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Further Generalizations of Eneström-Kakeya Theorem

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Abstract

Some extensions and generalizations of Eneström-Kakeya theorem are available in the literature. In this paper further generalized results are given.

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1 Introduction

The Eneström-Kakeya theorem [4] given below is well known in the theory of zero distribution of polynomials.

Theorem A. *For an nth-order polynomial* $P(z) = \sum_{i=0}^{n} a_i z^i$, *assume*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0.$$

Then $P(z)$ *has all its zeros in the disk* $|z| \leq 1$.

In the literature some attempts have been made to extend and generalize the Eneström-Kakeya theorem. Joyal et al [3] extended the Eneström-Kakeya theorem to the polynomials with general monotonic coefficients by showing that if

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0,$$

then all the zeros of $P(z)$ are contained in the disk

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}.$$

Aziz and Zargar [1] generalized the result of Joyal et al [3] as follows.

Theorem B. *If* $P(z) = \sum_{i=0}^{n} a_i z^i$ *is an nth-order polynomial such that for some* $\lambda \geq 1$,

$$\lambda a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0,$$

then $P(z)$ *has all its zeros in the disk*

$$|z + \lambda - 1| \leq \frac{\lambda a_n - a_0 + |a_0|}{|a_n|}.$$

On the other hand Govil and Rahman [2] investigated the zero distribution of polynomials such that the modulii of coefficients are monotonic, and proved the following theorem.

Theorem C. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be the nth-order polynomial such that for some* $a > 0$,

$$|a_n| \geq a|a_{n-1}| \geq a^2|a_{n-2}| \geq \cdots \geq a^{n-1}|a_1| \geq a^n|a_0|.$$

Then all the zeros of $P(z)$ *lie in the disk* $|z| \leq K_1/a$, *where* K_1 *is the greatest positive root of the equation*

$$K^{n+1} - 2K^n + 1 = 0.$$

Govil and Rahman [2] also proved that if $P(z) = \sum_{i=0}^{n} a_i z^i$ is an nth-order complex polynomial such that

$$|\arg a_i - \beta| \leq \alpha \leq \pi/2, \; i = 0, 1, 2, \cdots, n,$$

for some real β, and

$$|a_n| \geq |a_{n-1}| \geq \cdots \geq |a_1| \geq |a_0|,$$

then all the zeros of $P(z)$ lie in the disk

$$|z| \leq \cos\alpha + \sin\alpha + \frac{2\sin\alpha}{|a_n|} \sum_{i=0}^{n-1} |a_i|$$

.

Recently Shah and Liman [5] generalized Theorem B and the result of Govil and Rahman [2], and proved the following two theorems.

Theorem D. *Consider an nth-order complex polynomial* $P(z) = \sum_{i=0}^{n} a_i z^i$ *with* $\mathrm{Re}\{a_i\} = \alpha_i$ *and* $\mathrm{Im}\{a_i\} = \beta_i$, $i = 0, 1, 2, \cdots, n$. *If for some* $\lambda \geq 1$,

$$\lambda \alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \alpha_0,$$

$$\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_1 \geq \beta_0 > 0,$$

then all the zeros of $P(z)$ *lie in the disk*

$$\left| z + \frac{(\lambda - 1)\alpha_n}{a_n} \right| \leq \frac{\lambda\alpha_n - \alpha_0 + |\alpha_0| + \beta_n}{|a_n|}.$$

Theorem E. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be the nth-order complex polynomial such that*

$$|\arg a_i - \beta| \leq \alpha \leq \pi/2, \ i = 0, 1, 2, \cdots, n,$$

for some real β*, and*

$$\lambda |a_n| \geq |a_{n-1}| \geq \cdots \geq |a_1| \geq |a_0|,$$

for some $\lambda \geq 1$*. Then all the zeros of* $P(z)$ *lie in the disk*

$$|z + \lambda - 1| \leq \frac{1}{|a_n|} \left\{ (\lambda |a_n| - |a_0|)(\cos\alpha + \sin\alpha) + |a_0| + 2\sin\alpha \sum_{i=0}^{n-1} |a_i| \right\}.$$

This paper presents further generalizations of the Eneström-Kakeya theorem. To illustrate the motivation of this paper, consider a fifth-order real polynomial given by

$$\begin{aligned} P(z) &= \sum_{i=0}^{5} a_i z^i \\ &= 5z^5 + 4z^4 + 10z^3 + 3z^2 - z - 2. \end{aligned}$$

Obviously Theorem B is not applicable to this polynomial. However we have

$$a_5 \geq a_4 \geq \lambda a_3 \geq a_2 \geq a_1 \geq a_0,$$

for $0.3 \leq \lambda \leq 0.4$. Then it is natural to ask what happens in Theorem B if

$$a_{n-k+1} \geq \lambda a_{n-k} \geq a_{n-k-1},$$

for some $\lambda \neq 1$ and $1 \leq k \leq n$ ($a_{-1} = 0$). Similar questions can be raised for other theorems mentioned above. In the next section we present some solutions to such questions for Theorem B through Theorem E.

2 Theorems and proofs

Theorem 1. *Let* $P(z) = \sum_{i=0}^{n} a_i z^i$ *be the nth-order polynomial such that for some* $\lambda \neq 1$, $1 \leq k \leq n$ *and* $a_{n-k} \neq 0$,

$$a_n \geq a_{n-1} \geq \cdots \geq a_{n-k+1} \geq \lambda a_{n-k} \geq a_{n-k-1} \geq \cdots \geq a_1 \geq a_0.$$

If $a_{n-k-1} > a_{n-k}$, then all the zeros of $P(z)$ lie in the disk $|z| \leq K_1$, where K_1 is the greatest positive root of the equation

$$K^{k+1} - \delta_1 K^k - |\gamma_1| = 0,$$

where

$$\gamma_1 = \frac{(\lambda - 1)a_{n-k}}{a_n},$$
$$\delta_1 = \frac{a_n + (\lambda - 1)a_{n-k} - a_0 + |a_0|}{|a_n|}.$$

If $a_{n-k} > a_{n-k+1}$, then all the zeros of $P(z)$ lie in the disk $|z| \leq K_2$, where K_2 is the greatest positive root of the equation

$$K^k - \delta_2 K^{k-1} - |\gamma_2| = 0,$$

where

$$\gamma_2 = \frac{(1 - \lambda)a_{n-k}}{a_n},$$
$$\delta_2 = \frac{a_n + (1 - \lambda)a_{n-k} - a_0 + |a_0|}{|a_n|}.$$

Proof. Consider a polynomial

$$\begin{aligned} \Phi(z) &= (1 - z)P(z) \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \cdots + (a_1 - a_0)z + a_0. \end{aligned}$$

If $a_{n-k-1} > a_{n-k}$, then $a_{n-k+1} > a_{n-k}$ and $\Phi(z)$ can be written as

$$\begin{aligned} \Phi(z) &= -a_n z^{n+1} - (\lambda - 1)a_{n-k}z^{n-k} + (a_n - a_{n-1})z^n + \cdots \\ &\quad + (a_{n-k+1} - a_{n-k})z^{n-k+1} + (\lambda a_{n-k} - a_{n-k-1})z^{n-k} \\ &\quad + (a_{n-k-1} - a_{n-k-2})z^{n-k-1} + \cdots + (a_1 - a_0)z + a_0. \end{aligned}$$

For $|z| > 1$,

$$\begin{aligned} |\Phi(z)| &\geq |a_n z^{n+1} + (\lambda - 1)a_{n-k}z^{n-k}| \\ &\quad - |z|^n \Bigg\{ (a_n - a_{n-1}) + \cdots + \frac{(a_{n-k+1} - a_{n-k})}{|z|^{k-1}} + \frac{(\lambda a_{n-k} - a_{n-k-1})}{|z|^k} \\ &\quad + \frac{(a_{n-k-1} - a_{n-k-2})}{|z|^{k+1}} + \cdots + \frac{(a_1 - a_0)}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} \Bigg\} \\ &\geq |z|^{n-k}|a_n z^{k+1} + (\lambda - 1)a_{n-k}| - |z|^n \{a_n + (\lambda - 1)a_{n-k} - a_0 + |a_0|\} \\ &> 0 \end{aligned}$$

if

$$|z^{k+1} + \gamma_1| > \delta_1 |z|^k.$$

This inequality holds if

$$|z|^{k+1} - |\gamma_1| > \delta_1 |z|^k.$$

Hence all the zeros of $P(z)$ with modulus greater than one lie in the disk $|z| \leq K_1$, where K_1 is the greatest positive root of the equation

$$K^{k+1} - \delta_1 K^k - |\gamma_1| = 0.$$

But the zeros of $P(z)$ with modulus less than or equal to one are already contained in the disk $|z| \leq K_1$ since $K_1 > 1$ (see Remark 1 below).

The second part can be proved similarly. If $a_{n-k} > a_{n-k+1}$, then $a_{n-k} > a_{n-k-1}$ and $\Phi(z)$ can be written as

$$\begin{aligned}\Phi(z) &= -a_n z^{n+1} - (1-\lambda)a_{n-k}z^{n-k+1} + (a_n - a_{n-1})z^n + \cdots \\ &\quad +(a_{n-k+1} - \lambda a_{n-k})z^{n-k+1} + (a_{n-k} - a_{n-k-1})z^{n-k} \\ &\quad +(a_{n-k-1} - a_{n-k-2})z^{n-k-1} + \cdots + (a_1 - a_0)z + a_0.\end{aligned}$$

For $|z| > 1$,

$$\begin{aligned}|\Phi(z)| &\geq |z|^{n-k+1}||a_n z^k + (1-\lambda)a_{n-k}| \\ &\quad -|z|^n \Bigg\{(a_n - a_{n-1}) + \cdots + \frac{(a_{n-k+1} - \lambda a_{n-k})}{|z|^{k-1}} \\ &\quad + \frac{(a_{n-k} - a_{n-k-1})}{|z|^k} + \frac{(a_{n-k-1} - a_{n-k-2})}{|z|^{k+1}} + \cdots + \frac{(a_1 - a_0)}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\Bigg\} \\ &\geq |z|^{n-k+1}|a_n z^k + (1-\lambda)a_{n-k}| - |z|^n\{a_n + (1-\lambda)a_{n-k} - a_0 + |a_0|\} \\ &> 0\end{aligned}$$

if

$$|z^k + \gamma_2| > \delta_2 |z|^{k-1}.$$

This inequality holds if

$$|z|^k - |\gamma_2| > \delta_2 |z|^{k-1}.$$

Hence all the zeros of $P(z)$ with modulus greater than one lie in the disk $|z| \leq K_2$, where K_2 is the greatest positive root of the equation

$$K^k - \delta_2 K^{k-1} - |\gamma_2| = 0.$$

But the zeros of $P(z)$ with modulus less than or equal to one are already contained in the disk $|z| \leq K_2$ since $K_2 > 1$ (see Remark 2 below).

Remark 1. Let

$$f_1(K) = K^{k+1} - \delta_1 K^k - |\gamma_1|.$$

To prove $K_1 > 1$, it suffices to show that $f_1(1) < 0$. If $a_{n-k-1} > a_{n-k}$, then one of the following four cases happens.

(a) $a_{n-k+1} \geq a_{n-k-1} > a_{n-k} > 0$ and $\lambda > 1$.
(b) $a_{n-k+1} \geq a_{n-k-1} \geq 0 > a_{n-k}$ and $\lambda \leq 0$.
(c) $a_{n-k+1} \geq 0 \geq a_{n-k-1} > a_{n-k}$ and $\lambda < 1$.
(d) $0 \geq a_{n-k+1} \geq a_{n-k-1} > a_{n-k}$ and $0 < \lambda < 1$.

It is easily seen that $\gamma_1 > 0$ and $\delta_1 \geq 1 + \gamma_1$ for the cases (a), (b) and (c), and $\delta_1 \geq 1 + |\gamma_1|$ for the case (d). Then $f_1(1) = 1 - \delta_1 - |\gamma_1| < 0$ and we have $K_1 > 1$.

Remark 2. Let

$$f_2(K) = K^k - \delta_2 K^{k-1} - |\gamma_2|.$$

If $a_{n-k} > a_{n-k+1}$, then the following four cases are possible to occur.

(a) $a_{n-k} > a_{n-k+1} \geq a_{n-k-1} \geq 0$ and $0 \leq \lambda < 1$.
(b) $a_{n-k} > a_{n-k+1} \geq 0 \geq a_{n-k-1}$ and $\lambda < 1$.
(c) $a_{n-k} > 0 \geq a_{n-k+1} \geq a_{n-k-1}$ and $\lambda \leq 0$.
(d) $0 > a_{n-k} > a_{n-k+1} \geq a_{n-k-1}$ and $\lambda > 1$.

Then $\gamma_2 > 0$ and $\delta_2 \geq 1 + \gamma_2$ for the first three cases, and $\delta_2 \geq 1 + |\gamma_2|$ for the last case. Hence $f_2(1) = 1 - \delta_2 - |\gamma_2| < 0$ and we have $K_2 > 1$.

Theorem 2. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be the nth-order polynomial such that for some $a > 0$, $\lambda(\neq 1) > 0$, $1 \leq k \leq n$ and $a_{n-k} \neq 0$,*

$$|a_n| \geq a|a_{n-1}| \geq \cdots \geq a^{k-1}|a_{n-k+1}| \geq \lambda a^k |a_{n-k}| \geq a^{k+1}|a_{n-k-1}| \geq \cdots \geq a^n |a_0|.$$

If $|a_{n-k}| < a|a_{n-k-1}|$ (i.e., $\lambda > 1$), then all the zeros of $P(z)$ lie in the disk $|z| \leq K_1/a$, where K_1 is the greatest positive root of the equation

$$K^{n+1} - 2K^n + 1 = 0.$$

If $a|a_{n-k}| > |a_{n-k+1}|$ (i.e., $0 < \lambda < 1$), then $P(z)$ has all its zeros in the disk $|z| \leq K_2/a$, where K_2 is the greatest positive root of the equation

$$K^{n+1} - 2K^n + \frac{\lambda - 1}{\lambda} K^{n-k+1} + \frac{1}{\lambda} = 0.$$

Proof. If $|a_{n-k}| < a|a_{n-k-1}|$, then $a^{k-1}|a_{n-k+1}| \geq a^k |a_{n-k}|$, and we obtain the same result as Theorem B following the proof of Theorem 1 in [2]. Now suppose

$a|a_{n-k}| > |a_{n-k+1}|$, and let $Q(z) = \sum_{i=0}^{n-1} a_i z^i$. Then for $|z| = R(> 1/a)$,

$$\begin{aligned}
|Q(z)| &\leq |a_{n-1}|R^{n-1}\Big\{1 + \frac{1}{aR} + \frac{1}{(aR)^2} + \cdots + \frac{1}{(aR)^{k-2}} \\
&\quad + \frac{1}{\lambda(aR)^{k-1}} + \frac{1}{(aR)^k} + \cdots + \frac{1}{(aR)^{n-1}}\Big\} \\
&\leq |a_{n-1}|R^{n-1}\Big\{1 + \frac{1}{aR} + \frac{1}{(aR)^2} + \cdots + \frac{1}{(aR)^{k-2}} \\
&\quad + \frac{1}{\lambda(aR)^{k-1}} + \frac{1}{\lambda(aR)^k} + \cdots + \frac{1}{\lambda(aR)^{n-1}}\Big\} \\
&= |a_{n-1}|R^{n-1}\Big\{\frac{(aR)^{k-1} - 1}{(aR)^{k-2}(aR-1)} + \frac{(aR)^{n-k+1} - 1}{\lambda(aR)^{n-1}(aR-1)}\Big\} \\
&= |a_{n-1}|R^{n-1}\Big\{\frac{\lambda(aR)^n + (1-\lambda)(aR)^{n-k+1} - 1}{\lambda(aR)^{n-1}(aR-1)}.\Big\}
\end{aligned}$$

Hence

$$\begin{aligned}
|P(z)| &\geq |a_n|R^n - |a_{n-1}|R^{n-1}\Big\{\frac{\lambda(aR)^n + (1-\lambda)(aR)^{n-k+1} - 1}{\lambda(aR)^{n-1}(aR-1)}\Big\} \\
&> 0
\end{aligned}$$

if

$$\frac{|a_n|}{a|a_{n-1}|} > \frac{\lambda(aR)^n + (1-\lambda)(aR)^{n-k+1} - 1}{\lambda(aR)^{n-1}(aR-1)}.$$

Since $|a_n|/a|a_{n-1}| \geq 1$ by hypothesis, the above inequality holds if

$$\lambda(aR)^{n-1}(aR-1) > \lambda(aR)^n + (1-\lambda)(aR)^{n-k+1} - 1.$$

Replacing aR by K, we obtain the result.

Theorem 3. *Consider an nth-order complex polynomial* $P(z) = \sum_{i=0}^{n} a_i z^i$ *with* $\mathrm{Re}\{a_i\} = \alpha_i$ *and* $\mathrm{Im}\{a_i\} = \beta_i$ $i = 0, 1, 2, \cdots, n$, *and assume that for some* $\lambda \neq 1$, $1 \leq k \leq n$ *and* $\alpha_{n-k} \neq 0$,

$$\alpha_n \geq \cdots \geq \alpha_{n-k+1} \geq \lambda\alpha_{n-k} \geq \alpha_{n-k-1} \geq \cdots \geq \alpha_1 \geq \alpha_0,$$

$$\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_1 \geq \beta_0.$$

If $\alpha_{n-k-1} > \alpha_{n-k}$, *then all the zeros of* $P(z)$ *lie in the disk* $|z| \leq K_1$, *where* K_1 *is the greatest positive root of the equation*

$$K^{k+1} - \delta_1 K^k - |\gamma_1| = 0,$$

where

$$\gamma_1 = \frac{(\lambda-1)\alpha_{n-k}}{a_n},$$

$$\delta_1 = \frac{\alpha_n + (\lambda - 1)\alpha_{n-k} - \alpha_0 + |\alpha_0| + \beta_n - \beta_0 + |\beta_0|}{|a_n|}.$$

If $\alpha_{n-k} > \alpha_{n-k+1}$, then all the zeros of $P(z)$ lie in the disk $|z| \le K_2$, where K_2 is the greatest positive root of the equation

$$K^k - \delta_2 K^{k-1} - |\gamma_2| = 0,$$

where

$$\gamma_2 = \frac{(1-\lambda)\alpha_{n-k}}{a_n},$$

$$\delta_2 = \frac{\alpha_n + (1 - \lambda)\alpha_{n-k} - \alpha_0 + |\alpha_0| + \beta_n - \beta_0 + |\beta_0|}{|a_n|}.$$

Proof. Consider a polynomial

$$\begin{aligned} \Phi(z) &= (1-z)P(z) \\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \cdots + (a_1 - a_0)z + a_0 \\ &= -a_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + \cdots + (\alpha_1 - \alpha_0)z + \alpha_0 \\ &\quad + i\{(\beta_n - \beta_{n-1})z^n + \cdots + (\beta_1 - \beta_0)z + \beta_0\}. \end{aligned}$$

If $\alpha_{n-k-1} > \alpha_{n-k}$, then $\alpha_{n-k+1} > \alpha_{n-k}$ and $\Phi(z)$ can be written as

$$\begin{aligned} \Phi(z) &= -a_n z^{n+1} - (\lambda - 1)\alpha_{n-k} z^{n-k} \\ &\quad + (\alpha_n - \alpha_{n-1})z^n + \cdots + (\alpha_{n-k+1} - \alpha_{n-k})z^{n-k+1} \\ &\quad + (\lambda\alpha_{n-k} - \alpha_{n-k-1})z^{n-k} \cdots + (\alpha_1 - \alpha_0)z + \alpha_0 \\ &\quad + i\{(\beta_n - \beta_{n-1})z^n + \cdots + (\beta_1 - \beta_0)z + \beta_0\}. \end{aligned}$$

If $|z| > 1$, then

$$\begin{aligned} |\Phi(z)| &\ge |a_n z^{n+1} + (\lambda - 1)\alpha_{n-k} z^{n-k}| \\ &\quad - |z|^n \left\{ (\alpha_n - \alpha_{n-1}) + \cdots + \frac{(\alpha_{n-k+1} - \alpha_{n-k})}{|z|^{k-1}} \right. \\ &\quad \left. + \frac{(\lambda\alpha_{n-k} - \alpha_{n-k-1})}{|z|^k} + \cdots + \frac{(\alpha_1 - \alpha_0)}{|z|^{n-1}} + \frac{|\alpha_0|}{|z|^n} \right\} \\ &\quad - |z|^n \left\{ (\beta_n - \beta_{n-1}) + \cdots + \frac{(\beta_1 - \beta_0)}{|z|^{n-1}} + \frac{|\beta_0|}{|z|^n} \right\} \\ &\ge |a_n z^{n+1} + (\lambda - 1)\alpha_{n-k} z^{n-k}| \\ &\quad - |z|^n |\{\alpha_n + (\lambda - 1)\alpha_{n-k} - \alpha_0 + |\alpha_0| + \beta_n - \beta_0 + |\beta_0|)\} \\ &> 0 \end{aligned}$$

if

$$|z^{k+1}+\gamma_1|>\delta_1|z|^k.$$

But this inequality holds if

$$|z^{k+1}|-|\gamma_1|>\delta_1|z|^k.$$

As a result all the zeros of $P(z)$ with modulus greater than one are contained in the disk $|z|\leq K_1$, where K_1 is the greatest positive root of the equation

$$K^{k+1}-\delta_1 K^k-|\gamma_1|=0.$$

As in the case of Theorem 1, it can be shown that $K_1>1$. Hence all the zeros of $P(z)$ with modulus less that or equal to one are already lie in the disk $|z|\leq K_1$, and the proof of the first part is completed.

Now assume $\alpha_{n-k}>\alpha_{n-k+1}$. Then $\alpha_{n-k}>\alpha_{n-k-1}$, and $\Phi(z)$ can be written as

$$\begin{aligned}\Phi(z) &= -a_n z^{n+1}-(1-\lambda)\alpha_{n-k}z^{n-k+1}\\ &\quad +(\alpha_n-\alpha_{n-1})z^n+\cdots+(\alpha_{n-k+1}-\lambda\alpha_{n-k})z^{n-k+1}\\ &\quad +(\alpha_{n-k}-\alpha_{n-k-1})z^{n-k}\cdots+(\alpha_1-\alpha_0)z+\alpha_0\\ &\quad +i\{(\beta_n-\beta_{n-1})z^n+\cdots+(\beta_1-\beta_0)z+\beta_0\}.\end{aligned}$$

If $|z|>1$, then

$$\begin{aligned}|\Phi(z)| &\geq |a_n z^{n+1}+(1-\lambda)\alpha_{n-k}z^{n-k+1}|\\ &\quad -|z|^n\Big\{(\alpha_n-\alpha_{n-1})+\cdots+\frac{(\alpha_{n-k+1}-\lambda\alpha_{n-k})}{|z|^{k-1}}\\ &\quad +\frac{(\alpha_{n-k}-\alpha_{n-k-1})}{|z|^k}+\cdots+\frac{(\alpha_1-\alpha_0)}{|z|^{n-1}}+\frac{|\alpha_0|}{|z|^n}\Big\}\\ &\quad -|z|^n\Big\{(\beta_n-\beta_{n-1})+\cdots+\frac{(\beta_1-\beta_0)}{|z|^{n-1}}+\frac{|\beta_0|}{|z|^n}\Big\}\\ &\geq |a_n z^{n+1}+(1-\lambda)\alpha_{n-k}z^{n-k}|\\ &\quad -|z|^n|\{\alpha_n+(1-\lambda)\alpha_{n-k}-\alpha_0+|\alpha_0|+\beta_n-\beta_0+|\beta_0|\}\\ &> 0\end{aligned}$$

if

$$|z^k+\gamma_2|>\delta_1|z|^{k-1}.$$

But above inequality holds if

$$|z^k|-|\gamma_2|>\delta_2|z|^{k-1}.$$

Hence all the zeros of $P(z)$ with modulus greater than one lie in the disk $|z| \leq K_2$, where K_2 is the greatest positive root of the equation

$$K^k - \delta_2 K^{k-1} - |\gamma_2| = 0.$$

Again it is easily seen that $K_2 > 1$, and all the zeros of $P(z)$ with modulus less that or equal to one are already contained in the disk $|z| \leq K_2$, and the proof of the second part is completed.

Theorem 4. *Let $P(z) = \sum_{i=0}^{n} a_i z^i$ be the nth-order complex polynomial such that for some real β,*

$$|\arg a_i - \beta| \leq \alpha \leq \pi/2, \; i = 0, 1, 2, \cdots, n$$

and for some $\lambda \neq 1$ and $a_{n-k} \neq 0$,

$$|a_n| \geq |a_{n-1}| \geq \cdots |a_{n-k+1}| \geq \lambda |a_{n-k}| \geq |a_{n-k-1}| \geq \cdots \geq |a_1| \geq |a_0|.$$

If $|a_{n-k}| < |a_{n-k-1}|$ (i.e., $\lambda > 1$), then all the zeros of $P(z)$ lie in the disk $|z| \leq K_1$, where K_1 is the greatest positive root of the equation

$$K^{k+1} - \delta_1 K^k - |\gamma_1| = 0,$$

where

$$\gamma_1 = \frac{(\lambda - 1) a_{n-k}}{a_n},$$

$$\delta_1 = \frac{\{(|a_n| + (\lambda - 1)|a_{n-k}|)\}(\cos\alpha + \sin\alpha) + 2\sin\alpha \sum_{i=0}^{n-1} |a_i|}{|a_n|}.$$

If $|a_{n-k}| > |a_{n-k+1}|$ (i.e., $0 < \lambda < 1$), then all the zeros of $P(z)$ lie in the disk $|z| \leq K_2$, where K_2 is the greatest positive root of the equation

$$K^k - \delta_2 K^{k-1} - |\gamma_2| = 0,$$

where

$$\gamma_2 = \frac{(1 - \lambda) a_{n-k}}{a_n},$$

$$\delta_2 = \frac{\{(|a_n| + (1 - \lambda)|a_{n-k}|)\}(\cos\alpha + \sin\alpha) + 2\sin\alpha \sum_{i=0}^{n-1} |a_i|}{|a_n|}.$$

Proof. Consider a polynomial

$$\begin{aligned} \Phi(z) &= (1 - z) P(z) \\ &= -a_n z^{n+1} + (a_n - a_{n-1}) z^n + \cdots + (a_1 - a_0) z + a_0. \end{aligned}$$

If $|a_{n-k-1}| > |a_{n-k}|$, then $|a_{n-k+1}| > |a_{n-k}|$ and $\Phi(z)$ can be written as

$$\begin{aligned}\Phi(z) &= -a_n z^{n+1} - (\lambda-1)a_{n-k}z^{n-k} + (a_n - a_{n-1})z^n + \cdots \\ &\quad + (a_{n-k+1} - a_{n-k})z^{n-k+1} + (\lambda a_{n-k} - a_{n-k-1})z^{n-k} \\ &\quad + (a_{n-k-1} - a_{n-k-2})z^{n-k-1} + \cdots + (a_1 - a_0)z + a_0.\end{aligned}$$

If $|z| > 1$, then

$$\begin{aligned}|\Phi(z)| &\geq |a_n z^{n+1} + (\lambda-1)a_{n-k}z^{n-k}| \\ &\quad - |z|^n \Big\{ |a_n - a_{n-1}| + \cdots + \frac{|a_{n-k+1} - a_{n-k}|}{|z|^{k-1}} \\ &\quad + \frac{|\lambda a_{n-k} - a_{n-k-1}|}{|z|^k} + \cdots + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} \Big\} \\ &\geq |a_n z^{n+1} + (\lambda-1)a_{n-k}z^{n-k}| \\ &\quad - |z|^n \Big\{ |a_n - a_{n-1}| + \cdots + |a_{n-k+1} - a_{n-k}| \\ &\quad + |\lambda a_{n-k} - a_{n-k-1}| + \cdots + |a_1 - a_0| + |a_0| \Big\}.\end{aligned}$$

It was shown in [2] that, for two complex numbers b_0 and b_1, if $|b_0| \geq |b_1|$ and $|\arg b_i - \beta| \leq \alpha \leq \pi/2$, $i = 0, 1$, for some β, then

$$|b_0 - b_1| \leq (|b_0| - |b_1|)\cos\alpha + (|b_0| + |b_1|)\sin\alpha.$$

Using this fact, we have

$$\begin{aligned}|\Phi(z)| &\geq |a_n z^{n+1} + (\lambda-1)a_{n-k}z^{n-k}| \\ &\quad - |z|^n \Big\{ \{(|a_n| + (\lambda-1)|a_{n-k}|)\}(\cos\alpha + \sin\alpha) \\ &\quad - |a_0|(\cos\alpha + \sin\alpha - 1) + 2\sin\alpha \sum_{i=0}^{n-1} |a_i| \Big\} \\ &\geq |a_n z^{n+1} + (\lambda-1)a_{n-k}z^{n-k}| \\ &\quad - |z|^n \Big\{ \{(|a_n| + (\lambda-1)|a_{n-k}|)\}(\cos\alpha + \sin\alpha) + 2\sin\alpha \sum_{i=0}^{n-1} |a_i| \Big\} \\ &> 0\end{aligned}$$

if

$$|z^{k+1} + \gamma_1| > \delta_1 |z|^k.$$

This inequality holds if

$$|z|^{k+1} - |\gamma_1| > \delta_1 |z|^k,$$

and all the zeros of $P(z)$ with modulus greater than one lie in the disk $|z| \leq K_1$, where K_1 is the greatest positive root of the equation

$$K^{k+1} - \delta_1 K^k - |\gamma_1| = 0.$$

It is easily seen that $K_1 > 1$, and all the zeros of $P(z)$ with modulus less than or equal to one are already contained in the disk $|z| \leq K_1$.

Now consider the case $|a_{n-k}| > |a_{n-k+1}|$. Then $|a_{n-k}| > |a_{n-k-1}|$ and $\Phi(z)$ can be written as

$$\begin{aligned}\Phi(z) &= -a_n z^{n+1} - (1-\lambda)a_{n-k}z^{n-k+1} + (a_n - a_{n-1})z^n + \cdots \\ &\quad +(a_{n-k+1} - \lambda a_{n-k})z^{n-k+1} + (a_{n-k} - a_{n-k-1})z^{n-k} + \cdots \\ &\quad +(a_1 - a_0)z + a_0.\end{aligned}$$

If $|z| > 1$, then

$$\begin{aligned}|\Phi(z)| &\geq |a_n z^{n+1} + (1-\lambda)a_{n-k}z^{n-k+1}| \\ &\quad -|z|^n\Big\{|a_n - a_{n-1}| + \cdots + \frac{|a_{n-k+1} - \lambda a_{n-k}|}{|z|^{k-1}} \\ &\quad +\frac{|a_{n-k} - a_{n-k-1}|}{|z|^k} + \cdots + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\Big\} \\ &\geq |a_n z^{n+1} + (1-\lambda)a_{n-k}z^{n-k+1}| \\ &\quad -|z|^n\Big\{(|a_n| + (1-\lambda)|a_{n-k}|)(\cos\alpha + \sin\alpha) \\ &\quad -|a_0|(\cos\alpha + \sin\alpha - 1) + 2\sin\alpha\sum_{i=0}^{n-1}|a_i|\Big\} \\ &\geq |a_n z^{n+1} + (1-\lambda)a_{n-k}z^{n-k+1}| \\ &\quad -|z|^n\Big\{(|a_n| + (1-\lambda)|a_{n-k}|)(\cos\alpha + \sin\alpha) + 2\sin\alpha\sum_{i=0}^{n-1}|a_i|\Big\} \\ &> 0\end{aligned}$$

if

$$|z^k + \gamma_2| > \delta_2 |z|^{k-1}.$$

This inequality holds if

$$|z|^k - |\gamma_2| > \delta_2 |z|^{k-1},$$

and all the zeros of $P(z)$ lie in the disk $|z| \leq K_2$, where K_2 is the greatest positive root of the equation

$$K^k - \delta_2 K^{k-1} - |\gamma_2| = 0.$$

Again it can be shown that $K_2 > 1$, and all the zeros of $P(z)$ lie in the disk $|z| \leq K_2$.

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