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Certain classes of series associated with the Zeta function and multiple gamma functions

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Abstract

The authors apply the theory of multiple Gamma functions, which was recently revived in the study of the determinants of the Laplacians, in order to evaluate some families of series involving the Riemann Zeta function. By introducing a certain mathematical constant, they also systematically evaluate this constant and some definite integrals of the triple Gamma function. Various classes of series associated with the Zeta function are expressed in closed forms. Many of these results are also used here to compute the determinant of the Laplacian on the four-dimensional unit sphere $\mathbf{S}^4$ explicitly.

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1. Introduction, definitions, and preliminaries

The double Gamma function was defined and studied by Barnes [3–5] and others in about 1900. Although this function did not appear in the tables of the most well-known special functions, yet it was cited in the exercises by Whittaker and Watson [30, p. 264] and recorded also by Gradshteyn and Ryzhik [16, p. 661, Entry 6.441(4); p. 937, Entry 8.333]. Recently, this function was revived in the study of the determinants of the Laplacians on the n-dimensional unit sphere $\mathbf{S}^n$ (see [9,18,22,23,27,29]). Shintani [24] also used this function to prove the classical Kronecker limit formula. Its p-adic analytic extension appeared in a formula of Cassou-Noguès [7] for the p-adic L-functions at the point 0. More recently, Choi et al. [10–13] used this function to evaluate the

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sums of several classes of series involving the Riemann Zeta function. Matsumoto [21], on the other hand, proved asymptotic expansions of the Barnes double Zeta function and the double Gamma function, and presented an application to the Hecke L-functions of real quadratic fields. Before their investigation by Barnes, these functions had been introduced in a different form by (for example) Hölder [19], Alexeiewsky [1], and Kinkelin [20]. The theory of multiple Gamma functions was also developed in yet another paper by Barnes [6].

In this paper we aim at presenting an explicit form of the triple Gamma function Γ_3 by reducing the recurrence formula for the multiple Gamma function introduced by Vignéras [26] and a general form of a definite integral of the double Gamma function evaluated in terms of Γ_3. We also show that various series involving the Riemann Zeta function can be evaluated by using the theory of multiple Gamma functions (Γ_2 and Γ_3) and introducing a certain mathematical constant. Finally, by making use of some of our closed-form evaluations of series involving the Zeta function, we compute the determinant of the Laplacian on the four-dimensional unit sphere $\mathbf{S}^4$ with the standard metric.

In accordance with Barnes's definition [3], the double Gamma function $\Gamma_2 = 1/G$ satisfies each of the following properties:

(a) $G(z+1) = \Gamma(z)G(z)$ $(z \in \mathbb{C})$;

(b) $G(1) = 1$;

(c) Asymptotically,

$$\log G(z+n+2) = \frac{n+1+z}{2}\log(2\pi) + \left[\frac{n^2}{2} + n + \frac{5}{12} + \frac{z^2}{2} + (n+1)z\right]\log n - \frac{3n^2}{4} - n(1+z) - \log A + \frac{1}{12} + \mathrm{O}\left(\frac{1}{n}\right) \quad (n \to \infty), \tag{1.1}$$

where Γ is the familiar Gamma function:

$$\{\Gamma(z+1)\}^{-1} = \mathrm{e}^{\gamma z}\prod_{k=1}^{\infty}\left\{\left(1+\frac{z}{k}\right)\mathrm{e}^{-(z/k)}\right\} \tag{1.2}$$

and A is the Glaisher–Kinkelin constant defined by

$$\log A = \lim_{N\to\infty}\left\{\sum_{k=1}^{N} k\log k - \left(\frac{N^2}{2} + \frac{N}{2} + \frac{1}{12}\right)\log N + \frac{N^2}{4}\right\}, \tag{1.3}$$

the numerical value of A being $1.282427130\ldots$.

From this definition, Barnes [3] deduced several explicit Weierstrass canonical product forms of the double Gamma function Γ_2, one of which is recalled here in the form

$$\{\Gamma_2(z+1)\}^{-1} = G(z+1) = (2\pi)^{z/2}\mathrm{e}^{-(1/2)[(1+\gamma)z^2+z]}\prod_{k=1}^{\infty}\left\{\left(1+\frac{z}{k}\right)^k \mathrm{e}^{-z+z^2/2k}\right\}, \tag{1.4}$$

where γ denotes the Euler–Mascheroni constant given by

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) \cong 0.577\,215\,664\,901\,532\,5\ldots\ . \tag{1.5}$$

Voros [29] (see also Vardi [27]) showed for the Glaisher–Kinkelin constant A that

$$\log A = -\zeta'(-1) + \tfrac{1}{12}, \tag{1.6}$$

in terms of the Riemann Zeta function $\zeta(s)$ defined by

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} = \frac{1}{1-2^{-s}} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^s} \quad (\Re(s) > 1). \tag{1.7}$$

Indeed the Zeta function $\zeta(s)$ satisfies the functional equation (see [30, p. 269]):

$$\zeta(s) = 2^s \pi^{s-1} \Gamma(1-s)\zeta(1-s) \sin\frac{\pi s}{2} \tag{1.8}$$

and takes on the following special or limiting values (see [30, p. 271]):

$$\zeta(-1) = -\tfrac{1}{12}, \quad \zeta(0) = -\tfrac{1}{2}, \quad \zeta'(0) = -\tfrac{1}{2}\log(2\pi) \tag{1.9}$$

and

$$\lim_{s\to 1}\left(\zeta(s) - \frac{1}{s-1}\right) = \gamma. \tag{1.10}$$

The generalized (or Hurwitz) Zeta function $\zeta(s,a)$ is defined by

$$\zeta(s,a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^s} \quad (\Re(s) > 1;\ a \neq 0, -1, -2, \ldots), \tag{1.11}$$

which, just as $\zeta(s)$, can be continued meromorphically everywhere in the complex s-plane except for a simple pole (with residue 1). It is not difficult to see from definitions (1.7) and (1.11) that

$$\zeta(s, m+1) = \zeta(s) - \sum_{k=0}^{m-1} \frac{1}{(k+1)^s} \quad (m \in \mathbb{N} := \{1, 2, 3, \ldots\}) \tag{1.12}$$

and

$$\zeta(s, 1) = \zeta(s) = (2^s - 1)^{-1}\, \zeta(s, \tfrac{1}{2}). \tag{1.13}$$

There exists a relationship between the generalized Zeta function $\zeta(s,a)$ and the Bernoulli polynomials $B_n(a)$ (see [2, pp. 264–266]):

$$\zeta(-n, a) = -\frac{B_{n+1}(a)}{n+1} \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}), \tag{1.14}$$

which, for $a = 1$, yields

$$\zeta(0) = B_1 \quad \text{and} \quad \zeta(-n) = -\frac{B_{n+1}}{n+1} \quad (n \in \mathbb{N}), \tag{1.15}$$

where B_n denotes the Bernoulli numbers given by

$$B_n := B_n(0) = (-1)^n B_n(1) \quad (n \in \mathbb{N}_0)$$

or, more conveniently, by

$$B_n = B_n(1) \quad (n \in \mathbb{N}_0 \setminus \{1\}),$$

since

$$B_{2n+1} = 0 \quad (n \in \mathbb{N}).$$

The Digamma (or Psi) function $\psi(z)$ defined by

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \log\Gamma(z) = \int_1^z \psi(t)\,\mathrm{d}t \tag{1.16}$$

is meromorphic in the complex z-plane with simple poles at $z = 0, -1, -2, \ldots$ (with residue -1). We recall here some known identities involving $\psi(z)$ (see [14, pp. 31–40]):

$$\psi(n) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k} \quad (n \in \mathbb{N}) \tag{1.17}$$

and

$$\psi\left(n + \frac{1}{2}\right) = -\gamma - 2\log 2 + 2\sum_{k=0}^{n-1} \frac{1}{2k+1} \quad (n \in \mathbb{N}), \tag{1.18}$$

it being understood (as usual) that an empty sum is nil.

2. An explicit form for the triple gamma function Γ_3

Vignéras [28, p. 241] introduced the multiple Gamma function Γ_n by means of a recurrence formula, which can be applied here in order to evaluate the following Weierstrass canonical product form of the triple Gamma function Γ_3 explicitly:

$$\begin{aligned}\Gamma_3(1+z) &= G_3(1+z)\\ &= \exp\left[-\frac{1}{6}\left(\gamma + \frac{\pi^2}{6} + \frac{3}{2}\right)z^3 + \frac{1}{4}\left(\gamma + \log(2\pi) + \frac{1}{2}\right)z^2 + \Omega z\right]\\ &\quad \times \prod_{m \in \mathbb{N}_0^2 \times \mathbb{N}} \left\{\left(1 + \frac{z}{L(m)}\right)^{-1} \exp\left[\frac{z}{L(m)} - \frac{1}{2}\left(\frac{z}{L(m)}\right)^2 + \frac{1}{3}\left(\frac{z}{L(m)}\right)^3\right]\right\},\end{aligned} \tag{2.1}$$

where

$$\Omega = \frac{1}{12}\left(\frac{3}{2} - \gamma - 3\log(2\pi) + \frac{\pi^2}{12}\right) + \frac{1}{2}\sum_{n=1}^{\infty} (-1)^n \frac{\zeta(n+2)}{(n+3)(n+4)}$$

and

$$L(m) = m_1 + m_2 + m_3 \quad \text{with } m = (m_1, m_2, m_3) \in \mathbb{N}_0^2 \times \mathbb{N}.$$

Now the infinite sum in Ω can be evaluated explicitly by using a known formula [10, p. 116, Eq. (2.63)]:

$$\sum_{k=3}^{\infty} (-1)^k \frac{\zeta(k)}{(k+1)(k+2)} = \frac{1}{2} + \frac{\gamma}{6} - \frac{\pi^2}{72} - 2\log A. \tag{2.2}$$

We thus find that

$$\Omega = \tfrac{3}{8} - \tfrac{1}{4}\log(2\pi) - \log A \tag{2.3}$$

in terms of the Glaisher–Kinkelin constant A defined by (1.3).

Observe that, if

$$L(m) = m_1 + \cdots + m_n \quad \text{with } m = (m_1, \ldots, m_n) \in \mathbb{N}_0^{n-1} \times \mathbb{N},$$

then the number of solutions of

$$L(m) = k \quad (m \in \mathbb{N}_0^{n-1} \times \mathbb{N})$$

is

$$\binom{n+k-2}{n-1} \quad (k \in \mathbb{N}). \tag{2.4}$$

If we set $n = 3$ in (2.4), we observe that $\{\Gamma_3(z)\}^{-1}$ is an entire function with zeros at $z = -k$ $(k \in \mathbb{N}_0)$ whose multiplicity is

$$\tfrac{1}{2}(k^2 + 3k + 2) \quad (k \in \mathbb{N}_0).$$

Furthermore, (2.1) can be written in the following equivalent form analogous to (1.4):

$$\begin{aligned}\Gamma_3(1+z) &= G_3(1+z)\\ &= \exp\left[-\frac{1}{6}\left(\gamma + \frac{\pi^2}{6} + \frac{3}{2}\right)z^3 + \frac{1}{4}\left(\gamma + \log(2\pi) + \frac{1}{2}\right)z^2\right.\\ &\quad \left. + \left(\frac{3}{8} - \frac{1}{4}\log(2\pi) - \log A\right)z\right]\prod_{k=1}^{\infty}\left\{\left(1 + \frac{z}{k}\right)^{-(1/2)k(k+1)}\right.\\ &\quad \left.\times \exp\left[\frac{1}{2}(k+1)z - \frac{1}{4}\left(1 + \frac{1}{k}\right)z^2 + \frac{1}{6k}\left(1 + \frac{1}{k}\right)z^3\right]\right\}.\end{aligned} \tag{2.5}$$

It follows that Γ_3 satisfies several basic properties and characteristics, which are summarized here in

Theorem 2.1. *The triple Gamma function Γ_3 is the unique meromorphic function satisfying each of the following properties*:
(a) $\Gamma_3(1) = 1$;
(b) $\Gamma_3(z+1) = G(z)\Gamma_3(z) \quad (z \in \mathbb{C})$;
(c) *For $x \geqslant 1$, $\Gamma_3(x)$ is infinitely differentiable and*

$$\frac{d^4}{dx^4}\{\log \Gamma_3(x)\} \geqslant 0.$$

3. A set of mathematical constants

In this section, we shall introduce two interesting mathematical constants, in addition to the Glaisher–Kinkelin constant A, by means of the Euler–Maclaurin summation formula (cf. Hardy [17, p. 318]; see also Edwards [15, p. 117]):

$$\sum_{k=1}^{n} f(k) \sim C_0 + \int_a^n f(x)\,dx + \frac{1}{2}f(n) + \sum_{r=1}^{\infty}\frac{B_{2r}}{(2r)!}f^{(2r-1)}(n), \tag{3.1}$$

where C_0 is an arbitrary constant to be determined in each special case and

$$B_0 = 1, \quad B_1 = -\tfrac{1}{2}, \quad B_2 = \tfrac{1}{6}, \quad B_4 = -\tfrac{1}{30}, \quad B_6 = \tfrac{1}{42}, \quad B_8 = -\tfrac{1}{30},$$
$$B_{10} = \tfrac{5}{66}, \ldots, \quad \text{and} \quad B_{2n+1} = 0 \quad (n \in \mathbb{N})$$

are the Bernoulli numbers. Letting $f(x)=x^2 \log x$ and $f(x)=x^3 \log x$ in (3.1) with $a=1$, respectively, we obtain

$$\log B = \lim_{n\to\infty} \left[\sum_{k=1}^{n} k^2 \log k - \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \right) \log n + \frac{n^3}{9} - \frac{n}{12} \right] \tag{3.2}$$

and

$$\log C = \lim_{n\to\infty} \left[\sum_{k=1}^{n} k^3 \log k - \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} - \frac{1}{120} \right) \log n + \frac{n^4}{16} - \frac{n^2}{12} \right], \tag{3.3}$$

where B and C are constants whose approximate numerical values are given by

$$B \cong 1.030\,916\,75\ldots \tag{3.4}$$

and

$$C \cong 0.979\,557\,46\ldots\ . \tag{3.5}$$

The constant B was first considered by Choi and Srivastava [11, p. 102].

Moreover, using the Euler–Maclaurin summation formula (3.1) again, we can obtain a number of analytical representations of $\zeta(s)$, such as (cf. [17, p. 333])

$$\zeta(s) = \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} k^{-s} - \frac{n^{1-s}}{1-s} - \frac{1}{2} n^{-s} \right\} \quad (\Re(s) > -1), \tag{3.6}$$

$$\zeta(s) = \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} k^{-s} - \frac{n^{1-s}}{1-s} - \frac{1}{2} n^{-s} + \frac{1}{12} s n^{-s-1} \right\} \quad (\Re(s) > -3), \tag{3.7}$$

and

$$\zeta(s) = \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} k^{-s} - \frac{n^{1-s}}{1-s} - \frac{1}{2} n^{-s} + \frac{1}{12} s n^{-s-1} - \frac{1}{720} s(s+1)(s+2) n^{-s-3} \right\}$$
$$(\Re(s) > -5). \tag{3.8}$$

Now it is not difficult to express the mathematical constants B and C as

$$\log B = -\zeta'(-2) \tag{3.9}$$

and

$$\log C = -\zeta'(-3) - \tfrac{11}{720}, \tag{3.10}$$

respectively, in terms of special values of the derivative $\zeta'(s)$.

It is clear from (3.3) that $\log C$ must be the finite part of the divergent sum $\sum k^3 \log k$ according to some regularization; hence $\log C$ must be related to $\zeta'(s)$ for some special value of s. By differentiating both sides of (3.8) with respect to s and letting $s = -3$, we obtain

$$-\zeta'(-3) = \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} k^3 \log k - \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} - \frac{1}{120} \right) \log n + \frac{n^4}{16} - \frac{n^2}{12} \right\} + \frac{11}{720}, \tag{3.11}$$

which, when compared with (3.3), yields the desired expression (3.10).

With a view to obtaining the assertion (3.9) in a markedly different manner, we recall a result of Choi and Srivastava [11, p. 111, Eq. (4.24)] in the following *corrected* form:

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{(k+1)2^{2k}} = \frac{1}{2} - \log 2 + 14 \log B. \tag{3.12}$$

Comparing (3.12) with another known result (cf., e.g., [8, p. 191, Eq. (3.19)]):

$$\zeta(3) = \frac{2\pi^2}{7} \left(\log 2 + \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(k+1)2^{2k}} \right), \tag{3.13}$$

we immediately obtain the relationship:

$$\log B = \frac{\zeta(3)}{4\pi^2}, \tag{3.14}$$

which is precisely the same as our assertion (3.9), since (cf., e.g., [26, p. 387, Eq. (1.15)])

$$\zeta(2n+1) = (-1)^n \frac{2(2\pi)^{2n}}{(2n)!} \zeta'(-2n) \quad (n \in \mathbb{N}). \tag{3.15}$$

4. A special value of $\Gamma_3(\frac{1}{2})$

It is known from the work of Cassou-Noguès [7] (see also Barnes [3, p. 288, Section 17]) that

$$G\left(\frac{1}{2}\right) = 2^{1/24} \pi^{-(1/4)} e^{1/8} A^{-(3/2)}, \tag{4.1}$$

which may be compared with the well-known result:

$$\Gamma\left(\frac{1}{2}\right) = \pi^{1/2}. \tag{4.2}$$

We now proceed to express the value of $\Gamma_3(\frac{1}{2})$ in terms of the mathematical constants π, e, A, and B. We begin by recalling the following known asymptotic formulas:

$$\frac{1}{2}\log(2\pi) = \lim_{n\to\infty} \left[\sum_{k=1}^{n} \log k - \left(n + \frac{1}{2} \right) \log n + n \right] \tag{4.3}$$

and

$$\log\left(1 + \frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \mathrm{O}\left(\frac{1}{n^4}\right) \quad (n \to \infty). \tag{4.4}$$

By taking logarithms on both sides of (2.5) and setting $z = \frac{1}{2}$ in the resulting equation, if we make use of (1.5), we obtain

$$\log \Gamma_3\left(1+\frac{1}{2}\right) = \frac{3}{16} - \frac{1}{16}\log(2\pi) - \frac{1}{2}\log A + \lim_{n\to\infty}\left[-\sum_{k=1}^{n}\frac{k(k+1)}{2}\log\left(1+\frac{1}{2k}\right) + \frac{n^2}{8} + \frac{5}{16}n - \frac{1}{24}\log n\right]. \tag{4.5}$$

We first consider the following sum:

$$\begin{aligned} S_n &:= \sum_{k=1}^{n}\frac{k(k+1)}{2}\log\left(1+\frac{1}{2k}\right) \\ &= \frac{1}{2}\sum_{k=1}^{n} k(k+1)\log(2k+1) \\ &\quad - \frac{\log 2}{2}\sum_{k=1}^{n} k(k+1) - \frac{1}{2}\sum_{k=1}^{n} k^2\log k - \frac{1}{2}\sum_{k=1}^{n} k\log k \\ &= \frac{1}{8}\left(\sum_{k=1}^{n}(2k+1)^2\log(2k+1) - \sum_{k=1}^{n}\log(2k+1)\right) \\ &\quad - \frac{\log 2}{2}\sum_{k=1}^{n} k(k+1) - \frac{1}{2}\sum_{k=1}^{n} k^2\log k - \frac{1}{2}\sum_{k=1}^{n} k\log k. \end{aligned}$$

We thus have

$$\begin{aligned} S_n &= \frac{1}{8}\left(\sum_{k=1}^{2n+1} k^2\log k - 4\sum_{k=1}^{n} k^2\log k - 4\log 2\sum_{k=1}^{n} k^2 \right. \\ &\quad \left. - \sum_{k=1}^{2n+1}\log k + \sum_{k=1}^{n}\log k + n\log 2\right) \\ &\quad - \frac{\log 2}{2}\sum_{k=1}^{n} k(k+1) - \frac{1}{2}\sum_{k=1}^{n} k^2\log k - \frac{1}{2}\sum_{k=1}^{n} k\log k, \end{aligned}$$

which immediately leads us to

$$\begin{aligned} S_n &= \frac{1}{8}\sum_{k=1}^{2n+1} k^2\log k - \frac{1}{8}\sum_{k=1}^{2n+1}\log k - \sum_{k=1}^{n} k^2\log k - \frac{1}{2}\sum_{k=1}^{n} k\log k \\ &\quad + \frac{1}{8}\sum_{k=1}^{n}\log k - \left(\frac{n^3}{3} + \frac{3}{4}n^2 + \frac{7}{24}n\right)\log 2. \end{aligned} \tag{4.6}$$

Upon substituting from (4.6) into (4.5), if we apply (1.3), (3.2), and (4.3), we obtain

$$\log \Gamma_3\left(1+\frac{1}{2}\right) = \frac{3}{16} - \frac{1}{16}\log(2\pi) + \frac{7}{8}\log B$$
$$+ \lim_{n\to\infty}\left[-\left(\frac{n^3}{3}+\frac{3}{4}n^2+\frac{7}{24}n-\frac{1}{16}\right)\log\left(1+\frac{1}{2n}\right)\right.$$
$$\left. + \frac{n^2}{6}+\frac{n}{3}-\frac{35}{288}+\frac{1}{16}\log 2\right].$$

We therefore have

$$\log \Gamma_3\left(1+\frac{1}{2}\right) = \frac{3}{16} - \frac{1}{16}\log\pi + \frac{7}{8}\log B$$
$$+ \lim_{n\to\infty}\left[-\frac{n^2}{6}-\frac{n}{3}-\frac{19}{288}+\mathrm{O}\left(\frac{1}{n}\right)+\frac{n^2}{6}+\frac{n}{3}-\frac{35}{288}\right]$$
$$= -\frac{1}{16}\log\pi + \frac{7}{8}\log B,$$

where we have also used (4.4) for the second equality. Thus, we find that

$$\Gamma_3(1+\tfrac{1}{2}) = \pi^{-(1/16)}B^{7/8}, \tag{4.7}$$

which, in view of (4.1) and the assertion (b) of Theorem 2.1, yields

$$\Gamma_3(\tfrac{1}{2}) = 2^{-(1/24)}\pi^{3/16}e^{-(1/8)}A^{3/2}B^{7/8}. \tag{4.8}$$

5. Integral expressions for $\log G(z+a)$ and $\log \Gamma_3(z+a)$

Barnes [3, p. 283] expressed $\log G(z+a)$ as an integral of $\log\Gamma(t+a)$:

$$\int_0^z \log\Gamma(t+a)\,\mathrm{d}t = \frac{1}{2}[\log(2\pi)+1-2a]z - \frac{z^2}{2} + (z+a-1)\log\Gamma(z+a) - \log G(z+a)$$
$$+(1-a)\log\Gamma(a)+\log G(a), \tag{5.1}$$

which, in the special case when $a=1$, reduces at once to Alexeweisky's theorem:

$$\int_0^z \log\Gamma(t+1)\,\mathrm{d}t = \frac{1}{2}[\log(2\pi)-1]z - \frac{z^2}{2} + z\log\Gamma(z+1) - \log G(z+1). \tag{5.2}$$

Barnes's integral formula (5.1) was derived also by Choi et al. [9, p. 385, Eq. (2.4)].

Setting $z=t+a-1$ in (1.2) and (1.4), and taking the logarithmic derivatives of the resulting equations, we obtain

$$\sum_{k=1}^{\infty}\left(\frac{1}{t+a-1+k}-\frac{1}{k}\right) = -\frac{\Gamma'(t+a)}{\Gamma(t+a)} - \gamma \tag{5.3}$$

and

$$\sum_{k=1}^{\infty}\left(\frac{k}{t+a-1+k}-1+\frac{t+a-1}{k}\right)=\frac{G'(t+a)}{G(t+a)}-\frac{1}{2}\log(2\pi)+\frac{1}{2}+(1+\gamma)(t+a-1), \tag{5.4}$$

respectively.

Next we set $z=t+a-1$ in (2.5) and take the logarithmic derivative of the resulting equation. We thus find that

$$\begin{aligned}\frac{\Gamma_3'(t+a)}{\Gamma_3(t+a)}&=\left(\frac{3}{8}-\frac{1}{4}\log(2\pi)-\log A\right)\\&\quad+\frac{1}{2}\left(\gamma+\log(2\pi)+\frac{1}{2}\right)(t+a-1)-\frac{1}{2}\left(\gamma+\frac{\pi^2}{6}+\frac{3}{2}\right)(t+a-1)^2\\&\quad+\frac{t+a-1}{2}\left[\sum_{k=1}^{\infty}\left(\frac{k}{t+a-1+k}-1+\frac{t+a-1}{k}\right)\right.\\&\quad\left.+\sum_{k=1}^{\infty}\left(\frac{1}{t+a-1+k}-\frac{1}{k}\right)+\frac{\pi^2}{6}(t+a-1)\right],\end{aligned} \tag{5.5}$$

which, by virtue of (5.3) and (5.4), becomes

$$\begin{aligned}\frac{\Gamma_3'(t+a)}{\Gamma_3(t+a)}&=\left(\frac{3}{8}-\frac{1}{4}\log(2\pi)-\log A\right)\\&\quad+\left(\frac{1}{2}+\frac{1}{4}\log(2\pi)\right)(t+a-1)-\frac{(t+a-1)^2}{4}\\&\quad+\frac{1}{2}(t+a-1)\frac{G'(t+a)}{G(t+a)}-\frac{1}{2}(t+a-1)\frac{\Gamma'(t+a)}{\Gamma(t+a)}.\end{aligned} \tag{5.6}$$

Integrating both sides of (5.6) with respect to t from $t=0$ to $t=z$ with the aid of Barnes's integral formula (5.1), we obtain

$$\begin{aligned}\int_0^z \log G(t+a)\,\mathrm{d}t&=\left[\frac{1}{2}(a-1)\log(2\pi)-2\log A-\frac{a^2}{2}+a-\frac{1}{4}\right]z\\&\quad+\frac{1}{4}[\log(2\pi)+2-2a]z^2-\frac{1}{6}z^3\\&\quad+(z+a-2)\log G(z+a)-2\log\Gamma_3(z+a)\\&\quad+(2-a)\log G(a)+2\log\Gamma_3(a).\end{aligned} \tag{5.7}$$

In their special cases when $a=1$, if we further set $z=1$ and $z=\frac{1}{2}$, and make use of the aforecited known recurrence relations for $\Gamma(z), G(z)$, and $\Gamma_3(z)$, together with (4.1) and (4.8), we obtain the following definite integral formulas:

$$\int_0^1 \log \Gamma(t+1)\,\mathrm{d}t = \frac{1}{2}\log(2\pi) - 1, \tag{5.8}$$

$$\int_0^{1/2} \log \Gamma(t+1)\,\mathrm{d}t = -\frac{1}{2} - \frac{7}{24}\log 2 + \frac{1}{4}\log \pi + \frac{3}{2}\log A, \tag{5.9}$$

$$\int_0^1 \log G(t+1)\,\mathrm{d}t = \frac{1}{12} + \frac{1}{4}\log(2\pi) - 2\log A \tag{5.10}$$

and

$$\int_0^{1/2} \log G(t+1)\,\mathrm{d}t = \frac{1}{24}(1+\log 2) + \frac{1}{16}\log \pi - \frac{1}{4}\log A - \frac{7}{4}\log B, \tag{5.11}$$

of which (5.11) was derived directly from (1.4) by Choi and Srivastava (cf. [11, p. 105, Eq. (3.8)]). The first term on the right-hand side of (5.11) appears erroneously with a negative sign in the work of Choi and Srivastava [11, p. 105, Eq. (3.8)].

We can also evaluate each of the following integrals by direct use of the triple Gamma function Γ_3 in (2.5):

$$\int_0^1 \log \Gamma_3(t+1)\,\mathrm{d}t = -\frac{1}{24}\log(2\pi) + \frac{3}{2}\log B \tag{5.12}$$

and

$$\int_0^{1/2} \log \Gamma_3(t+1)\,\mathrm{d}t = -\frac{1}{256} - \frac{29}{1920}\log 2 - \frac{1}{48}\log \pi + \frac{1}{16}\log A + \frac{3}{4}\log B + \frac{15}{16}\log C. \tag{5.13}$$

For example, in order to evaluate the integral in (5.13), we take the logarithms on both sides of the equation (2.5) and integrate the resulting equation from $t=0$ to $t=\frac{1}{2}$. We then obtain

$$\int_0^{1/2} \log \Gamma_3(t+1)\,\mathrm{d}t = \frac{37}{768} + \frac{1}{128}\gamma - \frac{1}{48}\log(2\pi) - \frac{1}{8}\log A + \frac{1}{16}\lim_{n\to\infty} S_n, \tag{5.14}$$

where

$$S_n = \sum_{k=1}^{n}\Bigg[-\{(2k+1)^3\log(2k+1) + (2k)^3\log(2k)\} + \{(2k+1)\log(2k+1) + (2k)\log(2k)\}$$
$$+ (16k^3 + 12k^2 + 2k)\log(2k) + (4k^2+5k) + \frac{5}{6} - \frac{1}{8k}\Bigg],$$

which immediately yields

$$S_n = -\sum_{k=1}^{2n+1} k^3\log k + \sum_{k=1}^{2n+1} k\log k + 16\sum_{k=1}^{n} k^3\log k + 12\sum_{k=1}^{n} k^2\log k$$

$$+2\sum_{k=1}^{n} k\log k + 16(\log 2)\sum_{k=1}^{n} k^3 + (4+12\log 2)\sum_{k=1}^{n} k^2$$

$$+(5+2\log 2)\sum_{k=1}^{n} k - \frac{1}{8}\sum_{k=1}^{n}\frac{1}{k} + \frac{5}{6}n.$$

Using (1.3), (1.5), and (3.2), we have

$$S_n = \frac{11}{120}\log 2 - \frac{1}{8}\gamma + 3\log A + 12\log B + 15\log C$$

$$-\left(4n^4 + 12n^3 + 11n^2 + 3n - \frac{11}{120}\right)\log\left(1+\frac{1}{2n}\right)$$

$$+2n^3 + \frac{11}{2}n^2 + \frac{25}{6}n - \frac{13}{48} + \mathrm{O}\left(\frac{1}{n}\right) \quad (n\to\infty),$$

which, by means of the following asymptotic formula [cf. Eq. (4.4)]:

$$\log\left(1+\frac{1}{2n}\right) = \frac{1}{2n} - \frac{1}{8n^2} + \frac{1}{24n^3} - \frac{1}{64n^4} + \mathrm{O}\left(\frac{1}{n^5}\right) \quad (n\to\infty),$$

yields

$$\lim_{n\to\infty} S_n = -\tfrac{5}{6} + \tfrac{11}{120}\log 2 - \tfrac{1}{8}\gamma + 3\log A + 12\log B + 15\log C. \tag{5.15}$$

Finally, by substituting from (5.15) into (5.14), we obtain the desired formula (5.13).

6. Integrals involving the Psi function

In this section, we shall show that integrals of the forms:

$$\int_0^z t^k \psi(t+a)\,\mathrm{d}t \quad (k\in\mathbb{N})$$

can be expressed in terms of multiple Gamma functions.

First of all, integrating by parts with the aid of (5.1), we obtain

$$\int_0^z t\,\psi(t+a)\,\mathrm{d}t = \frac{1}{2}[2a-1-\log(2\pi)]z + \frac{z^2}{2} + (1-a)\log\Gamma(z+a)$$

$$+\log G(z+a) + (a-1)\log\Gamma(a) - \log G(a). \tag{6.1}$$

On the other hand, integrating by parts with the aid of (5.1) and (5.7), we obtain

$$2\int_0^z t\log\Gamma(t+a)\,\mathrm{d}t = \left(\frac{1}{4} - \frac{1}{2}a + \frac{1}{2}a^2 - 2\log A\right)z + \left(\frac{1}{2}\log(2\pi) - \frac{a}{2} + \frac{1}{4}\right)z^2$$

$$-\frac{z^3}{2} + [z^2-(a-1)^2]\log\Gamma(z+a) + (2a-3)\log G(z+a)$$

$$-2\log\Gamma_3(z+a)+(a-1)^2\log\Gamma(a)+(3-2a)\log G(a)$$
$$+2\log\Gamma_3(a). \tag{6.2}$$

Next, integrating by parts with the aid of (6.2), we obtain

$$\int_0^z t^2\psi(t+a)\,\mathrm{d}t=\left(-\frac{1}{4}+\frac{1}{2}a-\frac{1}{2}a^2+2\log A\right)z+\left(-\frac{1}{2}\log(2\pi)+\frac{a}{2}-\frac{1}{4}\right)z^2$$
$$+\frac{z^3}{2}+(a-1)^2\log\Gamma(z+a)+(3-2a)\log G(z+a)$$
$$+2\log\Gamma_3(z+a)-(a-1)^2\log\Gamma(a)+(2a-3)\log G(a)-2\log\Gamma_3(a). \tag{6.3}$$

Furthermore, integrating by parts with the aid of (5.7), we obtain

$$2\int_0^z t\log G(t+a)\,\mathrm{d}t=(2-a)\left(-\frac{1}{4}+\frac{1}{2}(a-1)\log(2\pi)-2\log A-\frac{a^2}{2}+a\right)z$$
$$+\frac{1}{2}\left(\frac{7}{4}+\frac{1}{2}\log(2\pi)-2\log A+\frac{a^2}{2}-2a\right)z^2$$
$$+\frac{1}{6}[\log(2\pi)-a]z^3-\frac{1}{8}z^4+(z^2-a^2+4a-4)\log G(z+a)$$
$$+2(a-2-z)\log\Gamma_3(z+a)+(a-2)^2\log G(a)$$
$$+2(2-a)\log\Gamma_3(a)+2\int_0^z\log\Gamma_3(t+a)\,\mathrm{d}t. \tag{6.4}$$

Finally, integrating by parts with the aid of (6.2) and (6.4), we obtain

$$3\int_0^z t^2\log\Gamma(t+a)\,\mathrm{d}t=\left[-a^2+\frac{3}{2}a-\frac{1}{4}+2(2a-3)\log A-\frac{1}{2}(a^2-3a+2)\log(2\pi)\right]z$$
$$+\left[\frac{9}{8}-\frac{7}{4}a+\frac{3}{4}a^2+\frac{1}{4}(3-2a)\log(2\pi)-\log A\right]z^2$$
$$+\frac{1}{3}[\log(2\pi)-1]z^3-\frac{3}{8}z^4+\{z^3+(a-1)^3\}\log\Gamma(z+a)$$
$$-(3a^2-9a+7)\log G(z+a)+2(2a-3-z)\log\Gamma_3(z+a)$$
$$+(1-a)^3\log\Gamma(a)+(3a^2-9a+7)\log G(a)+2(3-2a)\log\Gamma_3(a)$$
$$+2\int_0^z\log\Gamma_3(t+a)\,\mathrm{d}t \tag{6.5}$$

and

$$\int_0^z t^3\psi(t+a)\,\mathrm{d}t=\left[a^2-\frac{3}{2}a+\frac{1}{4}+2(3-2a)\log A+\frac{1}{2}(a^2-3a+2)\log(2\pi)\right]z$$

$$+\left[-\frac{3}{4}a^2+\frac{7}{4}a-\frac{9}{8}+\frac{1}{4}(2a-3)\log(2\pi)+\log A\right]z^2+\frac{1}{3}[1-\log(2\pi)]z^3$$
$$+\frac{3}{8}z^4+(1-a)^3\log\Gamma(z+a)+(3a^2-9a+7)\log G(z+a)$$
$$+2(z-2a+3)\log\Gamma_3(z+a)+(a-1)^3\log\Gamma(a)-(3a^2-9a+7)\log G(a)$$
$$+2(2a-3)\log\Gamma_3(a)-2\int_0^z\log\Gamma_3(t+a)\,\mathrm{d}t. \tag{6.6}$$

7. Series involving the Zeta functions

This subject has a long history and many techniques to evaluate various series involving the Zeta functions have been developed (see, for details, [25]). We show how beautifully the theory of multiple Gamma functions can be applied to evaluate certain classes of series associated with the Zeta functions. Many of our closed-form evaluations of series involving the Zeta function will be applied in Section 8 in order to compute the determinant of the Laplacian on the four-dimensional unit sphere $\mathbf{S}^4$ explicitly.

We begin by recalling the known result (cf., e.g., [25, p. 18]):

$$\sum_{n=2}^{\infty}(-1)^n\zeta(n,a)\frac{t^n}{n}=\log\Gamma(a+t)-\log\Gamma(a)-t\,\psi(a)\quad(|t|<|a|), \tag{7.1}$$

which readily yields

$$\sum_{n=2}^{\infty}\zeta(n,a)\frac{t^n}{n}=\log\Gamma(a-t)-\log\Gamma(a)+t\,\psi(a)\quad(|t|<|a|), \tag{7.2}$$

$$\sum_{n=1}^{\infty}\zeta(2n,a)\frac{t^{2n}}{n}=\log\Gamma(a+t)+\log\Gamma(a-t)-2\log\Gamma(a)\quad(|t|<|a|) \tag{7.3}$$

and

$$\sum_{n=2}^{\infty}\zeta(2n-1,a)\frac{t^{2n-1}}{2n-1}=\tfrac{1}{2}[\log\Gamma(a-t)-\log\Gamma(a+t)]+t\,\psi(a)\quad(|t|<|a|). \tag{7.4}$$

Differentiating both sides of (7.4) with respect to t and multiplying the resulting equation by t, we have

$$\sum_{n=2}^{\infty}\zeta(2n-1,a)t^{2n-1}=-\tfrac{1}{2}[t\,\psi(a-t)+t\,\psi(a+t)]+t\,\psi(a)\quad(|t|<|a|). \tag{7.5}$$

Integrating both sides of (7.5) with respect to t from $t=0$ to $t=z$, we obtain

$$\sum_{n=2}^{\infty}\zeta(2n-1,a)\frac{z^{2n}}{n}=-\int_0^z t\,\psi(a+t)\,\mathrm{d}t-\int_0^{-z}t\,\psi(a+t)\,\mathrm{d}t+\psi(a)z^2\quad(|z|<|a|). \tag{7.6}$$

In view of (6.1), we find from (7.6) that

$$\sum_{n=2}^{\infty}\zeta(2n-1,a)\frac{z^{2n}}{n}=[\psi(a)-1]z^2+(a-1)\log[\Gamma(a+z)\Gamma(a-z)]$$
$$-\log[G(a+z)G(a-z)]+2(1-a)\log\Gamma(a)+2\log G(a)\quad(|z|<|a|). \tag{7.7}$$

We now differentiate both sides of (7.4) with respect to t and multiply the resulting equation by t^3. Upon integrating this new equation with respect to t from $t=0$ to $t=z$, we obtain

$$\sum_{n=3}^{\infty}\zeta(2n-3,a)\frac{z^{2n}}{n}=-\int_0^z t^3\psi(a+t)\,\mathrm{d}t-\int_0^{-z} t^3\psi(a+t)\,\mathrm{d}t+\frac{\psi(a)}{2}z^4\quad(|z|<|a|), \tag{7.8}$$

which, by virtue of (6.6), yields

$$\sum_{n=3}^{\infty}\zeta(2n-3,a)\frac{z^{2n}}{n}=\left[\frac{3}{2}a^2-\frac{7}{2}a+\frac{9}{4}+\left(\frac{3}{2}-a\right)\log(2\pi)-2\log A\right]z^2$$
$$+\frac{1}{4}[2\psi(a)-3]z^4+(a-1)^3\log[\Gamma(a+z)\Gamma(a-z)]$$
$$-(3a^2-9a+7)\log[G(a+z)G(a-z)]$$
$$-2(z-2a+3)\log\Gamma_3(a+z)+2(z+2a-3)\log\Gamma_3(a-z)$$
$$+2(1-a)^3\log\Gamma(a)+2(3a^2-9a+7)\log G(a)$$
$$+4(3-2a)\log\Gamma_3(a)+2\int_0^z\log\Gamma_3(t+a)\,\mathrm{d}t$$
$$+2\int_0^{-z}\log\Gamma_3(t+a)\,\mathrm{d}t\quad(|z|<|a|). \tag{7.9}$$

Setting $a=2$ in (7.7), and applying (1.12) and (1.17), we obtain

$$\sum_{n=3}^{\infty}[\zeta(2n-1)-1]\frac{z^{2n}}{n}=-\gamma z^2+\frac{1}{2}[1-\zeta(3)]z^4$$
$$+\log[\Gamma(2+z)\Gamma(2-z)]-\log[G(2+z)G(2-z)]\quad(|z|<2). \tag{7.10}$$

Setting $z=\frac{3}{2}$ in (7.10), and making use of (1.12) and (4.1), we obtain

$$\sum_{n=3}^{\infty}\frac{1}{n}\left(\frac{3}{2}\right)^{2n}[\zeta(2n-1)-1]=\frac{73}{32}-\frac{9}{4}\gamma+3\log A-\frac{81}{32}\zeta(3)+\log(2^{-(1/12)}\times 5). \tag{7.11}$$

Setting $a=4$ in (7.7), and applying (1.12), (1.17), (1.8), and (1.16), we obtain

$$\sum_{n=3}^{\infty}\left[\zeta(2n-1)-1-\frac{1}{2^{2n-1}}-\frac{1}{3^{2n-1}}\right]\frac{z^{2n}}{n}=\left(\frac{5}{6}-\gamma\right)z^2+\frac{1}{2}\left[\frac{251}{216}-\zeta(3)\right]z^4$$
$$+3\log[\Gamma(4+z)\Gamma(4-z)]-\log[G(4+z)G(4-z)]+\log(2^{-4}\times 3^{-6})\quad(|z|<4). \tag{7.12}$$

Setting $z=3$ in (7.12), we obtain

$$\sum_{n=3}^{\infty}\frac{3^{2n}}{n}\left[\zeta(2n-1)-1-\frac{1}{2^{2n-1}}-\frac{1}{3^{2n-1}}\right]=\frac{873}{16}-9\gamma-\frac{81}{2}\zeta(3)+\log(3^{-3}\times 5^{2}). \tag{7.13}$$

Setting $a=4$ in (7.9), and applying (1.12) and (1.17), we obtain

$$\begin{aligned}\sum_{n=3}^{\infty}\left(\zeta(2n-3)-1-\frac{1}{2^{2n-3}}-\frac{1}{3^{2n-3}}\right)\frac{z^{2n}}{n}&=\left(\frac{49}{4}-\frac{5}{2}\log(2\pi)-2\log A\right)z^2\\&\quad+\frac{1}{2}\left(\frac{1}{3}-\gamma\right)z^4+27\log[\Gamma(4+z)\Gamma(4-z)]\\&\quad-19\log[G(4+z)G(4-z)]\\&\quad+2(5-z)\log\Gamma_3(4+z)\\&\quad+2(5+z)\log\Gamma_3(4-z)+\log(2^{-16}\times 3^{-54})\\&\quad+2\int_0^z\log\Gamma_3(t+4)\,\mathrm{d}t\\&\quad+2\int_0^{-z}\log\Gamma_3(t+4)\,\mathrm{d}t\quad(|z|<4).\end{aligned} \tag{7.14}$$

Using (5.8), (5.10), and (5.12), we readily obtain

$$\begin{aligned}\int_0^3\log\Gamma_3(t+4)\,\mathrm{d}t&=10\int_0^1\log(t+1)\,\mathrm{d}t+4\int_0^1\log(t+2)\,\mathrm{d}t+\int_0^1\log(t+3)\,\mathrm{d}t\\&\quad+19\int_0^1\log\Gamma(t+1)\,\mathrm{d}t+12\int_0^1\log G(t+1)\,\mathrm{d}t+3\int_0^1\log\Gamma_3(t+1)\,\mathrm{d}t\\&=-33+\frac{259}{8}\log 2+9\log 3+\frac{99}{8}\log\pi-24\log A+\frac{9}{2}\log B\end{aligned} \tag{7.15}$$

and

$$\begin{aligned}\int_0^{-3}\log\Gamma_3(t+4)\,\mathrm{d}t&=-\int_0^1\log\Gamma(t+1)\,\mathrm{d}t-3\int_0^1\log G(t+1)\,\mathrm{d}t-3\int_0^1\log\Gamma_3(t+1)\,\mathrm{d}t\\&=\frac{3}{4}-\frac{9}{8}\log(2\pi)+6\log A-\frac{9}{2}\log B.\end{aligned} \tag{7.16}$$

If we set $z=3$ in (7.14), and make use of (7.15) and (7.16), we obtain

$$\sum_{n=3}^{\infty}\frac{3^{2n}}{n}\left(\zeta(2n-3)-1-\frac{1}{2^{2n-3}}-\frac{1}{3^{2n-3}}\right)=\frac{237}{4}-\frac{81}{2}\gamma-54\log A+\log(2^{12}\times 3^{-27}\times 5^{8}). \tag{7.17}$$

Setting $a=2$ in (7.9), and applying (1.12) and (1.17), we obtain

$$\sum_{n=3}^{\infty}[\zeta(2n-3)-1]\frac{z^{2n}}{n}=\left(\frac{5}{4}-\frac{1}{2}\log(2\pi)-2\log A\right)z^2$$
$$-\tfrac{1}{4}(1+2\gamma)z^4+\log[\Gamma(2+z)\Gamma(2-z)]-\log[G(2+z)G(2-z)]$$
$$+2(1-z)\log\Gamma_3(2+z)+2(1+z)\log\Gamma_3(2-z)$$
$$+2\int_0^z\log\Gamma_3(t+2)\,\mathrm{d}t+2\int_0^{-z}\log\Gamma_3(t+2)\,\mathrm{d}t\quad(|z|<2). \tag{7.18}$$

Making use of (5.8) to (5.13), we find that

$$\int_0^{3/2}\log\Gamma_3(t+2)\,\mathrm{d}t=\int_0^1\log G(t+1)\,\mathrm{d}t+\int_0^1\log\Gamma_3(t+1)\,\mathrm{d}t$$
$$+\int_0^{1/2}\log\Gamma(t+1)\,\mathrm{d}t+2\int_0^{1/2}\log G(t+1)\,\mathrm{d}t+\int_0^{1/2}\log\Gamma_3(t+1)\,\mathrm{d}t$$
$$=-\frac{259}{768}-\frac{29}{1920}\log 2+\frac{9}{16}\log\pi-\frac{15}{16}\log A-\frac{5}{4}\log B+\frac{15}{16}\log C \tag{7.19}$$

and

$$\int_0^{-(3/2)}\log\Gamma_3(t+2)\,\mathrm{d}t=-2\int_0^1\log\Gamma_3(t+1)\,\mathrm{d}t+\int_0^{1/2}\log\Gamma_3(t+1)\,\mathrm{d}t$$
$$+\int_0^1\log G(t+1)\,\mathrm{d}t-\int_0^{1/2}\log G(t+1)\,\mathrm{d}t-\int_0^1\log\Gamma(t+1)\,\mathrm{d}t$$
$$+\int_0^{1/2}\log\Gamma(t+1)\,\mathrm{d}t-\int_0^{-(1/2)}\log(t+1)\,\mathrm{d}t$$
$$=\frac{29}{768}-\frac{29}{1920}\log 2-\frac{3}{16}\log A-\frac{1}{2}\log B+\frac{15}{16}\log C. \tag{7.20}$$

Setting $z=\frac{3}{2}$ in (7.18), and applying (4.1), (4.8), (7.19), and (7.20), we obtain

$$\sum_{n=3}^{\infty}\frac{1}{n}\left(\frac{3}{2}\right)^{2n}[\zeta(2n-3)-1]=-\frac{17}{96}-\frac{81}{32}\gamma+\frac{27}{4}\log A+\frac{15}{4}\log C+\log(2^{-(269/480)}\times 5). \tag{7.21}$$

8. The determinant of the Laplacian on $\mathbf{S}^4$

Choi [9] computed the determinants of the Laplacians on the n-dimensional unit sphere $\mathbf{S}^n$ ($n=1, 2, 3$) by factorizing the analogous Weierstrass canonical product form of a shifted sequence of eigenvalues of the Laplacians on $\mathbf{S}^n$ into multiple Gamma functions. Here we compute the determinant of the Laplacian on $\mathbf{S}^4$ by using the method proposed by Choi and Srivastava [12] for the computation of the determinants of the Laplacians on $\mathbf{S}^n$ ($n = 1, 2, 3$), together with the results given in Section 7.

Let $\{\lambda_n\}$ be a sequence such that

$$0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots, \quad \lambda_n \uparrow \infty \ (n \to \infty); \tag{8.1}$$

henceforth we consider only such nonnegative increasing sequences. Then we can show that

$$Z(s) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s}$$

converges absolutely in the half-plane $\Re(s) > \sigma$ for some real number σ.

Definition 8.1 (cf. Osgood et al. [22])**.** The determinant of the Laplacian Δ on the compact manifold M is defined to be

$$\det{}' \Delta := \prod_{\lambda_j \neq 0} \lambda_j,$$

where $\{\lambda_n\}$ is the sequence (8.1) of eigenvalues of the Laplacian Δ on M. But this is always divergent; so, in order for this expression to make sense, some sort of regularization procedure must be used. It is easily seen that, formally, $\mathrm{e}^{-Z'(0)}$ is the product of nonzero eigenvalues of Δ. This product does not converge, but $Z(s)$ can be continued analytically to a neighborhood of $s = 0$, and we define

$$\det{}' \Delta := \mathrm{e}^{-Z'(0)}$$

to be the *Functional Determinant of the Laplacian* Δ on M.

Definition 8.2. Let

$$\mu := \inf \left\{ \alpha > 0 \,\middle|\, \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{\alpha}} < \infty \right\}.$$

Then we call μ the *order* of the sequence $\{\lambda_k\}$. We also let

$$Z(s,a) := \sum_{k=1}^{\infty} \frac{1}{(\lambda_k + a)^s}$$

and the analogous Weierstrass canonical product:

$$E(\lambda) = \prod_{k=1}^{\infty} \left\{ \left(1 - \frac{\lambda}{\lambda_k}\right) \exp\left(\frac{\lambda}{\lambda_k} + \frac{\lambda^2}{2\lambda_k^2} + \cdots + \frac{\lambda^{[\mu]}}{[\mu]\lambda_k^{[\mu]}} \right) \right\},$$

where $[\mu]$ denotes the integer part of the order μ of the sequence $\{\lambda_n\}$. Let

$$D(\lambda) = \exp(-Z'(0, -\lambda)).$$

Formally, indeed, we have

$$Z'(0, -\lambda) = -\sum_{k=1}^{\infty} \log(\lambda_k - \lambda),$$

which implies that

$$D(\lambda) = \prod_{k=1}^{\infty} (\lambda_k - \lambda).$$

Voros [29] gave the formula:

$$D(\lambda) = \exp(-Z'(0)) \exp\left(-\sum_{m=1}^{[\mu]} \mathrm{FP} Z(m) \frac{\lambda^m}{m} \right) \times \exp\left(-\sum_{m=2}^{[\mu]} C_{-m} \left(1 + \cdots + \frac{1}{m-1} \right) \frac{\lambda^m}{m!} \right) E(\lambda), \tag{8.2}$$

where the *finite part* prescription is applied, as usual, as follows (cf. [27, p. 446]):

$$\mathrm{FP} f(s) = \begin{cases} f(s) & \text{if } s \text{ is not a pole,} \\ \lim_{\varepsilon \to 0} \left(f(s+\varepsilon) - \frac{\text{Residue}}{\varepsilon} \right) & \text{if } s \text{ is a simple pole,} \end{cases}$$

and

$$Z(-m) = (-1)^m m! C_{-m}. \tag{8.3}$$

Now consider the sequence of eigenvalues on the standard Laplacian Δ_n on $\mathbf{S}^n$. It is known from the work of Vardi [27] that the standard Laplacian Δ_n ($n \in \mathbb{N}$) has eigenvalues $\mu_k = k(k+n-1)$ with multiplicity

$$\binom{k+n}{n} - \binom{k+n-2}{n}.$$

Let us consider the sequence $\{\lambda_k\}$ as the spectrum shifted by $((n-1)/2)^2$. Then the shifted sequence $\{\lambda_k\}$ is written in the following simple and tractable form:

$$\lambda_k = \mu_k + \left(\frac{n-1}{2} \right)^2 = \left(k + \frac{n-1}{2} \right)^2 \tag{8.4}$$

with multiplicity

$$\binom{k+n}{n} - \binom{k+n-2}{n} \quad (k \in \mathbb{N}_0).$$

We will exclude the zero mode, *i.e.*, start the sequence at $k = 1$ for later use. Furthermore, with a view to emphasizing n on $\mathbf{S}^n$, we use the notations $Z_n(s)$, $Z_n(s,a)$, $E_n(\lambda)$, and $D_n(\lambda)$ instead of $Z(s)$, $Z(s,a)$, $E(\lambda)$, and $D(\lambda)$, respectively.

We readily observe from (8.2) that

$$D_n\left(\left(\frac{n-1}{2} \right)^2 \right) = \det{}' \Delta_n, \tag{8.5}$$

where $\det' \Delta_n$ are the determinants of the Laplacians on $\mathbf{S}^n$ ($n \in \mathbb{N}$).

Letting $n=4$ in the shifted sequence (8.4) of eigenvalues of Δ_4 on $\mathbf{S}^4$, we obtain a discrete sequence as follows:

$$(k+\tfrac{3}{2})^2 \text{ with multiplicity } \tfrac{1}{6}(k+1)(k+2)(2k+3) \quad (k\in\mathbb{N}). \tag{8.6}$$

We see that the sequence in (8.6) has the order $\mu=2$. Now it follows from (8.2) and (8.5) that

$$\det{}'\Delta_4 = D_4(\tfrac{9}{4}) = \exp(-Z_4'(0) - \tfrac{9}{4}\mathrm{FP}Z_4(1) - \tfrac{81}{32}\mathrm{FP}Z_4(2) - \tfrac{81}{32}C_{-2})E_4(\tfrac{9}{4}), \tag{8.7}$$

where $\det{}'\Delta_4$ denotes the determinant of the Laplacian on $\mathbf{S}^4$.

We can express $Z_4(s)$ for the sequence (8.6) in terms of the Riemann Zeta function as follows:

$$\begin{aligned} Z_4(s) &= \frac{1}{6}\sum_{k=1}^{\infty}\frac{(k+1)(k+2)(2k+3)}{(k+\frac{3}{2})^{2s}} \\ &= \frac{2^{2s}}{6}\sum_{k=1}^{\infty}\frac{(k+1)(k+2)}{(2k+3)^{2s-1}} = \frac{2^{2s}}{6}\sum_{k=2}^{\infty}\frac{k(k+1)}{(2k+1)^{2s-1}} \\ &= \frac{2^{2s}}{24}\left(\sum_{k=2}^{\infty}\frac{1}{(2k+1)^{2s-3}} - \sum_{k=2}^{\infty}\frac{1}{(2k+1)^{2s-1}}\right) \\ &= \frac{2^{2s}}{24}\left(\sum_{k=1}^{\infty}\frac{1}{(2k-1)^{2s-3}} - \sum_{k=1}^{\infty}\frac{1}{(2k-1)^{2s-1}} - \frac{1}{3^{2s-3}} + \frac{1}{3^{2s-1}}\right), \end{aligned} \tag{8.8}$$

which, in view of (1.7), becomes

$$Z_4(s) = \tfrac{1}{3}(2^{2s-3}-1)\zeta(2s-3) - \tfrac{1}{3}(2^{2s-3}-\tfrac{1}{4})\zeta(2s-1) - \tfrac{1}{3}(\tfrac{2}{3})^{2s-3} + \tfrac{1}{8}(\tfrac{2}{3})^{2s}. \tag{8.9}$$

It follows from (1.11) and (8.8) that $Z_4(s)$ has simple poles at $s=1$ and $s=2$ with their residues $-\frac{1}{24}$ and $\frac{1}{6}$, respectively.

Using (1.15) and (8.3), we obtain

$$C_{-2} = \frac{1}{2}Z_4(-2) = -\frac{9{,}801{,}047}{2^{12}\times 3^3\times 5\times 7} \tag{8.10}$$

and

$$Z_4'(0) = \log(2^{-(2869/1440)}\times 3^2) + \frac{1}{12}\zeta'(-1) - \frac{7}{12}\zeta'(-3). \tag{8.11}$$

Now we evaluate $\mathrm{FP}Z_4(1)$ and $\mathrm{FP}Z_4(2)$. Since $Z_4(s)$ has simple poles at $s=1$ and $s=2$, we have to use the second case of the definition of $\mathrm{FP}f(s)$ to compute the finite parts of $Z_4(s)$ for $s=1$ and $s=2$.

Using the expression in (8.9) for $Z_4(s)$ and (1.10), we easily see that

$$\begin{aligned} \mathrm{FP}Z_4(1) &= \lim_{\varepsilon\to 0}\left(Z_4(1+\varepsilon) + \frac{1}{24\varepsilon}\right) \\ &= -\frac{31}{72} - \frac{1}{3}\lim_{\varepsilon\to 0}\left[\left(2^{2\varepsilon-1} - \frac{1}{4}\right)\left\{\zeta(2\varepsilon+1) - \frac{1}{2\varepsilon}\right\} + \frac{2^{2\varepsilon}-1}{4\varepsilon}\right] \\ &= -\frac{31}{72} - \frac{\gamma}{12} - \frac{1}{6}\log 2. \end{aligned} \tag{8.12}$$

Similarly, we have

$$\begin{aligned}\mathrm{FP}Z_4(2) &= \lim_{\varepsilon\to 0}\left(Z_4(2+\varepsilon) - \frac{1}{6\varepsilon}\right)\\ &= -\frac{16}{81} - \frac{7}{12}\zeta(3) + \frac{1}{3}\lim_{\varepsilon\to 0}\left[(2^{2\varepsilon+1}-1)\left\{\zeta(2\varepsilon+1) - \frac{1}{2\varepsilon}\right\} + \frac{2^{2\varepsilon+1}-2}{2\varepsilon}\right]\\ &= -\frac{16}{81} + \frac{\gamma}{3} + \frac{2}{3}\log 2 - \frac{7}{12}\zeta(3). \qquad (8.13)\end{aligned}$$

Since the sequence in (8.6) has the order $\mu = 2$, the analogous Weierstrass canonical product $E_4(\lambda)$ of the sequence in (8.6) is

$$\begin{aligned}E_4(\lambda) &= \prod_{k=1}^{\infty}\left(1 - \frac{\lambda}{(k+(3/2))^2}\right)^{(1/6)(k+1)(k+2)(2k+3)}\\ &\times \exp\left\{\frac{1}{6}(k+1)(k+2)(2k+3)\left(\frac{\lambda}{(k+(3/2))^2} + \frac{\lambda^2}{2(k+(3/2))^4}\right)\right\}. \qquad (8.14)\end{aligned}$$

Upon setting $\lambda = \frac{9}{4}$ in (8.14) and taking the logarithms on both sides of the resulting equation, if we make use of (1.7) and the Maclaurin series of $\log(1+x)$, we obtain

$$\begin{aligned}\log E_4\left(\frac{9}{4}\right) &= -\frac{1}{6}\sum_{k=1}^{\infty}(k+1)(k+2)\left\{\sum_{n=3}^{\infty}\frac{3^{2n}}{n(2k+3)^{2n-1}}\right\}\\ &= -\frac{1}{24}\sum_{n=3}^{\infty}\frac{3^{2n}}{n}\left[\sum_{k=1}^{\infty}\frac{1}{(2k-1)^{2n-3}} - \sum_{k=1}^{\infty}\frac{1}{(2k-1)^{2n-1}} - \frac{1}{3^{2n-3}} + \frac{1}{3^{2n-1}}\right]\\ &= -\frac{1}{24}\sum_{n=3}^{\infty}\frac{1}{n}\left[3^{2n}\zeta(2n-3) - 8\left(\frac{3}{2}\right)^{2n}\zeta(2n-3) - 3^{2n}\zeta(2n-1)\right.\\ &\qquad \left. + 2\left(\frac{3}{2}\right)^{2n}\zeta(2n-1) - 24\right]. \qquad (8.15)\end{aligned}$$

Now let p_1, p_2, p_3, and p_4 denote the sums of the Zeta series occurring in (7.11), (7.13), (7.17), and (7.21), respectively. We then find from (8.15) that

$$\begin{aligned}\log E_4\left(\frac{9}{4}\right) &= -\frac{1}{24}(2p_1 - p_2 + p_3 - 8p_4)\\ &= -\frac{4}{9} + \frac{21}{32}\gamma - \frac{189}{128}\zeta(3)\\ &\quad + \frac{17}{4}\log A + \frac{5}{4}\log C + \log(2^{-(979/1440)} \times 3). \qquad (8.16)\end{aligned}$$

Finally, in view of (1.6) and (3.10), it follows from (8.7) and (8.9) to (8.16) that

$$\det{}' \varDelta_4 = \frac{1}{3}\exp\left(\frac{35,639,301}{2^{17}\times 5\times 7} - \frac{13}{3}\zeta'(-1) - \frac{2}{3}\zeta'(-3)\right), \tag{8.17}$$

which can be written in the following equivalent from:

$$\det{}' \varDelta_4 = \frac{1}{3}A^{13/3}\times C^{2/3}\exp\left(\frac{183,758,875}{2^{17}\times 3^3\times 7}\right). \tag{8.18}$$

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