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DERIVATIVES OF UNIVALENT FUNCTIONS AND THE HYPERBOLIC METRIC

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ABSTRACT. Let f be an analytic and univalent function on a simply connected domain D, and let λ_D be the hyperbolic metric on D. We prove the sharp inequality

$$\left|\frac{f^n(w)}{f(w)}\right| \leq n!4^{n-1}\lambda_D(w)^{n-1}, \qquad w \in D.$$

This can be viewed as a generalization of de Branges's famous result that $|a_n| \leq n$ for function in the class S. Our proof of the above also uses a generalization of K. Löwner's sharp estimate of the coefficients of the inverses of functions in S. We generalize Löwner's result to arbitrary powers of the inverse. We also consider the case when f is convex univalent and when D is convex.

1. Introduction. Let f be an analytic and univalent function on the unit disk U. It is well known that (see, for example, [**3**, p. 32]) f satisfies the following necessary condition

$$\left|\frac{f''(z)}{f'(z)}\right| \leq \frac{6}{1-|z|^2}. \tag{1}$$

On the other hand, B. Osgood [**7**] has generalized (1) to arbitrary simply connected domains. Osgood proved that if $f(w)$ is any univalent function on a simply connected domain D, and if $\lambda_D(w)$ is the hyperbolic metric on D, then

$$\left|\frac{f''(w)}{f'(w)}\right| \leq 8\lambda_D(w). \tag{2}$$

Moreover, the constant 8 above is sharp.

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The proofs of (1) and (2) above are based on the well-known coefficient bound $|a_2| \leq 2$ for the class S of normalized univalent functions with expansion $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ and should be viewed as a generalization of this estimate to points away from the origin of the unit disk and to an arbitrary simply connected domain. On the other hand, L. de Branges, in a celebrated paper [**2**], has settled Bieberbach's long standing conjecture that $|a_n| \leq n$ for functions in S. In this paper we use de Branges's estimates to generalize (2) to the nth derivative. Our main result is the following:

Theorem 1. *Let f be analytic and univalent on a proper simply connected domain D of $\mathbf{C}$, and let $\lambda_D(w)$ denote the hyperbolic metric on D. Then for all w in D,*

$$\left|\frac{f^n(w)}{f'(w)}\right| \leq n! 4^{n-1} \lambda_D(w)^{n-1}. \tag{3}$$

If, moreover, f is convex univalent (i.e., the image $f(D)$ is convex), then we have the stronger estimate

$$\left|\frac{f^n(w)}{f'(w)}\right| \leq n! \binom{2n-1}{n} \lambda_D(w)^{n-1}. \tag{4}$$

The constants above are sharp.

We note that we need not assume f is normalized in Theorem 1 since the expression f^n/f' is unchanged if f is replaced by $Af + B$, A and B complex numbers with $A \neq 0$.

A corresponding generalization of (1) to the nth derivative with sharp constants is implicit in the work of Z. Jakubowski [**4**]. Jakubowski proved that conditional on de Branges's theorem, we have

Theorem (Jakubowski [**4**]). *Let f be a univalent function on the unit disk U. Then for all z in U,*

$$\left|\frac{f^n(z)}{f'(z)}\right| \leq \frac{(n+1)! 2^{n-2}}{(1-|z|^2)^{n-1}}. \tag{5}$$

If, moreover, f is convex, then

$$\left|\frac{f^n(z)}{f'(z)}\right| \leq \frac{n!2^{n-1}}{(1-|z|^2)^{n-1}}. \tag{6}$$

The constants above are best possible in the sense that they are approached as z tends to 1 along the positive real axis for the Koebe function $k(z) = z/(1-z)^2$ and the convex function $l(z) = z/(1-z)$.

Recalling that for the unit disk U, $\lambda_U(z) = 1/(1-|z|^2)$, we see that Theorem 1 is a generalization of Jakubowski's theorem with sharp constants to arbitrary simply connected domains.

In attempting to generalize Osgood's proof of (2) to Theorem 1, in addition to the use of de Branges's estimates, we are led very naturally to consider a coefficient problem in the class S. We have the following:

Theorem 2. *Let* $w = g(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$. *For* $k = 1, 2, 3, \ldots$ *and* $n = k, k+1, k+2, \ldots$, *let* $B_{nk}(a_2, a_3, \ldots, a_n)$ *be the coefficient of* w^n *in the expansion of* $G_k(w) = [g^{-1}(w)]^k = \sum_{n=k}^{\infty} B_{nk} w^n$ *in a neighborhood of the origin where* g^{-1} *is the inverse of* g. *We then have the sharp inequality*

$$|B_{nk}(a_2, a_3, \ldots, a_n)| \leq \frac{k}{n}\binom{2n}{n-k} \tag{7}$$

for $n = k, k+1, k+2, \ldots$ *with equality precisely for rotations of the Koebe function* $k(z) = z/(1-z)^2$.

Theorem 2 in the case $k = 1$ was settled by K. Löwner in his classic 1923 paper [**6**] as an application of his parametric method. G. Schober in [**8**] has given four different proofs of Löwner's result. We will prove Theorem 2 using one of Schober's methods which is a consequence of A. Baernstein's powerful integral means estimate [**1**]. Our proof differs little from that of Schober. We include it here for completeness and to point out that it holds for any power of the inverse. It seems that our results represent an interesting application of our generalization of Löwner's result.

It is clear that Theorem 1 holds with smaller constants if we restrict the domains under consideration. We will consider the case of convex domains in Section 4. It appears that the estimates in Jakubowski's theorem for the unit disk hold more generally for all convex domains with the same best possible constants. We will prove this in the case $n \leq 4$. Our results for convex domains are less complete, as a result analogous to Theorem 2 for convex functions does not hold in general.

We end this introduction with a few words on our methods of proof. Apart from de Branges's estimate and our use of Baernstein's integral mean result to prove Theorem 2, our method is elementary. Roughly, we use the Riemann mapping theorem to transfer the estimate required at a given point of a simply connected domain to the origin of the unit disk where we can apply de Branges's estimate. The resulting transformation rule involves all lower derivatives which can be taken care of by induction. It turns out that the algebra involved can be managed explicitly, and it is possible to maintain a sharp estimate until the end.

2. A coefficient inequality in the class S. In this section we will prove Theorem 2 which is clearly an extremal coefficient problem in the class S. It is natural to hope that the Koebe function $k(z)$ again provides the extremal function for our problem, and we will prove that this is indeed the case. We first compute $B_{nk}(2,3,\dots,n)$ corresponding to the Koebe function. We have $G(w) = k^{-1}(w) = (1-\sqrt{1+4w})^2/4w$ so that

$$G_k(w) = \frac{(1-\sqrt{1+4w})^{2k}}{(4w)^k} = \sum_{n=k}^{\infty}(-1)^{n-k}\frac{k}{n}\binom{2n}{n-k}w^n. \tag{8}$$

This leads us to the correct upper bound in Theorem 2. We will now prove Theorem 2 following the method of [**8**].

Proof of Theorem 2. We set

$$\frac{wG_k'(w)}{G_k(w)} = k + \sum_{n=1}^{\infty} M_{nk}w^n, \tag{9}$$

which is valid in a neighborhood of $w = 0$. We have for $n = 1, 2, 3, \dots,$

$$\begin{aligned} M_{nk} &= \frac{1}{2\pi i} \int_{|w|=\rho} \frac{wG_k'(w)}{G_k(w)} \frac{dw}{w^{n+1}} \\ &= \frac{k}{2\pi i} \int_{|z|=r} g(z)^{-n} \frac{dz}{z} \\ &= \frac{k}{2\pi} \int_0^{2\pi} g(re^{i\theta})^{-n}\, d\theta, \end{aligned}$$

where we may set $r = 1$ since $z/g(z)$ is bounded. We now apply Baernstein's integral mean result [**1**] with negative exponent to obtain the inequality

$$\begin{aligned} |M_{nk}| &\le \frac{k}{2\pi} \int_0^{2\pi} |g(e^{i\theta})|^{-n}\, d\theta \\ &\le \frac{k}{2\pi} \int_0^{2\pi} |k(e^{i\theta})|^{-n}\, d\theta \\ &= k \binom{2n}{n} \end{aligned} \tag{10}$$

for $n = 1, 2, 3, \dots$. From (9) we have

$$wG_k'(w) - kG_k(w) = G_k(w) \sum_{n=1}^{\infty} M_{nk} w^n \tag{11}$$

from which we obtain the recursive formula

$$(n-k)B_{nk} = \sum_{j=k}^{n-1} B_{jk} M_{n-j\,k}$$

for $n = k, k+1, k+2, \dots$.

We can now prove our estimate by induction. Since, trivially, $B_{kk} = 1$ and if

$$|B_{jk}| \le \frac{k}{j} \binom{2j}{j-k} \quad \text{for } j = k, \dots, n-1,$$

then

$$\begin{aligned}(n-k)|B_{nk}| &\le \sum_{j=k}^{n-1} |B_{jk}|\,|M_{n-j\,k}| \\ &\le \sum_{j=k}^{n-1} \frac{k}{j}\binom{2j}{j-k} k \binom{2n-2j}{n-j} \\ &= (n-k)\frac{k}{n}\binom{2n}{n-k}\end{aligned}$$

where the last equality may be proven by substituting $G_k(w) = [k^{-1}(w)]^k$ for the inverse of the Koebe function in (11). Equality can occur in the above only if equality holds in (10) for Baernstein's estimate and by [**1**] this implies that g must be a rotation of the Koebe function. □

3. Proof of Theorem 1. Theorem 1 is essentially a generalization of de Branges's famous estimate $|a_n| \le n$ to a general point of an arbitrary simply connected domain. Clearly one can transfer the derivative estimate at a point w of a simply connected domain to the origin of the unit disk via the Riemann mapping theorem. In order to keep track of the resulting changes in derivatives (due to the chain rule), we need the following transformation formula due to Todorov (see [**9**, p. 224]):

Lemma 1. *Let $w = g(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be a conformal map from the unit disk onto a simply connected domain D. Let $G_k(w)$ and $B_{nk}(a_2, \dots, a_n)$ be as in Theorem 2. Suppose that f is any analytic function in D with $f(0) = 0$. We have*

$$\frac{f^n(0)}{n!} = \sum_{j=1}^{n} \frac{(f\circ g)^j(0)}{j!} B_{nj}(a_2, \dots, a_n). \tag{12}$$

Proof of Theorem 1. We first note that it is sufficient to prove inequalities (3) and (4) in the case $w = 0$, $\lambda_D(0) = 1$, $f(0) = 0$ and $f'(0) = 1$. For instance, since the inequalities are invariant under maps of D of the form $w \to aw + b$, $a \ne 0$, it is possible to assume that

$w = 0$ and $\lambda_D(0) = 1$. Similarly, the inequalities are invariant when f is replaced by $Af + B$, $A \neq 0$, so it is possible to assume $f(0) = 0$ and $f'(0) = 1$.

We now let $w = g(z)$ be a conformal map of the unit disk onto D with $g(0) = 0$ and $g'(0) = 1$. Then with the same notation as in Lemma 1 and Theorem 2, we have (12) by Lemma 1. Since g is in S, we also have inequality (7) by Theorem 2. Now $f \circ g$ is also in S because of our normalization, so we have by de Branges's theorem,

$$\left| \frac{(f \circ g)^j(0)}{j!} \right| \leq j. \tag{13}$$

It follows from (7), (12) and (13) that

$$\left| \frac{f^n(0)}{n!} \right| \leq \sum_{j=1}^{n} \frac{j^2}{n} \binom{2n}{n-j} = 4^{n-1} \tag{14}$$

and (3) follows by our earlier remarks.

We note incidentally that the last combinatorial equality in (14) can be proved using (12). We set in (12) $g(z) = k(z) = z/(1-z)^2$ so that

$$B_{nj} = (-1)^{n-j} \frac{j}{n} \binom{2n}{n-j}$$

by (8). Now choose f so that $h(z) = (f \circ g)(z) = (-1)^n k(-z)$. Then the righthand side of (12) gives exactly the same sum in (14). Now $f(w) = (h \circ g^{-1})(w) = (-1)^n k(-k^{-1}(w)) = (-1)^{n+1} w/(1 + 4w)$ so that $f^n(0)/n! = 4^{n-1}$.

To show that the constant is sharp, we let $D = \mathbf{C} \backslash [-1/4, -\infty)$ which is the image of U under k. We have $\lambda_D(k(z))|k'(z)| = \lambda_U(z)$ so that $z = 0$ gives $\lambda_D(0) = 1$. Let $f(w) = (h \circ k^{-1})(w) = (-1)^{n+1} w/(1+4w)$ be the function defined above which is clearly univalent, and we have $f^n(0)/n! = (-1)^{n+1} 4^{n-1}$.

The case when f is convex is similar except that now $f \circ g$ is also convex and we have the well-known elementary estimate (see, for example, [**3**, p. 45]),

$$\left| \frac{(f \circ g)^j(0)}{j!} \right| \leq 1. \tag{15}$$

By (7), (12) and (15), we have

$$\left|\frac{f^n(0)}{n!}\right| \le \sum_{j=1}^{n} \frac{j}{n} \binom{2n}{n-j} = \binom{2n-1}{n}, \tag{16}$$

thus proving (4).

Again to prove the last equality in (16), we set in (12) $g(z) = k(z)$ and choose f so that $h(z) = (f \circ g)(z) = (-1)^{n+1} z/(1+z)$ and the righthand side of (12) gives the required sum in (16). We also have $f(w) = (h \circ k^{-1})(w) = (-1)^{n+1}(1-(1+4w)^{-1/2})/2$ and

$$\frac{f^n(0)}{n!} = \binom{2n-1}{n}.$$

We now let $D = \mathbf{C}\backslash[-1/4, -\infty)$ as before and choose $f = h \circ k^{-1}$ as above. f is clearly convex since h is and

$$f^n(0)/n! = (-1)^{n+1} \binom{2n-1}{n}$$

so that the constant in (4) is sharp. □

4. Convex domains. It is clear that Theorem 1 holds with smaller constants if we restrict the domains under consideration. In the case of convex domains, in analogy with Theorem 2, it is tempting to guess that the $|B_{nk}|$ are maximized over the class of convex functions by $l(z) = z/(1-z)$. In this case we would have

$$[l^{-1}(w)]^k = w^k(1+w)^{-k} = \sum_{n=k}^{\infty} (-1)^{n-k} \binom{n-1}{k-1} w^n,$$

and it is thus tempting to conjecture that

$$|B_{nk}| \le \binom{n-1}{k-1}$$

for convex g. We note that as in the proof of Theorem 1, the conjecture implies that for convex domains, we have by (12),

$$\left|\frac{f^n(0)}{n!}\right| \le \sum_{k=1}^{n} k \binom{n-1}{k-1} = (n+1)2^{n-2}$$

and if, in addition, f is also convex, then

$$\left|\frac{f^n(0)}{n!}\right| \le \sum_{k=1}^{n} \binom{n-1}{k-1} = 2^{n-1}.$$

We note this means that the estimates in Jakubowski's theorem for the unit disk actually hold for all convex domains. Unfortunately, it has been observed in [**5**] that for large n, B_{n1} cannot be $O((2-\varepsilon)^n)$ for any $\varepsilon > 0$ so that the conjecture is certainly false for $k = 1$. However, it holds at least for all $k \ge n-3$, i.e., we have the following:

Lemma 2. *Let g be a normalized convex univalent function on the unit disk, and let B_{nk} be as defined in Theorem 2. Then, for $n-3 \le k \le n$, we have*

$$|B_{nk}| \le \binom{n-1}{k-1}.$$

Lemma 2 can be proved using an estimate of Trimble [**10**] for convex maps which we state as

Lemma 3 (Trimble). *Let $g(z) = z + a_2z^2 + a_3z^3 + \ldots$ be a convex univalent function on the unit disk. Then*

$$\text{(17)}\qquad \begin{aligned} &\text{(a)} \quad |a_3 - a_2^2| \le (1-|a_2|^2)/3 \\ &\text{(b)} \quad \left|a_4 - 5a_2a_3/2 + 3a_2^3/2\right| \le (1-|a_2|^2)/6. \end{aligned}$$

Trimble in [**10**] actually proved only (17)(a) but (17)(b) follows easily from his method by comparing the next coefficient of Φ in his proof.

Proof of Lemma 2. We let $g = z + a_2z^2 + a_3z^3 + \ldots$. By equating coefficients in

$$[g^{-1}(w)]^k = \sum_{n=k}^{\infty} B_{nk}w^n,$$

we have

$$\begin{aligned} B_{nn} &= 1, \\ B_{n\,n-1} &= -(n-1)a_2, \\ B_{n\,n-2} &= -(n-2)a_3 + [(n-2)(n+1)/2]a_2^2, \end{aligned}$$

and

$$B_{n\,n-3} = -(n-3)[a_4 - (n+1)a_2a_3 + (n+1)(n+2)a_2^3/6].$$

Since g is convex, we have $|a_j| \le 1$ so that the cases $k = n$ and $k = n-1$ are trivial. Now we can rewrite

$$B_{n\,n-2} = (n-2)[(a_2^2 - a_3) + (n-1)a_2^2/2],$$

so that by (17)(a),

$$\begin{aligned} |B_{n\,n-2}| &\le (n-2)[(1-|a_2|^2)/3 + (n-1)|a_2|^2/2] \\ &= (n-2)[1/3 + (3n-5)|a_2|^2/6] \\ &\le \binom{n-1}{2}. \end{aligned}$$

Similarly, we can rewrite

$$\begin{aligned} B_{n\,n-3} = -(n-3)[(a_4 - 5a_2a_3/2 + 3a_2^3/2) + (n-3/2)a_2(a_2^2 - a_3) \\ + (n-2)(n-1)a_2^3/6]. \end{aligned}$$

Using (17)(a), (17)(b) and setting $t = |a_2|$ gives

$$\begin{aligned} |B_{n\,n-3}| &\le [1 - t^2 + (2n-3)t(1-t^2) + (n-1)(n-2)t^3][n-3]/6 \\ &= [t^3(n^2-5n+5) - t^2 + (2n-3)t + 1][n-3]/6. \end{aligned}$$

Now it follows easily by calculus that the function

$$f(t) = t^3(n^2-5n+5) - t^2 + (2n-3)t + 1$$

satisfies $f(0) = 1 > 0$, $f(1) = (n-1)(n-2)$, and

$$f'(t) > 0 \quad \text{for } 0 \le t \le 1 \text{ and } n \ge 4.$$

It follows that we have $|B_{n\,n-3}| \leq \binom{n-1}{3}$. □

It follows from Lemma 2 and the discussion preceding it that we have the following:

Theorem 3. *Let f be an analytic and univalent function on a convex domain, and let $2 \leq n \leq 4$; then*

$$\left|\frac{f^n(w)}{f'(w)}\right| \leq (n+1)!2^{n-2}\lambda_D(w)^{n-1}.$$

If f is in addition convex, then

$$\left|\frac{f^n(w)}{f'(w)}\right| \leq n!2^{n-1}\lambda_D(w)^{n-1}.$$

It is possible that the estimates in Theorem 3 actually hold for all n. Even though the conjecture stated above is not true in general, all we need is that it holds on average, namely that

$$\sum_{k=1}^{n} k|B_{nk}| \leq \sum_{k=1}^{n} k \binom{n-1}{k-1}$$

and

$$\sum_{k=1}^{n} |B_{nk}| \leq \sum_{k=1}^{n} \binom{n-1}{k-1}$$

for Theorem 3 to hold for all n.

We will now observe a simple method of Osgood which proves the estimates of Theorems 1 and 3 for all n but with nonsharp constants. Osgood in [**7**, Theorem 2] gave a necessary and sufficient condition for a hyperbolic domain D such that the inequality (2) will hold for all univalent functions on D. We note that Osgood's criteria holds for all derivatives. The proof of [**7**, Theorem 2] with the obvious modification gives:

Theorem (Osgood). *Let D be a hyperbolic domain in the complex plane and $n = 2, 3, 4, \dots$. Then there exist constants c_n such that*

$$\left|\frac{f^n(w)}{f'(w)}\right| \le c_n \lambda_D(w)^{n-1}, \qquad w \in D,$$

for all univalent analytic functions on D if and only if there is a positive constant c such that

$$\lambda_D(w) \ge \frac{c}{d_D(w)}, \qquad w \in D$$

where $d_D(w)$ is the Euclidean distance from w to the boundary of D. Moreover, if c is given as above, then the c_n can be chosen to be $n!n/c^{n-1}$.

We now observe that for a simply connected domain $1/4 \le c \le 1/2$ (Koebe 1/4 theorem) and for a convex domain $c = 1/2$ ($|a_2| \le 1$). Osgood's theorem thus implies the estimate (3) with the constants off by a factor of n. For a convex domain, it implies the following

Corollary. *Let f be a univalent function on a convex domain. Then*

$$\left|\frac{f^n(w)}{f'(w)}\right| \le n!n2^{n-1} \lambda_D(w)^{n-1}.$$

For $n \le 4$, the constants are not sharp in view of Theorem 3. For all n, the constants in the Corollary are off the expected sharp constants by a factor of about 2.

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