}^3} = \mathbb{R}^3 \cup \{\infty\}$ that is actually real-analytic off $\partial\mathbb{D}$. Precise statements require some additional preparation, and for more background and details we refer to [3].

Let $f\colon \mathbb{D} \longrightarrow \mathbb{C}$ be a harmonic mapping. As is customary we write $f = h + \bar{g}$, where g and h are analytic. We assume that f is locally injective and that the dilatation $\omega = g'/h'$ is the square of a meromorphic function on $\mathbb{D}$. Under these assumptions there is a lift $\widetilde{f}\colon \mathbb{D} \longrightarrow \Sigma$, the Weierstrass-Enneper lift, onto a minimal surface $\Sigma \subset \mathbb{R}^3$. Furthermore, $\widetilde{f}$ is a conformal mapping of $\mathbb{D}$ to Σ, each with its Euclidean metric. We let $\mathbf{g}_0$ denote the Euclidean metric on $\mathbb{R}^3$, or the induced Euclidean metric on Σ. The pullback of $\mathbf{g}_0$ is a conformal metric on $\mathbb{D}$:

$$e^{2\sigma}|dz|^2 = \widetilde{f}^*(\mathbf{g}_0) \quad \text{where} \quad e^{\sigma} = |h'| + |g'|.$$

In terms of σ, the Gauss curvature of Σ at a point $\widetilde{f}(z)$ is

$$K(\widetilde{f}(z)) = -e^{-2\sigma(z)}\Delta\sigma(z).$$

For a minimal surface the curvature is ≤ 0. The Schwarzian of f (or of $\widetilde{f}$) is

$$\mathcal{S}f = 2(\sigma_{zz} - \sigma_z^2). \tag{3}$$

This becomes the usual Schwarzian when f is a analytic, in which case $\sigma = \log|f'|$.

Much of our work will go into defining an injective, continuous reflection of Σ across its boundary, $R\colon \Sigma \longrightarrow \Sigma^* \subset \overline{\mathbb{R}^3}$, with which we will extend $\widetilde{f}$ to

$$\widetilde{F}(z) = \begin{cases} \widetilde{f}(z), & z \in \overline{\mathbb{D}}, \\ R(\widetilde{f}(1/\bar{z})), & z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}. \end{cases}$$

The analysis will include a discussion of boundary values.

We state our results in parallel to Theorem 1, including the homeomorphic extension for the first part:

Theorem 2. *Let f be harmonic and locally injective in $\mathbb{D}$ with lift $\widetilde{f}\colon \mathbb{D} \longrightarrow \Sigma$.*

(a) If

$$|\mathcal{S}f(z)| + e^{2\sigma(z)}|K(\widetilde{f}(z))| \leq \frac{2}{(1-|z|^2)^2}, \quad z \in \mathbb{D}, \tag{4}$$

then $\widetilde{f}$ is injective in $\mathbb{D}$. If $\widetilde{f}(\partial\mathbb{D})$ is a Jordan curve then $\widetilde{F}$ is a continuous, injective extension to $\overline{\mathbb{C}}$.

(b) If for some $t < 1$

$$|\mathcal{S}f(z)| + e^{2\sigma(z)}|K(\widetilde{f}(z))| \leq \frac{2t}{(1-|z|^2)^2}, \quad z \in \mathbb{D}, \tag{5}$$

and if for some constant C

$$\|\nabla\sigma(z)\| \leq \frac{C}{1-|z|^2}, \quad z \in \mathbb{D}, \tag{6}$$

then $\widetilde{F}$ is a quasiconformal extension to $\overline{\mathbb{C}}$ with a bound depending only on t and C.

The injectivity in part (a) was proved in [3] in even greater generality, so the point here is the extension. It was also proved in [3] that if f satisfies (4) then f and $\widetilde{f}$ have spherically continuous extensions to $\partial\mathbb{D}$. Furthermore, we know exactly when $\widetilde{f}(\partial\mathbb{D})$ fails to be a simple closed curve in $\mathbb{R}^3$, namely when $\widetilde{f}$ maps $\overline{\mathbb{D}}$ into a catenoid and $\partial\Sigma$ is pinched by a *Euclidean circle* on the surface. More precisely, there is a Euclidean circle C on $\overline{\Sigma}$ and a point $P \in C$ with $\widetilde{f}(\zeta_1) = P = \widetilde{f}(\zeta_2)$ for a pair of points $\zeta_1, \zeta_2 \in \partial\mathbb{D}$. Equality holds in (4) along $\widetilde{f}^{-1}(C \setminus \{P\})$, and because of this a function satisfying the stronger inequality (5) is always injective on $\partial\mathbb{D}$.

Independent of its connection with injectivity, an enduring source of interest in the analytic Schwarzian stems from its invariance properties under Möbius transformations: if $T(z) = (az + b)/(cz + d)$ then

$$S(T \circ f) = Sf \quad \text{and} \quad S(f \circ T) = ((Sf) \circ T)(T')^2. \tag{7}$$

For harmonic mappings and the harmonic Schwarzian the former equation does not apply since $T \circ f$ is generally not harmonic. However, the latter equation continues to hold. As a consequence of this and Schwarz's Lemma, if a harmonic mapping f satisfies (4) or (5) and if T is a Möbius transformation of $\mathbb{D}$ onto itself, then $f \circ T$ also satisfies the inequalities. The equations (7) are contained in the more general chain rule for the Schwarzian,

$$S(g \circ f) = ((Sg) \circ f)(f')^2 + Sf. \tag{8}$$

By 'quasiconformal' we mean that $\widetilde{F}$ satisfies

$$\frac{\max_{\|X\|=1} \|D_X \widetilde{F}\|}{\min_{\|X\|=1} \|D_X \widetilde{F}\|} \leq A \tag{9}$$

at all points in $\mathbb{C} \setminus \partial\mathbb{D}$ for an A that depends only on t and C. The ratio is 1 at points in $\mathbb{D}$ because there $\widetilde{F}(z) = \widetilde{f}(z)$ is conformal.

The statements in Theorem 2 all reduce to their classical counterparts when f is analytic, including the formula for the extension. The condition (6) becomes

$$\left|\frac{f''(z)}{f'(z)}\right| \leq \frac{C}{1 - |z|^2}$$

and this is true with $C = 6$ when f is injective in $\mathbb{D}$, in particular when f satisfies (1). For harmonic mappings we must assume (6), but it is a mild restriction that holds for many cases of interest, for example when $\widetilde{f}$ is bounded.

The proof of Theorem 2 is in several parts. In Section 3 we will construct the reflection and show that it is injective when f satisfies (4). This is supported by lemmas on convexity and critical points proved in Section 2. In Section 4 we show that the extension matches up continuously along $\partial\mathbb{D}$, completing the proof of the first part of the theorem. In Section 5 we show that the reflection, and hence the extension, is quasiconformal when f satisfies the stronger inequality (5).

The reflection $w \mapsto w^*$ sews a surface $\Sigma^* = R(\Sigma)$ to the minimal surface Σ along the boundary. It would be interesting to study the geometry of Σ^*, both when R is simply injective and especially when it is quasiconformal. The latter provides a class of surfaces that are quasiconformaly equivalent to a sphere, about which there is limited knowledge. We hope to return to this topic on another occasion.

2. Three Lemmas on Convexity and Critical Points

In this section we borrow some results and techniques from [3], all having to do with convexity, to set the stage for constructing the reflection.

A real-valued function u on $\mathbb{D}$ is *hyperbolically convex* if

$$(u \circ \gamma)''(s) \geq 0 \tag{10}$$

for all hyperbolic geodesics $\gamma(s)$ in $\mathbb{D}$, where s is the hyperbolic arclength parameter. A special case of Theorem 4 in [3] tells us that when f satisfies the injectivity condition (4) the positive function

$$u_{\widetilde{f}}(z) = \frac{1}{\sqrt{(1-|z|^2)e^{\sigma(z)}}}, \quad z \in \mathbb{D}, \tag{11}$$

is hyperbolically convex. The principle is that an upper bound for the Schwarzian leads to a lower bound for the Hessian of $u_{\widetilde{f}}$, and from there to (10) when $u_{\widetilde{f}}$ is restricted to a geodesic. We will have some additional comments at the end of this section.

A second principle is to employ a version of the Schwarzian introduced by Ahlfors in [1] when studying conditions such as (4) along curves. Let $\varphi : (a,b) \to \mathbb{R}^n$ be of class C^3 with $\varphi'(x) \neq 0$. Ahlfors defined

$$S_1\varphi = \frac{\langle \varphi''', \varphi' \rangle}{|\varphi'|^2} - 3\frac{\langle \varphi'', \varphi' \rangle^2}{|\varphi'|^4} + \frac{3}{2}\frac{\|\varphi''\|^2}{\|\varphi'\|^2}, \tag{12}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product. If T is a Möbius transformation of $\overline{\mathbb{R}^n}$ then $S_1(T \circ \varphi) = S_1\varphi$, so this important invariance property is available.

Whereas Ahlfors' interest was in the relation of $S_1\varphi$ to the change in cross ratio under φ, another geometric property of $S_1\varphi$ was discovered by Chuaqui and Gevirtz in [4]. Namely, if

$$v = \|\varphi'\|$$

then

$$S_1\varphi = \left(\frac{v'}{v}\right)' - \frac{1}{2}\left(\frac{v'}{v}\right)^2 + \frac{1}{2}v^2\kappa^2, \tag{13}$$

where κ is the curvature of the curve $x \mapsto \varphi(x)$.

S_1 generalizes the real part of the analytic Schwarzian, while the connection we need between S_1 and the Schwarzian for harmonic maps is

$$S_1\widetilde{f}(x) \leq \mathrm{Re}\{\mathcal{S}f(x)\} + e^{2\sigma(x)}|K(\widetilde{f}(x))|, \quad -1 < x < 1;$$

see Lemma 1 in [3]. Thus if f satisfies (4) then

$$S_1\widetilde{f}(x) \leq \frac{2}{(1-x^2)^2}, \quad -1 < x < 1. \tag{14}$$

With all this as background, our first lemma is fairly straightforward.

Lemma 1. *Let f satisfy (4), with lift $\widetilde{f}$, and let T be a Möbius transformation of $\overline{\mathbb{R}^3}$. The function*

$$u_{T\circ\widetilde{f}}(z) = \frac{1}{\sqrt{(1-|z|^2)e^{\tau(z)}}}, \quad e^{\tau} = (\|T'\| \circ \widetilde{f})e^{\sigma},$$

is hyperbolically convex in $\mathbb{D}$.

While $(T \circ \widetilde{f})(\mathbb{D}) = T(\Sigma)$ is generally not a minimal surface, T *is* a conformal mapping of $\overline{\mathbb{R}^3}$ and $e^{2\tau}|dz|^2$ is the corresponding conformal metric on $\mathbb{D}$.

Proof of Lemma 1. Since $u_{\widetilde{f}}$ is hyperbolically convex and we can consider $\widetilde{f} \circ M$ for any Möbius transformation of $\mathbb{D}$ onto itself, it suffices to show that $u_{T\circ\widetilde{f}}$ is hyperbolically convex along the diameter $-1 < x < 1$. The argument proceeds by comparing coefficients in two second-order differential equations.

Let $\varphi(x) = (T \circ \widetilde{f})(x)$. From Möbius invariance and (14),

$$S_1\varphi(x) = S_1\widetilde{f}(x) \leq \frac{2}{(1-x^2)^2}, \quad -1 < x < 1.$$

Now with $v(x) = |\varphi'(x)| = e^{\tau(x)}$, as above, from (13)

$$\left(\frac{v'(x)}{v(x)}\right)' - \frac{1}{2}\left(\frac{v'(x)}{v(x)}\right)^2 \leq S_1\varphi(x) \leq \frac{2}{(1-x^2)^2}. \tag{15}$$

Let $2p$ denote the left-hand side, so that

$$2p(x) \leq \frac{2}{(1-x^2)^2}, \quad -1 < x < 1. \tag{16}$$

The function $V = v^{-1/2}$ satisfies the differential equation

$$V'' + pV = 0 \tag{17}$$

and the function

$$W(x) = \frac{V(x)}{\sqrt{1-x^2}} \tag{18}$$

is precisely $u_{T\circ\widetilde{f}}$ restricted to $-1 < x < 1$. If we give $-1 < x < 1$ its hyperbolic parametrization,

$$s = \frac{1}{2}\log\frac{1+x}{1-x}, \quad x(s) = \frac{e^{2s}-1}{e^{2s}+1}, \quad x'(s) = 1 - x(s)^2,$$

a calculation produces

$$\frac{d^2}{ds^2}W = \left(\frac{1}{(1-x^2)^2} - p(x)\right)(1-x^2)^2 W(x), \quad x = x(s),$$

and appealing to (16) shows this is nonnegative. □

The topological condition that $\widetilde{f}$ be injective on $\partial\mathbb{D}$ has an analytical consequence on critical points that is important for much of our work.

Lemma 2. *If f satisfies* (4) *and is injective on $\partial\mathbb{D}$ then function $u_{T\circ\widetilde{f}}$ has at most one critical point in $\mathbb{D}$.*

Proof. Suppose that $u_{T\circ\widetilde{f}}$ has two critical points. Composing $\widetilde{f}$ with a Möbius transformation of $\mathbb{D}$ onto itself we may locate the critical points at 0 and a, $0 < a < 1$. By convexity these must give absolute minima of $u_{T\circ\widetilde{f}}$ in $\mathbb{D}$, and the same must be true of $u_{T\circ\widetilde{f}}(x)$ for $0 \leq x \leq a$. Hence $u_{T\circ\widetilde{f}}$ is constant on $[0, a]$ and thus constant on $(-1, 1)$ because it is real analytic there.

It follows that the function $v(x) = e^{\tau(x)}$ is a constant multiple of $1/(1-x^2)^2$. But then $V(x) = v(x)^{-1/2}$ is constant multiple of $\sqrt{1-x^2}$, and from the differential equation (17) we conclude that $p(x) = 1/(1-x^2)^2$. In turn, from (13) and (15) this forces the curvature κ to vanish identically. Thus $T \circ \widetilde{f}$ maps the interval $(-1, 1)$ onto a line with speed $\|\varphi'(x)\| = v(x) = 1/(1-x^2)$, and so $\varphi(1) = \varphi(-1) = \infty$. This violates the assumption that $\widetilde{f}$, hence $T \circ \widetilde{f}$, is injective on $\partial\mathbb{D}$. □

Continuing with the same assumptions, we now show what happens when there is exactly one critical point.

Lemma 3. *Let f satisfy* (4) *and be injective on $\partial\mathbb{D}$. Let T be a Möbius transformation of $\overline{\mathbb{R}^3}$. The following are equivalent:*

(i) $u_{T\circ\widetilde{f}}$ has a critical point.

(ii) $(T \circ \widetilde{f})(\mathbb{D})$ is bounded.

(iii) $u_{T\circ\widetilde{f}}(re^{i\theta})$ is eventually increasing along each radius $[0, e^{i\theta})$.

(iv) $u_{T\circ\widetilde{f}}(z) \to \infty$ *as* $|z| \to 1$.

In the proof of this lemma, and elsewhere, we will have occasion to use Möbius inversions. Following Ahlfors we write

$$J(x) = \frac{x}{\|x\|^2}$$

and for the derivative

$$J'(x) = \frac{1}{\|x\|^4}(\|x\|^2\mathrm{Id} - 2Q(x)), \tag{19}$$

where

$$Q(x)_{ij} = x_i x_j.$$

and Id is the identity. From this and $Q(x)^2 = \|x\|^2 Q(x)$ one has

$$\|J'(x)\| = \frac{1}{\|x\|^2}. \tag{20}$$

Proof of Lemma 3. If *(iv)* holds there is an interior minimum so *(iv)* $\implies$ *(i)* is immediate.

Suppose *(i)* holds. We may assume the critical point is at the origin. The value $u_{T\circ\widetilde{f}}(0)$ is the absolute minimum for $u_{T\circ\widetilde{f}}$ in $\mathbb{D}$ and so

$$e^{\tau(z)} \le \frac{e^{\tau(0)}}{1-|z|^2}, \quad z \in \mathbb{D}.$$

Thus τ remains finite in $\mathbb{D}$ and ∞ cannot be a point on $T(\Sigma)$.

To show that $T(\Sigma)$ is bounded we first work along $[0,1)$. The hyperbolically convex function $W(x) = u_{T\circ\widetilde{f}}(x)$ in (18) cannot be constant because 0 is the unique critical point. Hence if $x(s)$ is the hyperbolic arclength parametrization of $[0,1)$ with $x(0) = 0$ then

$$\frac{d}{ds}W(x(s)) \ge a, \quad W(x(s)) \ge as + b,$$

for some $a, b > 0$ and all $s \ge s_0 > 0$. From this

$$v(x) = \frac{1}{V(x)^2} \le \frac{1}{(1-x^2)\left(\frac{a}{2}\log\frac{1+x}{1-x} + b\right)^2} = -\frac{1}{a}\frac{d}{dx}\left(\frac{1}{\frac{a}{2}\log\frac{1+x}{1-x} + b}\right).$$

Therefore

$$\int_0^1 e^{\tau(x)}\,dx = \int_0^1 v(x)\,dx < \infty,$$

with a bound depending only on a, b, s_0, and $(T\circ\widetilde{f})(1)$ is finite.

This argument can be applied on every radius $[0, e^{i\theta})$, and by compactness the corresponding numbers $a_\theta, b_\theta, s_\theta$ can be chosen positive independent of θ. This proves that $T\circ\widetilde{f}$ is bounded, and hence that *(i)* $\implies$ *(ii)*.

For *(ii)* $\implies$ *(iii)* we can first rotate and assume $e^{i\theta} = 1$. In the notation above, we need to show for some $x_0 > 0$ that $W(x)$ is increasing for $x_0 \le x < 1$.

We have to follow T by an inversion, so to simplify the notation let $\widetilde{f}_1 = T\circ\widetilde{f}$ and $u_{\widetilde{f}_1}(z) = ((1-|z|^2)e^{\tau(z)})^{-1/2}$. For w_0 to be determined let

$$I(w) = \frac{w - w_0}{\|w - w_0\|^2},$$

and write

$$\widetilde{f}_2 = I \circ \widetilde{f}_1, \quad u_{\widetilde{f}_2}(z) = \frac{1}{\sqrt{(1-|z|^2)e^{\nu(z)}}}, \quad \text{and} \quad W_2(x) = u_{\widetilde{f}_2}(x),\, x \in (-1,1).$$

Again, we know that $W_2(x(s))$ is convex, where s is the hyperbolic arclength parameter.

From (20),

$$e^{\nu(z)} = \frac{e^{\tau(z)}}{\|\widetilde{f}_1(z) - w_0\|^2} \quad \text{or} \quad \nu(z) = \tau(z) - \log\|\widetilde{f}_1(z) - w_0\|^2,$$

and therefore

$$\nabla\nu(0) = \nabla\tau(0) + \frac{2}{\|w_0\|^2}\left(\left\langle \frac{\partial \widetilde{f}_2}{\partial x}(0), w_0 \right\rangle, \left\langle \frac{\partial \widetilde{f}_2}{\partial y}(0), w_0 \right\rangle\right). \tag{21}$$

But also

$$\nabla u_{\widetilde{f}_2}(0) = -\frac{1}{2}\nabla\nu(0),$$

and from this equation and (21) it is clear we can choose w_0 to make

$$W_2'(0) = a > 0.$$

Convexity then ensures $W_2(x(s)) \geq as$.

To work back to W, write

$$\widetilde{f}_1 = \frac{\widetilde{f}_2}{\|\widetilde{f}_2\|^2} + w_0, \tag{22}$$

whence

$$\|D\widetilde{f}_1\| = \frac{\|D\widetilde{f}_2\|}{\|\widetilde{f}_2\|^2},$$

and

$$W = W_2\|\widetilde{f}_2\|.$$

The assumption we make in (ii) is that $\widetilde{f}_1(\mathbb{D}) = (T \circ \widetilde{f})(\mathbb{D})$ is bounded, and (22) thus implies that $\|\widetilde{f}_2\| \geq \delta > 0$. Therefore $W(x(s)) \geq a\delta s$. By convexity, there is an $x_0 > 0$ so that $W(x)$ is increasing for $x_0 \leq x < 1$. This completes the proof that $(ii) \implies (iii)$.

Finally, if (iii) holds then for each θ there exists $0 < r_\theta < 1$ such that

$$\frac{\partial}{\partial r} u_{T\circ\widetilde{f}}(r_\theta e^{i\theta}) \geq a_\theta > 0.$$

By compactness the r_θ can be chosen bounded away from 1 and the a_θ bounded away from 0. By hyperbolic convexity, along the tail of each radius $u_{T\circ\widetilde{f}}(r(s)e^{i\theta})$ is uniformly bounded below by a linear function of the hyperbolic arclength parameter s, which tends to ∞ as $r = r(s) \to 1$. $\square$

We conclude this section with some remarks on introducing the function $u_{\widetilde{f}}$. Let

$$\lambda_{\mathbb{D}}(z)^2|dz|^2 = \frac{1}{(1-|z|^2)^2}|dz|^2$$

be the Poincaré metric for $\mathbb{D}$ (curvature -4) and let $\lambda_\Sigma^2\, \mathbf{g}_0$ be the conformal metric on Σ with

$$\widetilde{f}^*(\lambda_\Sigma^2\, \mathbf{g}_0) = \lambda_{\mathbb{D}}^2|dz|,$$

so that $\widetilde{f}$ is an isometry. Since $\widetilde{f}^*(\mathbf{g}_0) = e^{2\sigma}|dz|^2$ we have

$$(\lambda_\Sigma \circ \widetilde{f})(z) = \frac{1}{(1-|z|^2)e^{\sigma(z)}} \quad \text{or} \quad \lambda_\Sigma \circ \widetilde{f} = e^{-\sigma}\lambda_{\mathbb{D}}, \tag{23}$$

and

$$u_{\widetilde{f}} = (\lambda_\Sigma \circ \widetilde{f})^{1/2}.$$

If f is analytic and injective in $\mathbb{D}$ and the plane domain $\Omega = f(\mathbb{D})$ replaces Σ, then $\lambda_\Sigma = \lambda_\Omega$ is the Poincaré metric on Ω and $u_f = (\lambda_\Omega \circ f)^{1/2}$. In [6] it was shown that the hyperbolic convexity of $\lambda_{T(\Omega)}^{1/2}$ for any Möbius transformation T is a characteristic property of functions satisfying the Nehari condition (1). Lemma 1 is an analog of this for harmonic maps. We will use λ_Σ to write various identities, inequalities, etc., in forms intrinsic to Σ.

3. Circles and Reflections

We continue to assume that f satisfies the injectivity condition (4) with a lift $\widetilde{f}$ mapping $\mathbb{D}$ to the minimal surface $\Sigma \subset \mathbb{R}^3$, and also that $\widetilde{f}$ is injective on $\partial\mathbb{D}$. The purpose of this section is to define a reflection $R\colon \Sigma \longrightarrow \overline{\mathbb{R}^3} \setminus \overline{\Sigma}$ that provides a continuous, injective extension of $\widetilde{f}$. To extend Σ beyond its boundary we use a family of Euclidean circles (possibly including a line) each orthogonal to Σ. They are defined by the following lemma, which depends on properties of the function $u_{T\circ\widetilde{f}}$ established in the preceding section.

Lemma 4. *For each $w \in \Sigma$ there is a unique Euclidean circle $C_w \subset \overline{\mathbb{R}^3}$ with the following properties:*

(i) C_w is orthogonal to Σ at w;

(ii) $C_w \cap \Sigma = \{w\}$;

(iii) Let $z_0 \in \mathbb{D}$ and $w_0 = f(z_0)$. A point w_1 lies on $C_{w_0} \setminus \{w_0\}$ if and only if $u_{I\circ\widetilde{f}}$ has a critical point at z_0, where

$$I(w) = \frac{w - w_1}{\|w - w_1\|^2}, \quad w \in \overline{\mathbb{R}^3}.$$

If $u_{\widetilde{f}}$ has a critical point at z_0 then C_{w_0} is a line satisfying (i) and (ii).

Briefly, when referring to part (iii) we say that inversion about any point in C_{w_0} other than w_0 produces a critical point for $u_{I\circ\widetilde{f}}$ at $z_0 = \widetilde{f}^{-1}(w_0) \in \mathbb{D}$. Observe that if $I \circ \widetilde{f}$ produces a critical point for $u_{I\circ\widetilde{f}}$ at z_0 then so does any further *affine* change $A \circ I \circ \widetilde{f}$. We will need this later.

Proof. We begin by determining the conditions under which $u_{I\circ\widetilde{f}}$ has a critical point when I is an inversion. This recapitulates some of the calculations in the proof of Lemma 3.

Consider first the case $z_0 = 0$. We can also assume that $\widetilde{f}(0) = 0$, and we let $T_0\Sigma$ denote the tangent plane to Σ at 0. Computing from the definition of $u_{T\circ\widetilde{f}}$ we have, as in (21), that $\nabla u_{T\circ\widetilde{f}}(0) = 0$ if and only if $\nabla\tau(0) = 0$, and this is for any Möbius transformation T. Specializing to the inversion

$$I(w) = \frac{w - w_1}{\|w - w_1\|^2}, \quad (I \circ \widetilde{f})(z) = \frac{\widetilde{f}(z) - w_1}{\|\widetilde{f}(z) - w_1\|^2}, \quad w_1 \neq 0, \tag{24}$$

gives for $u_{I\circ\widetilde{f}}$ that

$$e^{\tau(z)} = \frac{e^{\sigma(z)}}{\|\widetilde{f}(z) - w_1\|^2} \quad \text{or} \quad \tau(z) = \sigma(z) - \log\|\widetilde{f}(z) - w_1\|^2.$$

Thus

$$\nabla\tau(0) = \nabla\sigma(0) + \frac{2}{\|w_1\|^2}(\langle \widetilde{f}_x(0), w_1\rangle\, , \, \langle \widetilde{f}_y(0), w_1\rangle), \tag{25}$$

and $\nabla\tau(0) = 0$ when

$$\frac{1}{\|w_1\|^2}\langle \widetilde{f}_x(0), w_1\rangle = -\frac{1}{2}\sigma_x(0) \quad \text{and} \quad \frac{1}{\|w_1\|^2}\langle \widetilde{f}_y(0), w_1\rangle = -\frac{1}{2}\sigma_y(0). \tag{26}$$

Since $\widetilde{f}$ is conformal

$$\langle \widetilde{f}_x(0), \widetilde{f}_y(0) \rangle = 0 \quad \text{and} \quad \|\widetilde{f}_x(0)\|^2 = \|\widetilde{f}_y(0)\|^2 = e^{2\sigma(0)}.$$

Then (26) says exactly that the point $w_1/\|w_1\|^2$ lies on a line orthogonal to $T_0\Sigma$ through the point

$$\zeta = -\frac{1}{2}e^{-2\sigma(0)} \left\{ \sigma_x(0)\widetilde{f}_x(0) + \sigma_y(0)\widetilde{f}_y(0) \right\}.$$

on $T_0\Sigma$. Call this line L_0; it depends only on the various data at 0.

The inversion $J(w) = w/\|w\|^2$ leaves the tangent plane $T_0\Sigma$ invariant and interchanges 0 and ∞, where L_0 and $T_0\Sigma$ meet a second time orthogonally. That is, if we put $C_0 = J(L_0)$ then $w_1 \in C_0$ and C_0 is orthogonal to Σ at 0 and also, generically, orthogonal to $T_0\Sigma$ at some other finite point (which we will determine). The exceptional case is when $L \cap T_0\Sigma = \{0\}$, which occurs when $\nabla\sigma(0) = 0$. Then $\nabla\tau(0) = 0$ and $u_{\widetilde{f}}$ already has a critical point at 0. In this case $C_0 = J(L_0) = L_0$. This proves parts (i) and (iii) of the lemma for $z_0 = 0$.

Part (ii) of the lemma, for $z_0 = 0$, follows from Lemma 3. Indeed, if the inversion (24) produces a critical point for $u_{I\circ\widetilde{f}}$ (at 0) then $I(\Sigma)$ is bounded, and hence w_1 cannot lie on Σ.

Finally, to pass from 0 to an arbitrary point $z_0 \in \mathbb{D}$, consider

$$\widetilde{f}_1(z) = f\left(\frac{z + z_0}{1 + \overline{z_0}z}\right) - \widetilde{f}(z_0).$$

By Schwarz' lemma

$$u_{\widetilde{f}_1(z)} = u_{\widetilde{f}}\left(\frac{z + z_0}{1 + \overline{z_0}z}\right),$$

hence $u_{I\circ\widetilde{f}_1}$ has a critical point at 0 if and only if $u_{I\circ\widetilde{f}}$ has a critical point at z_0. The statements (i), (ii) and (iii) then follow from the previous analysis. $\square$

By means of this construction, each point w on Σ is associated to a point w^* outside Σ on the tangent plane $T_w\Sigma$, namely the other point where C_w meets $T_w\Sigma$. The points w and w^* are endpoints of the diameter of C_w that lies in $T_w\Sigma$. We write $w^* = R(w)$, or $R_\Sigma(w)$, and refer to w^* as the reflection of w. In Section 4 we will show that R fixes $\partial\Sigma$ pointwise. Note also that the arguments used to define the reflection of Σ can be applied to define the reflection $R_{\Sigma'}$ of any surface $\Sigma' = T(\Sigma)$, T a Möbius transformation, using the function $u_{T\circ\widetilde{f}}$. It is not true, however, that $R_{\Sigma'} \circ T = T \circ R_\Sigma$. We will return to this at the end of this section.

It is a consequence of Lemma 2 that R is injective.

Lemma 5. *If $w \neq w'$ then $C_w \cap C_{w'} = \emptyset$. Hence R is injective.*

Proof. The circles meet Σ only at the distinct points w and w'. If there is a point $w_1 \in C_w \cap C_{w'}$ it is not on Σ and the inversion $I(w) = (w - w_1)/(\|w - w_1\|^2)$ produces critical points for $u_{I\circ\widetilde{f}}$ at distinct points $z = \widetilde{f}^{-1}(w)$ and $z' = \widetilde{f}^{-1}(w')$ in $\mathbb{D}$. This is impossible by Lemma 2. $\square$

It is not difficult to find a formula for $w^* = R(w)$. The vectors

$$X(z) = e^{-\sigma(z)}\widetilde{f}_x(z), \quad Y(z) = e^{-\sigma(z)}\widetilde{f}_y(z)$$

are an orthonormal basis for $T_w\Sigma$, $w = \widetilde{f}(z)$. Again, first take $z = 0$ and $w = \widetilde{f}(z) = 0$. Since $w^* \in C_0$ the equations (26) apply to w^* and from these

$$\frac{1}{\|w^*\|^2}\langle w^*, X\rangle = -\frac{1}{2}e^{-\sigma(0)}\sigma_x(0), \quad \frac{1}{\|w^*\|^2}\langle w^*, Y\rangle = -\frac{1}{2}e^{-\sigma(0)}\sigma_y(0).$$

This leads easily to

$$w^* = -\frac{2e^{\sigma(0)}}{\|\nabla\sigma(0)\|^2}\left\{\sigma_x(0)X(0) + \sigma_y(0)Y(0)\right\}.$$

The formula when z is any point in $\mathbb{D}$ and $w = \widetilde{f}(z)$ is obtained by renormalizing $\widetilde{f}$ as in the proof of the lemma, including a translation by $\widetilde{f}(z)$. The result is

$$w^* = w + \frac{e^{\sigma(z)}\alpha(z)}{\alpha(z)^2 + \beta(z)^2}X(z) + \frac{e^{\sigma(z)}\beta(z)}{\alpha(z)^2 + \beta(z)^2}Y(z) \tag{27}$$

where

$$\alpha(z) = \frac{x}{1-|z|^2} - \frac{1}{2}\sigma_x(z), \quad \beta(z) = \frac{y}{1-|z|^2} - \frac{1}{2}\sigma_y(z), \quad z = x + iy. \tag{28}$$

One can also verify

$$\nabla \log u_{\widetilde{f}}(z) = (\alpha(z), \beta(z)).$$

The function $u_{\widetilde{f}}$ has a critical point at z precisely when $\alpha(z) = \beta(z) = 0$, in which case $w^* = \infty$. Note as well that the diameter of C_w is

$$\|w^* - w\| = \frac{e^{\sigma(z)}}{\|\nabla \log u_{\widetilde{f}}(z)\|}. \tag{29}$$

Furthermore, we can write the reflection in a form intrinsic to the surface.

Lemma 6. *The reflection $w^* = R(w)$ is given by*

$$R(w) = w + 2J(\nabla \log \lambda_\Sigma(w)), \tag{30}$$

where $J(w) = w/\|w\|^2$.

Proof. Recall from (23) the conformal metric $\lambda_\Sigma^2\, \mathbf{g}_0$ on Σ that is isometric to the Poincaré metric $|dz|^2/(1-|z|^2)^2$ on $\mathbb{D}$ and the relation

$$\log(\lambda_\Sigma \circ \widetilde{f})(z) = -\log(1-|z|^2) - \sigma(z).$$

Then with (28),

$$\frac{1}{2}\frac{\partial}{\partial x}\log(\lambda_\Sigma \circ \widetilde{f}) = \frac{x}{1-|z|^2} - \frac{1}{2}\sigma_x = \alpha$$
$$\frac{1}{2}\frac{\partial}{\partial y}\log(\lambda_\Sigma \circ \widetilde{f}) = \frac{y}{1-|z|^2} - \frac{1}{2}\sigma_y = \beta.$$

Now let $\nabla \log \lambda_\Sigma$ be the gradient with respect to the Euclidean metric on Σ. As a vector field on Σ we can write, with $w = \widetilde{f}(z)$,

$$\begin{aligned}\nabla \log \lambda_\Sigma(w) &= e^{-\sigma(z)}\left\{\frac{\partial}{\partial x}(\log \lambda_\Sigma \circ \widetilde{f})(z)X(z) + \frac{\partial}{\partial y}(\log \lambda_\Sigma \circ \widetilde{f})(z)Y(z)\right\} \\ &= 2e^{-\sigma(z)}\{\alpha(z)X(z) + \beta(z)Y(z)\}\end{aligned}$$

and

$$\|\nabla \log \lambda_\Sigma(w)\|^2 = 4e^{-2\sigma(z)}(\alpha(z)^2 + \beta(z)^2).$$

Using the inversion $J(w) = w/\|w\|^2$ we thus have

$$2J(\nabla \log \lambda_\Sigma(w)) = \frac{e^{\sigma(z)}\alpha(z)}{\alpha(z)^2 + \beta(z)^2}X(z) + \frac{e^{\sigma(z)}\beta(z)}{\alpha(z)^2 + \beta(z)^2}Y(z)$$

and

$$R(w) = w + 2J(\nabla \log \lambda_\Sigma(w)).$$

as stated. □

Finally we consider a conformal invariance property of the construction. This will be important in the next section when we show that $\widetilde{f}$ and its extension match on $\partial\mathbb{D}$.

One cannot expect R_Σ to be conformally natural, meaning that

$$R_{\Sigma'}(T(w)) = T(R_\Sigma(w))$$

for a Möbius transformation T with $T(\Sigma) = \Sigma'$, since $w^* = R_\Sigma(w)$ is defined at each point w via the tangent plane to the surface and under a Möbius transformation this plane may become a sphere. However, if the families of circles $\{C_w : w \in \Sigma\}$ and $\{C_\omega : \omega \in \Sigma'\}$ define the reflections for the surfaces Σ and Σ', respectively, then $T(C_w) = C_{T(w)}$. We can describe this degree of conformal invariance succinctly by introducing

$$\mathcal{C}_\Sigma = \bigcup_{w\in\Sigma} C_w.$$

Then

$$T(\mathcal{C}_\Sigma) = \mathcal{C}_{T(\Sigma)}. \tag{31}$$

To show this, note that as $w = \widetilde{f}(z)$ varies over Σ, the circles $T(C_w)$ clearly have properties (i) and (ii) of Lemma 4 for the surface Σ'. Take a point $w_0 = \widetilde{f}(z_0)$, determining the circle C_{w_0}, and let $\omega_0 = T(w_0)$. The question is whether for any $\omega_1 = T(w_1) \in T(C_{w_0}) \setminus \{\omega_0\}$ the inversion

$$I(\zeta) = \frac{\zeta - \omega_1}{\|\zeta - \omega_1\|^2}$$

produces a critical point for $u_{I\circ T\circ \widetilde{f}}$ at z_0. But the map

$$(I \circ T)(v) = \frac{T(v) - \omega_1}{\|T(v) - \omega_1\|^2}$$

is a Möbius transformation sending w_1 to ∞, as is the inversion

$$K(v) = \frac{v - w_1}{\|v - w_1\|^2}.$$

It follows that

$$(I \circ T)(v) = (A \circ K)(v)$$

for an affine transformation A. Now the circle C_{w_0} has the property that the inversion $K(v) = (v - w_1)/\|v - w_1\|^2$ produces a critical point for $u_{K\circ f}$ at z_0, and since A is affine, $I \circ T = A \circ K$ produces a critical point for $u_{I\circ T\circ \widetilde{f}}$ at z_0 as we were required to show. We conclude that the circles $T(C_w)$ for the surface Σ' have the properties of the circles in Lemma 4, and that $T(C_w) = C_{T(w)}$.

In addition to the conformal invariance expressed by (31) we have

$$\mathcal{C}_\Sigma \cup \partial\Sigma = \overline{\mathbb{R}^3}, \tag{32}$$

and by Lemma 5 this is a disjoint union. We will not need (32) but we consider it an important feature of the construction. To prove it, observe that if $w_1 \notin \overline{\Sigma} = \Sigma \cup \partial\Sigma$ then the inversion

$$I(w) = \frac{w - w_1}{\|w - w_1\|^2}$$

has the property that $I(\Sigma)$ is bounded. It follows by Lemma 3 that $u_{I\circ\widetilde{f}}$ has a critical point, and by Lemma 4 that w_1 lies on some $C_{w_0} \setminus \{w_0\}$.

4. Definition of the Extension and Proof of Theorem 2, Part (a)

With assumptions and notations as before, we define

$$\widetilde{F}(z) = \begin{cases} \widetilde{f}(z), & z \in \overline{\mathbb{D}}, \\ R(\widetilde{f}(1/\bar{z})), & z \in \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}. \end{cases} \tag{33}$$

To prove that $\widetilde{F}$ defines an extension of $\widetilde{f}$ we must show that $\widetilde{f}$ and $R \circ \widetilde{f}$ match continuously along $\partial\mathbb{D}$.

Lemma 7. *Let $z \in \mathbb{D}$ and let d denote the spherical metric on $\overline{\mathbb{R}^3}$. Then*

$$d(\widetilde{f}(z), R(\widetilde{f}(z))) \to 0, \quad |z| \to 1.$$

Proof. We divide the proof into the cases when $u_{\widetilde{f}}$ has one critical point and when it has none. We work in the spherical metric because, first, $\widetilde{f}$ has a spherically continuous extension, and second, when $u_{\widetilde{f}}$ has no critical points we have to allow for shifting $\widetilde{f}$ by a Möbius transformation.

Suppose $u_{\widetilde{f}}$ has a unique critical point, which we can take to be at 0. The proof of Lemma 3 shows that there is an $a > 0$ such that along any radius $[0, e^{i\theta})$

$$(1 - r^2)\frac{\partial}{\partial r} u_{\widetilde{f}}(re^{i\theta}) \geq a$$

for all $r \geq r_0 > 0$. (This corresponds to $dW/ds \geq a$ in the proof of Lemma 3, where s is the hyperbolic arclength parameter.) From this it follows that

$$(1 - |z|^2)\|\nabla u_{\widetilde{f}}(z)\| \geq a > 0,$$

for all $|z| \geq r_0 > 0$.

From (29)

$$\begin{aligned} \|R(\widetilde{f}(z)) - \widetilde{f}(z)\| &= \frac{e^{\sigma(z)}}{\|\nabla \log u_{\widetilde{f}}(z)\|} = \frac{u_{\widetilde{f}}(z) e^{\sigma(z)}}{\|\nabla u_{\widetilde{f}}(z)\|} \\ &= \frac{1}{u_{\widetilde{f}}(z)} \frac{1}{(1 - |z|^2)\|\nabla u_{\widetilde{f}}(z)\|}. \end{aligned}$$

This tends to 0 as $|z| \to 1$ because $u_{\widetilde{f}}$ becomes infinite (Lemma 3) and $(1 - |z|^2)\|\nabla u_{\widetilde{f}}(z)\|$ stays bounded below. Geometrically, the diameter of $C_{\widetilde{f}(z)}$ tends to 0 as $|z|$ increases to 1.

Next, supposing that $u_{\widetilde{f}}$ has no critical point we produce one. That is, let T be a Möbius transformation so that $u_{T \circ \widetilde{f}}$ has a critical point at 0. The preceding argument can be repeated verbatim to conclude that

$$\|R(T(\widetilde{f}(z))) - T(\widetilde{f}(z))\| \to 0 \quad \text{as} \quad |z| \to 1. \tag{34}$$

If R were conformally natural, if we knew that $R \circ T = T \circ R$, then we would be done. Instead, we argue as follows.

Let $z \in \mathbb{D}$, $z \neq 0$. The length $\|R(T(\widetilde{f}(z))) - T(\widetilde{f}(z))\|$ is the diameter of the circle $C_{T(\widetilde{f}(z))}$ based at $T(\widetilde{f}(z))$ that defines the reflection for the surface $T(\Sigma)$, and it tends to 0 by (34). But now, if $C_{\widetilde{f}(z)}$ is the circle based at $\widetilde{f}(z)$, for the surface Σ, then the reflected point $R(\widetilde{f}(z))$ is also on this circle (diametrically opposite $\widetilde{f}(z)$) and then $T(R(\widetilde{f}(z))) \in C_{T(\widetilde{f}(z))}$. Therefore $\|T(R(\widetilde{f}(z))) - T(\widetilde{f}(z))\| \to 0$ as $|z| \to 1$, whence in the spherical metric $d(R(\widetilde{f}(z)), \widetilde{f}(z))$ tends to 0 as well and the proof is complete. □

Combining Lemmas 5 and 7 proves part (a) of Theorem 2; the mapping $\widetilde{F}$ defined in (33) is a continuous injective extension of $\widetilde{f}$. Furthermore, the formulas make clear that $\widetilde{F}$ is real-analytic off $\partial\mathbb{D}$.

Remarks. When f is analytic in $\mathbb{D}$ the Ahlfors-Weill extension extension can be written as

$$F(z) = \begin{cases} f(z), & z \in \overline{\mathbb{D}}, \\ f(\zeta) + \dfrac{(1-|\zeta|^2)f'(z)}{\bar{\zeta} - \frac{1}{2}(1-|\zeta|^2)\frac{f''(\zeta)}{f'(\zeta)}}, & \zeta = 1/\bar{z},\ z \in \mathbb{C} \setminus \overline{\mathbb{D}}. \end{cases}$$

Ahlfors and Weill did not express it in this form; see [5]. Alternatively, if $\lambda_\Omega|dw|$ is the Poincaré metric on $\Omega = f(\mathbb{D})$ then

$$F(z) = \begin{cases} f(z), & z \in \overline{\mathbb{D}}, \\ f(\zeta) + \dfrac{1}{\partial_w \log \lambda_\Omega(f(\zeta))}, & \zeta = 1/\bar{z},\ z \in \mathbb{C} \setminus \overline{\mathbb{D}}. \end{cases}$$

The equation (30) for the reflection gives exactly

$$R(w) = w + \frac{1}{\partial_w \log \lambda_\Omega(w)} \tag{35}$$

when f is analytic.

The Ahlfors-Weill reflection is conformally natural: If T is a Möbius transformation of $\overline{\mathbb{C}}$ and $T(\Omega) = \Omega'$ then

$$R_{\Omega'} \circ T = T \circ R_\Omega.$$

From the perspective of the present paper this is because all tangent planes $T_z(\Omega)$ to Ω can be identified with $\mathbb{C}$, which is preserved by the extensions to $\overline{\mathbb{R}^3}$ of the Möbius transformations.

The reflection defining the Ahlfors-Weill extension was expressed in a form like (35) also by Epstein [7] in his penetrating geometric study of Nehari's and related theorems. Still another interesting geometric construction, using Euclidean circles of curvature, was given by Minda [9].

5. Quasiconformality of the Reflection and Proof of Theorem 2, Part (b)

We now assume that f satisfies

$$|\mathcal{S}f(z)| + e^{2\sigma(z)}|K(\widetilde{f}(z))| \le \frac{2t}{(1-|z|^2)^2}, \quad z \in \mathbb{D} \tag{36}$$

for some $t < 1$ and that

$$\|\nabla\sigma(z)\| \le \frac{C}{1-|z|^2}, \quad z \in \mathbb{D}, \tag{37}$$

for some $C < \infty$. Under these conditions we will show that the reflection $w^* = R(w)$ is quasiconformal.

Necessarily the analysis shifts to Σ and some of the geometric notions attached to Σ as a surface in $\mathbb{R}^3$ with its induced Euclidean metric $\mathbf{g}_0$, e.g., the gradient and the Hessian of a function, the covariant derivative and second fundamental form, and the curvature. As a reference we cite [11], whose notation we generally follow. If V is a vector field on Σ we let $\overline{D}_V$ be the Euclidean covariant derivative on $\mathbb{R}^3$ in the direction V, applied to a function or a vector field on Σ, and we let D_V be the covariant derivative on Σ. If ψ is a function on Σ then $\overline{D}_V\psi = D_V\psi = V\psi$. The gradient of ψ is the vector field defined by

$$\langle \nabla\psi, V \rangle = V\psi$$

and its Hessian is the symmetric, covariant 2-tensor defined by

$$\operatorname{Hess}\psi(V, W) = \langle D_V \nabla\psi, W \rangle.$$

If W is a vector field on Σ then

$$\overline{D}_V W = D_V W + II(V, W)$$

where $II(V, W)$ is the second fundamental form of Σ.

We can regard $w \mapsto R(w)$ as a vector field on Σ (not tangent to Σ) and we will compute its covariant derivative $\overline{D}_V R$ in the direction of a vector V, $\|V\| = 1$, tangent to Σ. At each $w \in \Sigma$ we seek upper and lower bounds

$$m(w) \leq \|\overline{D}_V R\| \leq M(w),$$

where $\sup_{w\in\Sigma} M(w)/m(w)$ is bounded by a quantity depending on t and C.

To do this we must translate the inequality (36) to one for functions defined on the surface. This requires the full differential-geometric definition of the conformal Schwarzian as a symmetric, traceless 2-tensor, and uses in particular a generalization of the chain rule (8) for the Schwarzian. We refer to [3] for the details as they are applicable here, and to [12] for a more general treatment.

Very briefly, the main points are these. For a function ψ defined on a 2-dimensional Riemannian manifold $(M, \mathbf{g})$ the Schwarzian tensor of ψ is

$$B_{\mathbf{g}}(\psi) = \operatorname{Hess}_{\mathbf{g}} \psi - d\psi \otimes d\psi - \frac{1}{2}(\Delta_{\mathbf{g}}\psi - \|\nabla_{\mathbf{g}}\psi\|^2_{\mathbf{g}})\mathbf{g} \tag{38}$$

where the Hessian, Laplacian, gradient, and norm are taken with respect to a Riemannian metric $\mathbf{g}$. The final term is the trace of $\operatorname{Hess}_{\mathbf{g}} \psi - d\psi \otimes d\psi$, so the full tensor is traceless. If f is a conformal mapping with conformal factor $e^{2\psi}\mathbf{g}$ then, by definition,

$$\mathcal{S}_{\mathbf{g}} f = B_{\mathbf{g}}(\psi).$$

In the case of a harmonic map f and its lift $\widetilde{f}\colon (\mathbb{D}, |dz|^2) \to (\Sigma, \mathbf{g_0})$, with conformal factor $\widetilde{f}^*(\mathbf{g}_0) = e^{2\sigma}|dz|^2$ as before, we have

$$\mathcal{S}f = \mathcal{S}\widetilde{f} = B(\sigma),$$

with respect to the Euclidean metric, i.e., computing the right-hand side produces $2(\sigma_{zz} - \sigma_z^2)$, which we took as the definition of the harmonic Schwarzian. Here, and below, when a quantity is calculated with respect to the Euclidean metric we drop the subscript $\mathbf{g}_0$.

The quantities defining $B_{\mathbf{g}}(\psi)$ which depend on the metric change in a not very complicated manner when the metric changes *conformally.* This is the basis for a generalized chain rule. It reads, in one form,

$$B_{\hat{\mathbf{g}}}(\psi - \rho) = B_{\mathbf{g}}(\psi) - B_{\mathbf{g}}(\rho), \quad \hat{\mathbf{g}} = e^{2\rho}\mathbf{g},$$

and (equivalently) in terms of conformal mappings, say $(M_1, \mathbf{g}_1) \xrightarrow{h} (M_2, \mathbf{g}_2) \xrightarrow{f} (M_3, \mathbf{g}_3)$,

$$\mathcal{S}_{\mathbf{g}_1}(f \circ h) = h^*(\mathcal{S}_{\mathbf{g}_2} f) + \mathcal{S}_{\mathbf{g}_1} h.$$

From the last equation, if f and h are inverse to each other then $\mathcal{S}_{\mathbf{g}_1} h = -h^*(\mathcal{S}_{\mathbf{g}_2} f)$.

Specializing to our case, but set up a little differently than before, we find the following. Recall from (23) the metric $\lambda_\Sigma^2\, \mathbf{g}_0$ with $\lambda_\Sigma \circ \widetilde{f} = e^{-\sigma}\lambda_{\mathbb{D}}$. Consider $\widetilde{f}\colon (\mathbb{D}, \mathbf{g}) \to (\Sigma, \mathbf{g}_0)$, $\mathbf{g} = \lambda_{\mathbb{D}}^2|dz|^2$, as a conformal mapping with conformal factor $e^{2\sigma}\lambda_{\mathbb{D}}^{-2}$. We take the Schwarzian tensor of $\widetilde{f}$ with respect to $\mathbf{g}$:

$$\mathcal{S}_{\mathbf{g}}\widetilde{f} = B_{\mathbf{g}}(\sigma - \log\lambda_{\mathbb{D}}).$$

Similarly, if $\widetilde{h} = \widetilde{f}^{-1}$ then $\widetilde{h}\colon (\Sigma, \mathbf{g}_0) \to (\mathbb{D}, \mathbf{g})$ is conformal with conformal factor λ_Σ^2. The Schwarzian tensor of $\widetilde{h}$ is with respect to the induced Euclidean metric on Σ and

$$\mathcal{S}\widetilde{h} = B(\log\lambda_\Sigma).$$

From the formulas above,

$$B(\log\lambda_\Sigma) = \mathcal{S}\widetilde{h} = -\widetilde{h}^*\mathcal{S}_{\mathbf{g}}\widetilde{f} = -\widetilde{h}^*(B_{\mathbf{g}}(\sigma - \log\lambda_{\mathbb{D}})).$$

while

$$B_{\mathbf{g}}(\sigma - \log \lambda_{\mathbb{D}}) = B(\sigma) - B(\log \lambda_{\mathbb{D}}) = B(\sigma),$$

the last equation holding because one has $B(\log \lambda_{\mathbb{D}}) = 0$ (computing in the Euclidean metric).

On the other hand, $h\colon (\Sigma, \widetilde{\mathbf{g}}) \to (\mathbb{D}, \mathbf{g})$ is an isometry for $\widetilde{\mathbf{g}} = \lambda_\Sigma^2 \mathbf{g}_0$, thus

$$\|B(\log \lambda_\Sigma)\|_{\widetilde{\mathbf{g}}} = \|B_{\mathbf{g}}(\sigma - \log \lambda_{\mathbb{D}})\|_{\mathbf{g}},$$

and in turn

$$\|B_{\mathbf{g}}(\sigma - \log \lambda_{\mathbb{D}})\|_{\mathbf{g}} = \|B(\sigma)\|_{\mathbf{g}} = \lambda_{\mathbb{D}}^{-2}\|B(\sigma)\| = \lambda_{\mathbb{D}}^{-2}|\mathcal{S}f|.$$

In the final term $\mathcal{S}f$ is the harmonic Schwarzian. Combining these with (36) we find

$$\begin{aligned}\|B(\log \lambda_\Sigma)\|_{\widetilde{\mathbf{g}}} + \lambda_\Sigma^{-2}|K| &= \|B(\log \lambda_\Sigma)\|_{\widetilde{\mathbf{g}}} + \lambda_{\mathbb{D}}^{-2} e^{2\sigma}|K| \\ &= \lambda_{\mathbb{D}}^{-2}(|\mathcal{S}f| + e^{2\sigma}|K|) \le 2t.\end{aligned}$$

Finally, we switch to the norm in the Euclidean metric and state the results of the calculations above as a lemma.

Lemma 8. *If f satisfies* (36) *then*

$$\|B(\log \lambda_\Sigma)\| + |K| \le 2t\lambda_\Sigma^2. \tag{39}$$

This is the inequality we use when working on Σ, eliminating direct mention of $\widetilde{f}$.

We proceed with the computation of $\overline{D}_V R$ using the formula (30),

$$R = \mathrm{Id} + 2J(\nabla \log \lambda_\Sigma),$$

and the formula (19),

$$J'(x) = \frac{1}{\|x\|^4}(\|x\|^2 \mathrm{Id} - 2Q(x)).$$

We have, first,

$$\overline{D}_V R = V + 2J'(\nabla \log \lambda_\Sigma)(\overline{D}_V \nabla \log \lambda_\Sigma),$$

and also the relation

$$\overline{D}_V \nabla \log \lambda_\Sigma = D_V \nabla \log \lambda_\Sigma + \mathit{II}(V, \nabla \log \lambda_\Sigma).$$

Hence

$$\overline{D}_V R = V + \frac{2}{\|\nabla \log \lambda_\Sigma\|^4}\left\{\|\nabla \log \lambda_\Sigma\|^2 \mathrm{Id} - 2Q(\nabla \log \lambda_\Sigma)(D_V \nabla \log \lambda_\Sigma + \mathit{II}(V, \nabla \log \lambda_\Sigma))\right\}$$

At this point it is prudent to simplify the notation somewhat. Let

$$\Lambda = \|\nabla \log \lambda_\Sigma\|, \quad Q = Q(\nabla \log \lambda_\Sigma), \quad \mathit{II} = \mathit{II}(V, \nabla \log \lambda_\Sigma).$$

Furthermore,

$$\mathrm{Hess}(\log \lambda_\Sigma)(V, W) = \langle D_V \nabla \log \lambda_\Sigma, W\rangle$$

so we identify the vector $D_V \nabla \log \lambda_\Sigma$ with the 1-tensor $\mathrm{Hess}(\log \lambda_\Sigma)(V, \cdot\,)$ and write

$$H = D_V \nabla \log \lambda_\Sigma.$$

The Schwarzian tensor enters through the Hessian terms, but this is not immediate.

The expression for $\overline{D}_V R$ now appears a little more manageable:

$$\overline{D}_V R = V + \frac{2}{\Lambda^2}\left\{H - \frac{2}{\Lambda^2}Q(H) + \mathit{II} - \frac{2}{\Lambda^2}Q(\mathit{II})\right\}.$$

To be clear, Λ is a scalar, II and H are vectors, and Q is a matrix operating on the vectors II and H.

To find the norm $\|\overline{D}_V R\|^2$ we are aided by several facts. First, H is tangent to Σ while $I\!I$ is normal to Σ. Second, Q is symmetric and

$$Q^2 = \Lambda^2 Q.$$

Finally, from its definition,

$$Q_{ij} = Q(\nabla \log \lambda_\Sigma)_{ij} = (\nabla \log \lambda_\Sigma)_i (\nabla \log \lambda_\Sigma)_j$$

and it is easy to see that for any vector X one has

$$Q(X) = \langle \nabla \log \lambda_\Sigma, X \rangle \nabla \log \lambda_\Sigma.$$

Hence

$$\langle Q(I\!I), V \rangle = \langle I\!I, Q(V) \rangle = \langle \nabla \log \lambda_\Sigma, V \rangle \langle I\!I, \nabla \log \lambda_\Sigma \rangle = 0,$$

because $I\!I$ is normal to Σ and so is orthogonal to $\nabla \log \lambda_\Sigma$. In expanding $\|\overline{D}_V R\|^2$ a number of terms then drop out and, at length, we obtain

$$\|\overline{D}_V R\|^2 = 1 + \frac{4}{\Lambda^2}\langle H, V \rangle + \frac{4}{\Lambda^4}\left\{\|H\|^2 - 2\langle Q(H), V \rangle + \|I\!I\|^2\right\} \tag{40}$$

where we have also used $\|V\| = 1$.

Referring to the definition (38) we have

$$B(\log \lambda_\Sigma) = \mathrm{Hess}(\log \lambda_\Sigma) - d\log \lambda_\Sigma \otimes d\log \lambda_\Sigma - \frac{1}{2}(\Delta \log \lambda_\Sigma - \|\nabla \log \lambda_\Sigma\|^2)\mathbf{g}_0.$$

Evaluate $B(\log \lambda_\Sigma)(V, \cdot)$ and treat this 1-tensor as a vector, which, continuing the pattern of notation, we will denote by B. With these abbreviations note that (39) implies

$$\|B\| + |K| \le 2t\lambda_\Sigma^2. \tag{41}$$

Next, in components the 2-tensor $d\log \lambda_\Sigma \otimes d\log \lambda_\Sigma$ is exactly $Q(\nabla \log \lambda_\Sigma)$, which we have denoted by Q. Finally we write

$$\rho = \frac{1}{2}(\Delta \log \lambda_\Sigma - \|\nabla \log \lambda_\Sigma\|^2) = \frac{1}{2}(\Delta \log \lambda_\Sigma - \Lambda^2).$$

for the trace. In these terms

$$H = B + Q(V) + \rho V.$$

and in (40),

$$\langle H, V \rangle = \langle B, V \rangle + \langle Q(V), V \rangle + \rho,$$

$$\|H\|^2 = \|B\|^2 + \Lambda^2 \langle Q(V), V \rangle + \rho^2 + 2\langle B, Q(V) \rangle + 2\rho \langle B, V \rangle + 2\rho \langle Q(V), V \rangle,$$
$$\langle Q(H), V \rangle = \langle H, Q(V) \rangle = \langle B, Q(V) \rangle + \Lambda^2 \langle Q(V), V \rangle + \rho \langle Q(V), V \rangle.$$

Substitution results in a quite compact expression:

$$\|\overline{D}_V R\|^2 = \frac{4}{\Lambda^4}\left\{\|B + \frac{1}{2}(\Delta \log \lambda_\Sigma)V\|^2 + \|I\!I\|^2\right\}.$$

This is the penultimate form. The final step, to bring in the inequality (41) for the Schwarzian, is to introduce the curvature.

The curvature of Σ with the metric $\lambda_\Sigma^2 \mathbf{g}_0$ is -4 since $(\Sigma, \lambda_\Sigma^2\, \mathbf{g}_0)$ is isometric to $(\mathbb{D}, \lambda_{\mathbb{D}}|dz|^2)$. For the curvature $K \le 0$ of Σ as a minimal surface one obtains

$$\Delta \log \lambda_\Sigma = 4\lambda_\Sigma^2 - |K|.$$

Hence

$$\|D_V R\|^2 = \frac{4}{\Lambda^4}\left\{\|B - \frac{1}{2}|K|V + 2\lambda_\Sigma^2 V\|^2 + \|I\!I\|^2\right\}. \tag{42}$$

We want to bound this from above and below.

To obtain a lower bound we drop the term $\|II\|^2$ and use (41):

$$\|D_V R\| \geq \frac{2}{\Lambda^2}\|B - \frac{1}{2}|K|V + 2\lambda_\Sigma^2 V\| \geq \frac{2}{\Lambda^2}\left\{2\lambda_\Sigma^2 - \| - B + \frac{1}{2}|K|V\|\right\}$$
$$\geq \frac{2}{\Lambda^2}\left\{2\lambda_\Sigma^2 - \|B\| - \frac{1}{2}|K|\right\} \geq \frac{4\lambda_\Sigma^2}{\Lambda^2}(1-t).$$

To obtain an upper bound we have to estimate the term $\|II\|$. On a minimal surface we always have $II(X,Y) \leq \sqrt{|K|}\|X\|\,\|Y\|$, and so for our case

$$\|II\| = \|II(V, \nabla \log \lambda_\Sigma)\| \leq \sqrt{|K|}\,\|\nabla \log \lambda_\Sigma\| = \sqrt{|K|}\Lambda.$$

We need estimates for each of the factors on the right, and this is where we use the assumption (37), that

$$\|\nabla \sigma(z)\| \leq \frac{C}{1-|z|^2}.$$

An inequality for the curvature follows simply from dropping the positive $\|B\|$ term in (41), giving

$$|K| \leq 2t\lambda_\Sigma^2.$$

Next, from $\log(\lambda_\Sigma \circ \widetilde{f}) = \log \lambda_{\mathbb{D}} - \sigma$ and the bound on $\|\nabla\sigma\|$ we have

$$e^{\sigma(z)}\Lambda = e^{\sigma(z)}\|\nabla \log \lambda_\Sigma(\widetilde{f}(z))\| = \|\nabla \lambda_{\mathbb{D}}(z) - \nabla\sigma(z)\|$$
$$\leq \|\nabla \log \lambda_{\mathbb{D}}(z)\| + \|\nabla\sigma(z)\| \leq \frac{2+C}{1-|z|^2}.$$

Multiplying through by $e^{-\sigma}$ brings back λ_Σ on the right:

$$\Lambda \leq (2+C)\lambda_\Sigma.$$

Finally,

$$\|II\|^2 \leq |K|\Lambda^2 \leq |K|(2+C)^2\lambda_\Sigma^2 \leq 2t(2+C)^2\lambda_\Sigma^4.$$

Back to the equation (42) for $\|D_V R\|^2$, we have

$$\|D_V R\| \leq \frac{2}{\Lambda^2}\left\{\|B - \frac{1}{2}|K|V + 2\lambda_\Sigma^2 V\| + \|II\|\right\}$$
$$\leq \frac{2}{\Lambda^2}\left\{\|B\| + \frac{1}{2}|K| + 2\lambda_\Sigma^2 + \|II\|\right\}$$
$$\leq \frac{2}{\Lambda^2}\left\{2t\lambda_\Sigma^2 + 2\lambda_\Sigma^2 + \sqrt{2t}(2+C)\lambda_\Sigma^2\right\}$$
$$= \frac{2\lambda_\Sigma^2}{\Lambda^2}\left\{2t + \sqrt{2t}(2+C) + 2\right\}.$$

Combining the upper and lower bounds for $\|D_V R\|$ gives

$$\frac{\max_{\|V\|=1}\|D_V R\|}{\min_{\|V\|=1}\|D_V R\|} \leq \frac{2t + \sqrt{2t}(1+C) + 2}{2(1-t)}. \tag{43}$$

This shows that R is quasiconformal as a mapping from Σ to its reflection Σ^*. The extension of $\widetilde{f}$ to a mapping $\widetilde{F} : \overline{\mathbb{C}} \longrightarrow \overline{\Sigma} \cup \Sigma^*$ is as in (33). It, too, is quasiconformal with the same bound for the distortion. This completes the proof of Theorem 2.

When f is analytic satisfying the Ahlfors-Weill condition the quasiconformality of the reflection is measured simply by the Beltrami coefficient, and this turns out to be

$$\mu(1/\bar{z}) = \frac{\partial_{\bar{z}} R(z)}{\partial_z R(z)} = -\frac{1}{2}(1-|z|^2)^2 Sf(z), \quad z \in \mathbb{D}.$$

Thus $|\mu| \leq t < 1$ and the extension of f is a $(1+t)/(1-t)$-quasiconformal mapping of $\overline{\mathbb{C}}$. In the general case it is a question what one might take as a substitute for the Beltrami coefficient, but specializing to the analytic, planar case the bound (43) becomes

$$\frac{\max_{\|V\|=1} \|D_V R\|}{\min_{\|V\|=1} \|D_V R\|} \leq \frac{2t+2}{2(1-t)} = \frac{1+t}{1-t}$$

because all estimates involving the curvature and the second fundamental form (and the upper bound for $\|\nabla\sigma\|$) need not enter at all.

Remark: We have one final comment on when the condition

$$\|\nabla\sigma(z)\| \leq \frac{C}{1-|z|^2}$$

is satisfied if f satisfies the injectivity condition (4).

Suppose $u_{\tilde{f}}$ has a critical point at 0. This means that $\sigma_z(0) = 0$ and we claim that

$$\|\nabla\sigma(z)\| \leq \frac{2|z|}{1-|z|^2}. \tag{44}$$

By applying a rotation of the disk it suffices to establish this on $[0,1)$. With $k(x) = \sigma_z(x)$ we find

$$k'(x) = \sigma_{zz}(x) + \sigma_{z\bar{z}}(x) = (\sigma_{zz}(x) - \sigma_z(x)^2) + \sigma_{z\bar{z}}(x) + k(x)^2,$$

The bound (4) says that

$$|\sigma_{zz}(x) - \sigma_z(x)^2| + 2|\sigma_{z\bar{z}}(x)| \leq \frac{1}{(1-|z|^2)^2},$$

whence

$$|k'(x)| \leq \frac{1}{(1-x^2)^2} + |k(x)^2|.$$

Now let $a(x) = |k(x)|$, $b(x) = x/(1-x^2)$. Then

$$a'(x) \leq |h'(x)| \leq \frac{1}{(1-x^2)^2} + a(x)^2 \quad \text{while} \quad b'(x) = \frac{1}{(1-x^2)^2} + b(x)^2.$$

A standard comparison argument gives $a(x) \leq b(x)$, which is our claim.

Suppose that the surface Σ is bounded, or equivalently that $u_{\tilde{f}}$ has a critical point somewhere in the disk. We may compose with a Möbius transformation of $\mathbb{D}$ onto itself to locate the critical point at the origin, and for the new conformal factor we will have (44). Since the new and original conformal factors are scaled by a factor that is smooth in the closed disk, (44) will also hold for the original conformal factor up to a constant multiple.

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