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Multiple Gamma and related functions

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Abstract

The authors give several new (and potentially useful) relationships between the multiple Gamma functions and other mathematical functions and constants. As by-products of some of these relationships, a classical definite integral due to Euler and other definite integrals are also considered together with closed-form evaluations of some series involving the Riemann and Hurwitz Zeta functions.

Keywords: Multiple Gamma functions; Riemann's ζ-function; Hurwitz Zeta function; Determinants of Laplacians; Clausen integrals; Series involving the Zeta function

1. Introduction and preliminaries

The multiple Gamma functions were defined and studied by Barnes (cf. [7,8]) and others in about 1900. Although these functions did not appear in the tables of the most well-known special functions, yet the double Gamma function was cited in the exercises by Whittaker and Watson [42, p. 264] and recorded also by Gradshteyn and Ryzhik [24, p. 661, Entry 6.441(4); p. 937, Entry 8.333]. Recently, these functions were revived in the study of the

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determinants of the Laplacians on the n-dimensional unit sphere $\mathbf{S}^n$ (see [11,17,18,30,39,41]), and in the evaluations of specific classes of definite integrals and infinite series involving, for example, the Riemann and Hurwitz Zeta functions (see [4,15,16,18,19,27]). The subject of some of these developments can be traced back to an over two-century old theorem of Christian Goldbach (1690–1764), as noted in the work of Srivastava [32, p. 1] who investigated this subject in a systematic and unified manner (see also [35], [36, Chapter 3], and [37]). More recently, Adamchik ([2,3]) presented new integral representations for the G-function, its asymptotic expansions, its various relations with other special functions, as well as a procedure for its efficient numerical computation (see also the works of Choi [12], Choi and Seo [14], and Vardi [39] for several closely related results). The theory of the double Gamma function has indeed found interesting applications in many other recent investigations (see, for details, [36]).

In this paper we aim at presenting several new (and potentially useful) relationships between other mathematical functions and the multiple Gamma functions. As by-products of some of these relationships, we shall derive a classical definite integral due to Euler and evaluate some other related integrals and series involving the Riemann and Hurwitz Zeta functions.

We begin by recalling the Barnes G-function ($1/G = \Gamma_2$ being the so-called double Gamma function) which has several equivalent forms including (for example) the Weierstrass canonical product:

$$\{\Gamma_2(z+1)\}^{-1} = G(z+1) = (2\pi)^{z/2}\exp\left(-\tfrac{1}{2}[(1+\gamma)z^2+z]\right) \cdot \prod_{k=1}^{\infty}\left\{\left(1+\frac{z}{k}\right)^k \exp\left(-z+\frac{z^2}{2k}\right)\right\}, \tag{1.1}$$

where γ denotes the Euler–Mascheroni constant given by

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) \cong 0.577215664901532860606512\cdots. \tag{1.2}$$

For sufficiently large real x and $a \in \mathbb{C}$, we have the Stirling formula for the G-function:

$$\log G(x+a+1) = \frac{x+a}{2}\log(2\pi) - \log A + \frac{1}{12} - \frac{3x^2}{4} - ax + \left(\frac{x^2}{2} - \frac{1}{12} + \frac{a^2}{2} + ax\right)\log x + \mathrm{O}(1/x) \quad (x\to\infty), \tag{1.3}$$

where A is the Glaisher–Kinkelin constant given by (cf. [39,41])

$$\log A = \frac{1}{12} - \zeta'(-1). \tag{1.4}$$

Based on the Binet integral representation, Adamchik [2] derived the complete asymptotic expansion of the Barnes G-function.

The G-function satisfies the following fundamental functional relationships:

$$G(1) = 1 \quad \text{and} \quad G(z+1) = \Gamma(z)G(z) \ \ (z \in \mathbb{C}), \tag{1.5}$$

where Γ denotes the familiar Gamma function. Barnes [7] generalized the functional relationships in (1.5) to the case of the *multiple* Gamma functions which are denoted usually by $G_n(z)$ or $\Gamma_n(z)$. *Throughout this paper*, we choose to follow the notations used (for example) by Vignéras [40] (see also [36]). Thus we have

$$\Gamma_n(z) := \{G_n(z)\}^{(-1)^{n-1}} \quad (n \in \mathbb{N}), \tag{1.6}$$

so that

$$G_1(z) = \Gamma_1(z) = \Gamma(z), \quad G_2(z) = \frac{1}{\Gamma_2(z)} = G(z), \quad G_3(z) = \Gamma_3(z), \tag{1.7}$$

and so on. In terms of the multiple Gamma functions $G_n(z)$ of order n $(n \in \mathbb{N})$, the aforementioned functional relationships of Barnes [7] can be rewritten as follows (cf. [40, p. 239]; see also [36, p. 14, Theorem 1.4]):

$$G_n(1) = 1 \quad \text{and} \quad G_{n+1}(z+1) = G_n(z)G_{n+1}(z) \ \ (z \in \mathbb{C};\ n \in \mathbb{N}) \tag{1.8}$$

or, equivalently,

$$\Gamma_n(1) = 1 \quad \text{and} \quad \Gamma_{n+1}(z+1) = \frac{\Gamma_{n+1}(z)}{\Gamma_n(z)} \ \ (z \in \mathbb{C};\ n \in \mathbb{N}), \tag{1.9}$$

which may be used to define the multiple Gamma functions $G_n(z)$ and $\Gamma_n(z)$ $(n \in \mathbb{N})$.

From (1.8) and (1.2) one can easily derive the Weierstrass canonical product form of the triple Gamma function Γ_3 (see [18]):

$$\begin{aligned}\Gamma_3(1+z) &= G_3(1+z)\\ &= \exp\left[-\frac{z^3}{6}\left(\gamma + \frac{\pi^2}{6} + \frac{3}{2}\right) + \frac{1}{4}\left(\gamma + \log(2\pi) + \frac{1}{2}\right)z^2\right.\\ &\qquad \left. + \left(\frac{3}{8} - \frac{\log(2\pi)}{4} - \log A\right)z\right]\\ &\quad \cdot \prod_{k=1}^{\infty}\left\{\left(1+\frac{z}{k}\right)^{-\frac{1}{2}k(k+1)} \exp\left[\frac{1}{2}(k+1)z - \frac{1}{4}\left(1+\frac{1}{k}\right)z^2\right.\right.\\ &\qquad \left.\left. + \frac{1}{6k}\left(1+\frac{1}{k}\right)z^3\right]\right\}.\end{aligned} \tag{1.10}$$

Another form of the triple Gamma function Γ_3 appeared in the work of Choi [11] who expressed, in terms of the double and triple Gamma functions, the analogous Weierstrass canonical product of the shifted sequence of the eigenvalues of the Laplacian on the unit sphere $\mathbf{S}^3$ with standard metric which was indispensable to evaluate the determinant of the Laplacian on $\mathbf{S}^3$ there.

The Riemann Zeta function $\zeta(s)$ is defined by

$$\zeta(s) := \begin{cases} \sum_{k=1}^{\infty} \frac{1}{k^s} = \frac{1}{1-2^{-s}} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^s} & (\Re(s) > 1), \\ (1-2^{1-s})^{-1} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^s} & (\Re(s) > 0;\ s \neq 1), \end{cases} \tag{1.11}$$

which can, except for a simple pole only at $s = 1$ with its residue 1, be continued analytically to the whole complex s-plane by means of the contour integral representation (cf. [42, p. 266]) or many other integral representations (cf. [21, p. 33]). It satisfies the functional equation (see [38, p. 13])

$$\zeta(s) = 2^s \pi^{s-1} \Gamma(1-s) \zeta(1-s) \sin\left(\frac{\pi s}{2}\right). \tag{1.12}$$

The generalized (or Hurwitz) Zeta function $\zeta(s,a)$ is defined by

$$\zeta(s,a) := \sum_{k=0}^{\infty} \frac{1}{(k+a)^s} \quad (\Re(s) > 1;\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}), \tag{1.13}$$

which can, just as $\zeta(s)$, be continued analytically to the whole complex s-plane except for a simple pole only at $s = 1$. Indeed, from the definitions (1.11) and (1.13), it is easily observed that

$$\zeta(s,1) = \zeta(s) = (2^s - 1)^{-1}\, \zeta\left(s, \frac{1}{2}\right) \quad \text{and} \quad \zeta(s,2) = \zeta(s) - 1, \tag{1.14}$$

$$\zeta(s, ma) = \frac{1}{m^s} \sum_{j=0}^{m-1} \zeta\left(s, a + \frac{j}{m}\right) \quad (m \in \mathbb{N}), \tag{1.15}$$

and

$$\zeta(s) = \frac{1}{m^s - 1} \sum_{j=1}^{m-1} \zeta\left(s, \frac{j}{m}\right) \quad (m \in \mathbb{N} \setminus \{1\}). \tag{1.16}$$

Many elementary identities such as those derivable especially from (1.15) and (1.16) will be required in our present investigation.

2. A family of generalized Glaisher–Kinkelin constants

Following the works of Glaisher [22] and Alexeiewsky [5] on the asymptotic behavior of the product:

$$1^{1^p} \cdot 2^{2^p} \cdot 3^{3^p} \cdots n^{n^p}$$

when $n \to \infty$, Bendersky [9] considered the limit formula:

$$\log A_k = \lim_{n\to\infty} \left(\sum_{m=1}^{n} m^k \log m - p(n,k) \right), \tag{2.1}$$

where

$$p(n,k) = \frac{n^k}{2} \log n + \frac{n^{k+1}}{k+1} \left(\log n - \frac{1}{k+1} \right) + k! \sum_{j=1}^{k} \frac{n^{k-j} B_{j+1}}{(j+1)!(k-j)!} \left(\log n + (1-\delta_{j,k}) \sum_{l=1}^{j} \frac{1}{k-l+1} \right) \tag{2.2}$$

and $\delta_{j,k}$ is the Kronecker delta. For $n = 0$, (2.1) yields

$$\log A_0 = \lim_{n\to\infty} \left[\sum_{m=1}^{n} \log m - \left(\frac{1}{2} + n \right) \log n + n \right]. \tag{2.3}$$

This limit formula is fairly well-known. The finite sum on the right-hand side of (2.3) is equal to $\log n!$, and thus, using the Stirling formula, we immediately obtain

$$\log A_0 = \frac{\log(2\pi)}{2}. \tag{2.4}$$

For $n = 1$, the limit (2.1) defines the Glaisher–Kinkelin constant A [cf. Eq. (1.4) above]:

$$\log A_1 = \log A = \lim_{n\to\infty} \left[\sum_{m=1}^{n} m \log m - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12} \right) \log n + \frac{n^2}{4} \right]. \tag{2.5}$$

Using the functional relation (1.5), we express the finite sum in (2.5) in terms of the Barnes G-function:

$$\sum_{m=1}^{n} m \log m = \log \left(\prod_{m=1}^{n} m^m \right) = n \log n! - \log G(n+1). \tag{2.6}$$

By combining (2.5) and (2.6), and the Stirling formula, we can establish the relationship:

$$\log A = \lim_{n\to\infty}\left[-\log G(n+1) + \left(\frac{n^2}{2} - \frac{1}{12}\right)\log n - \frac{3n^2}{4} + \frac{n}{2}\log(2\pi) + \frac{1}{12}\right]. \tag{2.7}$$

In a similar way, we can demonstrate that the higher-order Glaisher–Kinkelin constant A_n is related to the multiple Barnes function. Upon setting $n = 3$ in (2.1) and observing that

$$\sum_{m=1}^{n} m^2 \log m = n^2 \log n! - (2n-1)\log G(n+1) - 2\log G_3(n+1), \tag{2.8}$$

we find that

$$\log A_2 = \lim_{n\to\infty}\left[-2\log G_3(n+1) - \left(\frac{n^3}{3} - \frac{n^2}{2} + \frac{1}{12}\right)\log n + \left(\frac{11n^3}{18} - \frac{3n^2}{4} - \frac{n}{6} + \frac{1}{12}\right) - \left(\frac{n^2}{2} - \frac{n}{2}\right)\log(2\pi) + (2n-1)\log A\right]. \tag{2.9}$$

The generalized Glaisher–Kinkelin constants A_n appear naturally in the asymptotic expansion of the multiple Barnes function. With a view to circumventing the problem of evaluation of the limits in (2.7) and (2.9), Adamchik [1] found the following closed-form representation for the generalized Glaisher–Kinkelin constant A_n:

$$\log A_n = \frac{B_{n+1}H_n}{n+1} - \zeta'(-n) \quad (n \in \mathbb{N}_0), \tag{2.10}$$

where B_k and H_k are the Bernoulli and harmonic numbers, respectively. The proof is based on the following analytical property of the Hurwitz Zeta function $\zeta(s, a)$:

$$\sum_{m=1}^{n} m^k \log m = \zeta'(-k, n+1) - \zeta'(-k) \tag{2.11}$$

and its known asymptotic expansion at infinity. In fact, in light of the following consequence of the functional equation (1.12) [33, p. 387, Eq. (1.15)]:

$$\zeta(2n+1) = (-1)^n \frac{2\cdot(2\pi)^{2n}}{(2n)!}\zeta'(-2n) \quad (n \in \mathbb{N}), \tag{2.12}$$

it is easily seen from (2.10) that

$$\log A_{2n} = (-1)^{n+1}\frac{(2n)!}{2\cdot(2\pi)^{2n}}\zeta(2n+1) \quad (n \in \mathbb{N}). \tag{2.13}$$

The constants A_1, A_2, and A_3 (denoted commonly by A, B, and C, respectively) were also considered by Voros [41], Vardi [39], Choi and Srivastava (cf. [16,18]; see also [36, Chapter 1]).

3. Special cases of the multiple Barnes function

The Barnes G-function, given by the functional equation in (1.5), is a generalization of the Euler Gamma function. Thus it is not surprising to anticipate that the G-function would have closed-form representations for particular values of its argument. As a matter of fact, the G-function can be expressed in finite terms by means of the generalized Glaisher–Kinkelin constants A_n, as well as other known constants.

Barnes [7, p. 283] evaluated the following integral (see also [36, p. 207, Eq. 3.4(444)]):

$$\int_0^z \log\Gamma(t+a)\,\mathrm{d}t = \frac{1}{2}[\log(2\pi)+1-2a]z - \frac{z^2}{2} + (z+a-1) \cdot \log\Gamma(z+a) - \log G(z+a) + (1-a) \cdot \log\Gamma(a) + \log G(a), \tag{3.1}$$

which, in the *special* case when $a=1$, reduces immediately to Alexeiewsky's theorem (cf. [5] and [36, p. 32, Eq. 1.3 (42)]):

$$\int_0^z \log\Gamma(t+1)\,\mathrm{d}t = \frac{1}{2}[\log(2\pi)-1]z - \frac{z^2}{2} + z\log\Gamma(z+1) - \log G(z+1) \tag{3.2}$$

or, equivalently,

$$\int_0^z \log\Gamma(t)\,\mathrm{d}t = \frac{z(1-z)}{2} + \frac{z}{2}\log(2\pi) - (1-z)\log\Gamma(z) - \log G(z), \tag{3.3}$$

since

$$\int_0^z \log t\,\mathrm{d}t = z\log z - z. \tag{3.4}$$

The integral formula (3.3), evaluated recently by Gosper [23] and Adamchik [1], happens to be a source for many closed-form representations of the Barnes G-function. The simplest non-trivial case $z=\frac{1}{2}$ is due to Barnes [7, p. 288, Section 7]:

$$G\left(\tfrac{1}{2}\right) = 2^{1/24} \cdot \pi^{-1/4} \cdot \mathrm{e}^{1/8} \cdot A^{-3/2}. \tag{3.5}$$

By recalling the following special value of the Gamma function (see [31, p. 1]):

$$\Gamma\left(\tfrac{1}{4}\right) \cong 3.625609908221908\cdots \tag{3.6}$$

and making use of a duplication formula for the G-function (cf. [7, p. 291] for the general case; see also [39]), Choi and Srivastava [16] showed that

$$G\left(\tfrac{1}{4}\right) = \mathrm{e}^{(3/32)-(\mathbf{G}/4\pi)} \cdot A^{-9/8} \cdot \left\{\Gamma\left(\tfrac{1}{4}\right)\right\}^{-3/4} \cong 0.293756\cdots \tag{3.7}$$

or, equivalently, that

$$G\left(\tfrac{3}{4}\right) = 2^{-1/8} \cdot \pi^{-1/4} \cdot \mathrm{e}^{(3/32)+(\mathbf{G}/4\pi)} \cdot A^{-9/8} \cdot \left\{\Gamma\left(\tfrac{1}{4}\right)\right\}^{1/4} \cong 0.848718\cdots, \tag{3.8}$$

where $\mathbf{G}$ is the Catalan constant defined by

$$\mathbf{G} := \frac{1}{2}\int_0^1 \mathbf{K}(k)\,\mathrm{d}k = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} \cong 0.915965594177219015\cdots, \tag{3.9}$$

and $\mathbf{K}$ is the complete elliptic integral of the first kind, given by

$$\mathbf{K}(k) := \int_0^{\pi/2} \frac{\mathrm{d}t}{\sqrt{1-k^2\sin^2 t}}. \tag{3.10}$$

The Catalan constant $\mathbf{G}$ becomes a special case of many other functions (one of which will be considered in the following section) and has, among other things, been used to evaluate integrals, such as (see [24, p. 526, Entry 4.224(2)]):

$$\int_0^{\pi/4} \log(\sin t)\,\mathrm{d}t = -\frac{\pi}{4}\log 2 - \frac{1}{2}\mathbf{G}, \tag{3.11}$$

and to give closed-form evaluations of a certain class of series involving the Zeta function (cf. [16]).

Other particular cases of the Barnes function $G(z)$ when

$$z = \frac{1}{6},\ \frac{1}{3},\ \frac{2}{3},\ \text{and}\ \frac{5}{6} \tag{3.12}$$

were recorded by (for example) Srivastava and Choi [36, p. 269, Problem 1], who also provided references to relevant *earlier* works dealing with these evaluations (see also [2,3]).

The identity (3.5) was generalized by Vardi [39] to the multiple Barnes function. The formula is much too complicated to be reproduced here. Here is one particular case when $n = 3$ and $z = \frac{1}{2}$ (see also [18, p. 95, Eq. (4.8)]):

$$G_3\left(\tfrac{1}{2}\right) = 2^{-1/24} \cdot \pi^{3/16} \cdot \mathrm{e}^{-1/8} \cdot A_1^{3/2} \cdot A_2^{7/8} = \Gamma_3\left(\tfrac{1}{2}\right), \tag{3.13}$$

where A_k denotes the generalized Glaisher–Kinkelin constants given by (2.1) with, of course,

$$A_1 = A \quad \text{and} \quad A_2 = B.$$

Other (known or new) closed-form representations for the triple Barnes function should emerge from the following integral (see [1] *and* Eq. (4.12) below with $a = 1$):

$$\int_0^z \log G(t)\,\mathrm{d}t = -2\log G_3(z+1) + z\log G(z) + \frac{z(z-2)}{4}\log(2\pi) - 2z\log A_1 - \frac{z(2z^2-6z+3)}{12}. \tag{3.14}$$

The integral (3.14) has an interesting application to the constants A_1 and A_2 (or, alternatively, A and B):

$$7\log(A_1\cdot A_2) = \frac{1}{6} - \frac{\log(2^8\cdot\pi^9)}{12} - 4\int_0^{1/2}\log G(t)\,\mathrm{d}t. \tag{3.15}$$

4. Relationships between multiple Gamma and other functions

The *Clausen function* (or the *Clausen integral*) $\mathrm{Cl}_2(t)$ (see Lewin [28, p. 101]; see also Chen and Srivastava [10, p. 184]) defined by

$$\mathrm{Cl}_2(t) := -\int_0^t \log\left[2\sin\left(\tfrac{1}{2}u\right)\right]\mathrm{d}u = \sum_{n=1}^{\infty}\frac{\sin(nt)}{n^2} \tag{4.1}$$

stems actually from the imaginary part of the *Dilogarithm function* $\mathrm{Li}_2(\mathrm{e}^{\mathrm{i}t})$ defined by

$$\mathrm{Li}_2(z) := -\int_0^z \frac{\log(1-u)}{u}\,\mathrm{d}u = \sum_{n=1}^{\infty}\frac{z^n}{n^2} \quad (|z|\leqslant 1). \tag{4.2}$$

Now, by employing integration by parts in the following well-known integral formula (see [7, p. 279]; see also [16, p. 94, Eq. (2.1)]) due originally to Kinkelin:

$$\int_0^t \pi u\cot(\pi u)\,\mathrm{d}u = t\log(2\pi) + \log\left(\frac{G(1-t)}{G(1+t)}\right), \tag{4.3}$$

we have (cf. [16, p. 95, Eq. (2.2)]):

$$\int_0^t \log[\sin(\pi u)]\,\mathrm{d}u = t\log\left(\frac{\sin(\pi t)}{2\pi}\right) + \log\left(\frac{G(1+t)}{G(1-t)}\right). \tag{4.4}$$

If we combine (4.1) and (4.4), we obtain the following relationship between the Clausen function and the G-function:

$$\mathrm{Cl}_2(2\pi t) = \sum_{n=1}^{\infty} \frac{\sin(2\pi n t)}{n^2}$$
$$= -2\pi t \log\left(\frac{\sin(\pi t)}{\pi}\right) - 2\pi \log\left(\frac{G(1+t)}{G(1-t)}\right), \tag{4.5}$$

which, for $t = \frac{1}{4}$, readily yields the Catalan constant:

$$\mathrm{Cl}_2\left(\frac{\pi}{2}\right) = \mathbf{G} \tag{4.6}$$

by just observing, for the third member of (4.5), the following identity derivable from Eqs. (3.7), (3.8), and (1.5):

$$\frac{G\left(\frac{5}{4}\right)}{G\left(\frac{3}{4}\right)} = 2^{1/8} \cdot \pi^{1/4} \cdot \exp\left(-\frac{\mathbf{G}}{2\pi}\right). \tag{4.7}$$

Next we recall a relationship (cf. [20, p. 210]):

$$S_2(\alpha) = \mathrm{Cl}_2(\alpha) - \tfrac{1}{4}\,\mathrm{Cl}_2(2\alpha), \tag{4.8}$$

where $S_\nu(\alpha)$ denotes the classical trigonometric series defined by (cf. [20,34])

$$S_\nu(\alpha) := \sum_{k=0}^{\infty} \frac{\sin(2k+1)\alpha}{(2k+1)^\nu}. \tag{4.9}$$

A combination of (4.5) and (4.8) gives the following relationship between $S_2(\alpha)$ and the G-function:

$$S_2(2\pi t) = \pi t \log[2\cot(\pi t)] + \frac{\pi}{2}\log\left(\frac{G(1+2t)}{G(1-2t)}\right) - 2\pi \log\left(\frac{G(1+t)}{G(1-t)}\right). \tag{4.10}$$

Integrating both sides of the second equality of (4.5) from 0 to t, we find an interesting relationship among a series involving a cosine function, a definite integral, and integrals of the G-function:

$$\frac{1}{4\pi^2}\sum_{n=1}^{\infty} \frac{\cos(2n\pi t) - 1}{n^3} = \int_0^t u \log[\sin(\pi u)]\,\mathrm{d}u - \frac{t^2}{2}\log \pi$$
$$+ \int_0^t \log G(1+u)\,\mathrm{d}u + \int_0^{-t} \log G(1+u)\,\mathrm{d}u. \tag{4.11}$$

Now we recall an integral formula related to the multiple Gamma functions (see [18, p. 96, Eq. (5.7)]) in the form:

$$\int_0^t \log G(u+a)\,\mathrm{d}u = \left[\frac{1}{2}(a-1)\log(2\pi) - 2\log A - \frac{a^2}{2} + a - \frac{1}{4}\right]t + \frac{1}{4}[\log(2\pi)+2-2a]t^2 - \frac{1}{6}t^3 + (t+a-2)\log G(t+a) - 2\log\Gamma_3(t+a) + (2-a)\log G(a) + 2\log\Gamma_3(a), \tag{4.12}$$

which, for $a = 1$, yields [cf. Eq. (3.14) above]

$$\int_0^t \log G(u+1)\,\mathrm{d}u = \left(\frac{1}{4} - 2\log A\right)t + \frac{t^2}{4}\log(2\pi) - \frac{t^3}{6} + (t-1)\log G(t+1) - 2\log\Gamma_3(t+1), \tag{4.13}$$

by virtue of Eqs. (1.5) and (1.8) *with* $n = 2$. If we apply (4.13) in (4.11), we obtain an interesting identity involving a series and an integral through multiple Gamma functions:

$$\frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n\pi t)-1}{n^3} = \frac{1}{\pi^2}\int_0^{\pi t} u\log(\sin u)\,\mathrm{d}u + \frac{t^2}{2}\log 2 + (t-1)\log G(t+1) - (1+t)\log G(1-t) - 2\log[\Gamma_3(1+t)\Gamma_3(1-t)], \tag{4.14}$$

the left-hand summation part of which can, in terms of the higher-order Clausen function $\mathrm{Cl}_n(t)$ be defined, for all $n \in \mathbb{N}\setminus\{1\}$, by (cf. [28, p. 191])

$$\mathrm{Cl}_n(t) := \begin{cases} \sum_{k=1}^{\infty}\frac{\sin(kt)}{k^n} & \text{if } n \text{ is even,} \\ \sum_{k=1}^{\infty}\frac{\cos(kt)}{k^n} & \text{if } n \text{ is odd,} \end{cases} \tag{4.15}$$

be expressed as follows:

$$\sum_{n=1}^{\infty}\frac{\cos(2n\pi t)-1}{n^3} = \mathrm{Cl}_3(2\pi t) - \zeta(3). \tag{4.16}$$

It is known that (cf., e.g., [26, p. 356, Entry (54.5.4)]; see also [10, p. 184, Eq. (2.23)])

$$\frac{1}{4\pi t}\mathrm{Cl}_2(2\pi t) = \frac{1}{2} - \frac{1}{2}\log[2\sin(\pi t)] - \sum_{k=1}^{\infty}\frac{\zeta(2k)}{2k+1}t^{2k}. \tag{4.17}$$

A combination of (4.5) and (4.17) gives another proof of a special case of the following known identity for series involving the Riemann Zeta function (see [16, p. 107, Eq. (4.10)]):

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{2k+1} t^{2k+1} = \frac{1}{2}[1 - \log(2\pi)]t + \frac{1}{2}\log\left(\frac{G(1+t)}{G(1-t)}\right) \quad (|t| < 1). \tag{4.18}$$

The special case of (4.14) when $t = \frac{1}{2}$, with the aid of (1.5), (1.6) *with* $n = 2$, (1.8), (2.13) *with* $n = 1$, (3.2) and (3.10), yields a known integral formula:

$$\int_0^{\pi/2} u \log(\sin u)\, \mathrm{d}u = \frac{7}{16}\zeta(3) - \frac{\pi^2}{8}\log 2, \tag{4.19}$$

which is due to Euler (1772) who proved it through a striking and elaborate scheme as noted by Ayoub [6, p. 1084].

We now give here another short proof of (4.19). Indeed, if we multiply the well-known formula:

$$u \cot u = -2\sum_{k=0}^{\infty} \zeta(2k)\left(\frac{u}{\pi}\right)^{2k} \quad (|u| < \pi) \tag{4.20}$$

by u and integrate the resulting equation with respect to u from 0 to $\pi/2$, we find that (cf. [37, p. 832, Eq. (2.4) *with* $\omega = s - 1 = 2$])

$$\int_0^{\pi/2} u^2 \cot u\, \mathrm{d}u = -2\int_0^{\pi/2} u \log(\sin u)\, \mathrm{d}u = -\frac{\pi^2}{4}\sum_{k=0}^{\infty} \frac{\zeta(2k)}{(k+1)\cdot 2^{2k}}, \tag{4.21}$$

which, when combined with a known result (cf., e.g., [10, p. 191, Eq. (3.19)]), proves (4.19).

From (4.17) and a result of Grosjean [25, p. 334, Eq. (12)] we readily obtain a closed-form evaluation of a class of series involving the Riemann Zeta function:

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{2k+1}\left(\frac{p}{2q}\right)^{2k} = \frac{1}{2} - \frac{1}{2}\log\left[2\sin\left(\frac{p\pi}{2q}\right)\right] - \frac{1}{8pq\pi}\sum_{k=1}^{q-1}\left[\zeta\left(2, \frac{k}{2q}\right) - \zeta\left(2, 1 - \frac{k}{2q}\right)\right]\sin\left(\frac{pk\pi}{q}\right)$$
$$(p, q \in \mathbb{N};\ 1 \leqslant p \leqslant 2q - 1;\ (p, q) = 1), \tag{4.22}$$

which, in the special cases when (respectively) $p = q = 1$, $p = q - 1 = 1$, and $p = q - 2 = 1$, yields

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{(2k+1)\cdot 2^{2k}} = \frac{1}{2} - \frac{1}{2}\log 2, \tag{4.23}$$

which is equivalent to a known result [11, p. 109, Eq. (4.22)];

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{(2k+1)\cdot 4^{2k}} = \frac{1}{2} - \frac{1}{4}\log 2 - \frac{\mathbf{G}}{\pi}, \tag{4.24}$$

which is precisely the same as the known result [16, p. 114, Eq. (5.9)];

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{(2k+1)\cdot 6^{2k}} = \frac{1}{2} + \frac{\pi}{2\sqrt{3}} - \frac{\sqrt{3}}{4\pi}\zeta\left(2, \frac{1}{3}\right), \tag{4.25}$$

which corresponds to the special case of the known result (4.17) above when $t = \frac{1}{6}$ (see also [36, Problem 2 (Chapter 3) and Problem 13 (Chapter 4)]).

In fact, closed-form evaluations of series involving the Zeta function have an over two-century old history as commented in Section 1 and have attracted many mathematicians ever since then (cf. [32]; see also [36, Chapter 3]). Here we also give a general formula for this subject by recalling (see [13, p. 224, Eq. (1.11)])

$$\sum_{k=2}^{\infty} \frac{(-1)^k a^{k+\beta}}{k+\beta}\zeta(k,\alpha) = \int_0^a t^{\beta}[\psi(t+\alpha) - \psi(\alpha)]\,\mathrm{d}t \quad (\beta \geqslant 0), \tag{4.26}$$

where the Digamma function (or the ψ-function) is the logarithmic derivative of the Gamma function Γ, and a closed-form solution to the integral (see [1] and [2]):

$$\begin{aligned}\int_0^a t^n \psi(t)\,\mathrm{d}t &= (-1)^{n-1}\zeta'(-n) + \frac{(-1)^n}{n+1}B_{n+1}H_n \\ &\quad + \sum_{k=0}^{n}(-1)^k \binom{n}{k} a^{n-k}\left(\zeta'(-k,a) - \frac{B_{k+1}(a)H_k}{k+1}\right),\end{aligned} \tag{4.27}$$

where B_n and $B_n(a)$ are the Bernoulli numbers and Bernoulli polynomials (see [21, pp. 35–36]), respectively, and H_n are the harmonic numbers defined by [cf. Eq. (2.10) above]

$$H_n := \sum_{k=1}^{n} \frac{1}{k} \quad (n \in \mathbb{N}). \tag{4.28}$$

Combining (4.26) and (4.28), we obtain the desired result:

$$\begin{aligned}\sum_{k=2}^{\infty} \frac{(-1)^k a^{k+n}}{k+n}\zeta(k) &= \frac{a^n}{n} + \frac{\gamma a^{n+1}}{n+1} + (-1)^{n-1}\zeta'(-n) + \frac{(-1)^n}{n+1}B_{n+1}H_n \\ &\quad + \sum_{k=0}^{n}(-1)^k \binom{n}{k} a^{n-k}\left(\zeta'(-k,a) - \frac{B_{k+1}(a)H_k}{k+1}\right) \\ &\quad (\Re(a) > 0;\ |a| < 1;\ n \in \mathbb{N}),\end{aligned} \tag{4.29}$$

where γ is the Euler–Mascheroni constant given by (1.2).

5. Evaluations of some definite integrals

We begin by combining formulas (4.14) and (4.16) in the form:

$$\frac{1}{4\pi^2}[\mathrm{Cl}_3(2\pi t) - \zeta(3)] = \frac{1}{\pi^2}\mathscr{I}(t) + \frac{t^2}{2}\log 2 + (t-1)\log G(t+1) - (1+t)\log G(1-t) - 2\log\mathscr{A}(t), \tag{5.1}$$

where, for convenience, $\mathscr{I}(t)$ and $\mathscr{A}(t)$ are defined by

$$\mathscr{I}(t) := \int_0^{\pi t} u\log(\sin u)\,\mathrm{d}u \tag{5.2}$$

and

$$\mathscr{A}(t) := \Gamma_3(1+t)\Gamma_3(1-t), \tag{5.3}$$

so that, obviously, Euler's integral (4.19) is precisely $\mathscr{I}(\frac{1}{2})$. Now we evaluate the integral $\mathscr{I}(\frac{1}{4})$. First of all, it follows from (1.7) that

$$\log\mathscr{A}(t) = \frac{1}{2}\left[\gamma + \log(2\pi) + \frac{1}{2}\right]t^2 + \frac{1}{2}\left(\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k+1}t^{2k+2} + \sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{k+1}t^{2k+2}\right). \tag{5.4}$$

Furthermore, the special case of a known result [16, p. 107, Eq. (4.11)] when $a = 1$ gives

$$\sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{k+1}t^{2k+2} = -(\gamma+1)t^2 - \log[G(1+t)G(1-t)] \quad (|t| < 1), \tag{5.5}$$

which is a companion formula of (5.4) for the G-function. We thus find from (1.5), (3.7), and (3.8) that

$$\sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{(k+1)\cdot 4^{2k}} = -4 - \gamma + \log\left[4\cdot\pi^4\cdot A^{36}\cdot\left\{\Gamma\left(\tfrac{1}{4}\right)\right\}^{-8}\right]. \tag{5.6}$$

The identity of Chen and Srivastava [10, p. 191, Eq. (3.20)] is rewritten here in the following form:

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{(k+1)\cdot 4^{2k}} = \frac{1}{2} - \frac{1}{2}\log 2 - \frac{4\mathbf{G}}{\pi} + \frac{35}{4\pi^2}\zeta(3). \tag{5.7}$$

Next, the special case of another known result [16, p. 108, Eq. (4.14)] when $a = 1$ yields

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{k+1}z^{2k+2}=[1-\log(2\pi)]\frac{z^2}{2}+z\log\left[\frac{G(1+z)}{G(1-z)}\right]$$
$$-\int_0^z \log G(t+1)\,\mathrm{d}t-\int_0^{-z}\log G(t+1)\,\mathrm{d}t \quad (|t|<1), \tag{5.8}$$

which, upon setting $z=\frac{1}{4}$ and using (5.7), gives the following interesting identity:

$$\int_0^{1/4}\log G(t+1)\,\mathrm{d}t+\int_0^{-1/4}\log G(t+1)\,\mathrm{d}t$$
$$=\frac{1}{32}\left(\log(2\pi)+\frac{4\mathbf{G}}{\pi}-\frac{35}{2\pi^2}\zeta(3)\right). \tag{5.9}$$

In order to illustrate the usefulness of (5.9), we integrate the special case of [16, p. 107, Eq. (4.10)] when $a=1$ from 0 to $\frac{1}{4}$, and we readily find from (5.9) that

$$\zeta(3)=\frac{4\pi^2}{35}\left(\frac{1}{2}+\frac{2\mathbf{G}}{\pi}-\sum_{k=1}^{\infty}\frac{\zeta(2k)}{(k+1)(2k+1)\cdot 4^{2k}}\right), \tag{5.10}$$

which is an interesting addition to the results recorded by Chen and Srivastava [10] who treated the subject of series representations for $\zeta(3)$ in a rather systematic manner and presented new results and some relevant connections with other functions.

From the definitions (4.15) and (1.11), it is easy to see that (cf. [28, p. 300])

$$\mathrm{Cl}_{2n+1}\left(\frac{\pi}{2}\right)=\frac{1-2^n}{2^{4n+1}}\zeta(2n+1)\quad(n\in\mathbb{N}). \tag{5.11}$$

Finally, upon setting $t=\frac{1}{4}$ in (5.1), and making use of (5.4), (5.6), (5.7), and (5.11) (with $n=1$), we obtain the desired integral evaluation (cf., e.g., [29, p. 539, Entry 2.6.34.2]):

$$\int_0^{\pi/4}u\log(\sin u)\,\mathrm{d}u=\mathscr{I}\left(\tfrac{1}{4}\right)=\frac{35}{128}\zeta(3)-\frac{\pi\mathbf{G}}{8}-\frac{\pi^2}{32}\log 2, \tag{5.12}$$

which, upon employing integration by parts, yields

$$\int_0^{\pi/4}u^2\cot u\,\mathrm{d}u=-\frac{35}{64}\zeta(3)+\frac{\pi\mathbf{G}}{4}+\frac{\pi^2}{32}\log 2. \tag{5.13}$$

We evaluate some more definite integrals and series involving the Zeta function. By applying the known result [20, p. 210, Eq. (13c)]:

$$S_2\left(\tfrac{\pi}{4}\right)=\frac{1}{32}\left[\sqrt{2}\zeta\left(2,\frac{1}{8}\right)-2\left(\sqrt{2}+1\right)\pi^2-16\sqrt{2}\mathbf{G}\right] \tag{5.14}$$

in (4.10) with $t = \frac{1}{8}$, and using the relationship (4.7), we obtain

$$\log\left(\frac{G\left(\frac{9}{8}\right)}{G\left(\frac{7}{8}\right)}\right) = \frac{1}{16}\log\left[2\left(2+\sqrt{2}\right)\pi\right] + \frac{1}{8\pi}\left(2\sqrt{2}-1\right)\mathbf{G} + \frac{1}{64\pi}\left[2\left(\sqrt{2}+1\right)\pi^2 - \sqrt{2}\zeta\left(2,\tfrac{1}{8}\right)\right]. \tag{5.15}$$

If we set $t = \frac{1}{8}$ in (4.3), (4.4), and (4.18), and make use of (5.15), we find that

$$\int_0^{\pi/8} u\cot u\,\mathrm{d}u = \frac{\pi}{16}\log\left[\left(2-\sqrt{2}\right)\pi\right] + \frac{1}{8}\left(1-2\sqrt{2}\right)\mathbf{G} + \frac{1}{64}\left[\sqrt{2}\zeta\left(2,\tfrac{1}{8}\right) - 2\left(\sqrt{2}+1\right)\pi^2\right], \tag{5.16}$$

$$\int_0^{\pi/8} \log(\sin u)\,\mathrm{d}u = -\frac{\pi}{16}\log(4\pi) + \frac{1}{8}\left(2\sqrt{2}-1\right)\mathbf{G} + \frac{1}{64}\left[2\left(\sqrt{2}+1\right)\pi^2 - \sqrt{2}\zeta\left(2,\tfrac{1}{8}\right)\right], \tag{5.17}$$

and

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)}{(2k+1)\cdot 8^{2k}} = \frac{1}{2} - \frac{1}{4}\log\left[\left(2-\sqrt{2}\right)\pi\right] + \frac{1}{2\pi}\left(2\sqrt{2}-1\right)\mathbf{G} + \frac{1}{16\pi}\left[2\left(\sqrt{2}+1\right)\pi^2 - \sqrt{2}\zeta\left(2,\tfrac{1}{8}\right)\right]. \tag{5.18}$$

By setting $t = \frac{1}{6}$ in (4.18) and applying (4.25), we obtain

$$\log\left(\frac{G\left(\frac{7}{6}\right)}{G\left(\frac{5}{6}\right)}\right) = \frac{\sqrt{3}}{18}\pi + \frac{1}{6}\log(2\pi) - \frac{\sqrt{3}}{12\pi}\zeta\left(2,\tfrac{1}{3}\right). \tag{5.19}$$

In view of (5.19) and the relationship (cf. [20, p. 210, Eq. (13e)]):

$$S_2\left(\tfrac{\pi}{6}\right) = \frac{2\mathbf{G}}{3}, \tag{5.20}$$

(4.10) with $t = \frac{1}{12}$ yields

$$\log\left(\frac{G\left(\frac{13}{12}\right)}{G\left(\frac{11}{12}\right)}\right) = -\frac{\mathbf{G}}{3\pi} + \frac{1}{24}\log\left[4\left(2+\sqrt{3}\right)\pi\right] + \frac{\sqrt{3}}{24}\left[\frac{\pi}{3} - \frac{1}{2\pi}\zeta\left(2,\tfrac{1}{3}\right)\right]. \tag{5.21}$$

If we set $t = \frac{1}{12}$ in (4.3), (4.4), and (4.18), and apply (5.21), we see that

$$\int_0^{\pi/12} u\cot u\,\mathrm{d}u = \frac{\mathbf{G}}{3} - \frac{\pi}{24}\log\left(\frac{2+\sqrt{3}}{\pi}\right) + \frac{\sqrt{3}}{24}\left[\frac{1}{2}\zeta\left(2,\tfrac{1}{3}\right) - \frac{\pi^2}{3}\right], \tag{5.22}$$

$$\int_0^{\pi/12} \log(\sin u)\,\mathrm{d}u = -\frac{\mathbf{G}}{3} - \frac{\pi}{24}\log(4\pi) + \frac{\sqrt{3}}{24}\left[\frac{\pi^2}{3} - \frac{1}{2}\zeta\left(2,\tfrac{1}{3}\right)\right], \tag{5.23}$$

and

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{(2k+1)\cdot 12^{2k}} = \frac{1}{2} - \frac{2\mathbf{G}}{\pi} + \frac{1}{4}\log\left(\frac{2+\sqrt{3}}{\pi}\right) + \frac{\sqrt{3}}{4}\left[\frac{\pi}{3} - \frac{1}{2\pi}\zeta\left(2,\tfrac{1}{3}\right)\right]. \tag{5.24}$$

Just as in getting (4.23), (4.24), and (4.25), the special cases of (4.22) when

$$p = q - 3 = 1 \quad \text{and} \quad p = q - 5 = 1$$

can also be seen to yield (5.18) and (5.24), respectively. In view of the rôles played by (5.15) and (5.21) in closed-form evaluations of definite integrals, we choose to record here the following general identity (cf. [36, p. 352, Problem 9]):

$$\log\left(\frac{G\left(1+\frac{p}{2q}\right)}{G\left(1-\frac{p}{2q}\right)}\right) = -\frac{p}{2q}\log\left[\frac{1}{\pi}\sin\left(\frac{p\pi}{q}\right)\right] + \frac{1}{8q^2\pi}\sum_{k=1}^{q-1}\left[\zeta\left(2,1-\frac{k}{2q}\right) - \zeta\left(2,\frac{k}{2q}\right)\right]\sin\left(\frac{pk\pi}{q}\right)$$
$$(p,q \in \mathbb{N};\ 1 \leqslant p \leqslant 2q-1;\ (p,q)=1), \tag{5.25}$$

which follows immediately from (4.18) (with $t = p/(2q)$) and (4.22).

We conclude this paper by remarking that (*easily computable*) closed-form evaluations of $\mathscr{I}(1/2^n)$ $(n \in \mathbb{N} \setminus \{1,2\})$ do not seem to have been carried out in the mathematical literature so far. The following integral formula, which is *complementary* to the identity (5.25), is worthy of note in this connection:

$$\begin{aligned}\mathscr{I}\left(\frac{p}{q}\right) := &\int_0^{p\pi/q} u\log(\sin u)\,\mathrm{d}u \\ &= -\frac{p^2\pi^2}{2q^2}\log 2 + \frac{1}{4}\left(1-\frac{1}{q^3}\right)\zeta(3) - \frac{p\pi}{2q^3}\sum_{k=1}^{q-1}\sin\left(\frac{2pk\pi}{q}\right)\zeta\left(2,\frac{k}{q}\right) \\ &\quad - \frac{1}{4q^3}\sum_{k=1}^{q-1}\cos\left(\frac{2pk\pi}{q}\right)\zeta\left(3,\frac{k}{q}\right) \quad (p,q \in \mathbb{N};\ 1 \leqslant p \leqslant q),\end{aligned} \tag{5.26}$$

which generalizes (4.19) *as well as* (5.12), since

$$\mathbf{G} = \frac{1}{16}\left[\zeta\left(2,\frac{1}{4}\right) - \zeta\left(2,\frac{3}{4}\right)\right]. \tag{5.27}$$

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