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Certain Classes of Series Involving the Zeta Function

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The authors apply the theory of the double gamma function, which was recently revived in the study of determinants of Laplacians, to evaluate some families of series involving the Riemann zeta function. Introducing a (presumably new) mathematical constant in the theory of the double gamma function, they also systematically evaluate a definite integral of the double gamma function and various classes of series associated with zeta functions. Some of these definite integrals are expressed in terms of quotients of double gamma functions.

Key Words: double gamma function, Maclaurin summation formula, Riemann's ζ-function.

1. INTRODUCTION AND PRELIMINARIES

The double gamma function was defined and studied by Barnes ([3], [4], and [5]) and others in about 1900, not appearing in the tables of the most well-known special functions, but cited in the exercise by Whittaker and Watson [27, p. 264] and recorded by Gradshteyn and Ryzhik [13, p. 661, Entry 6.441(4); p. 937, Entry 8.333]. Recently, this function was revived in

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the study of determinants of Laplacians (see [7], [19], [20], [25], and [26]). Shintani [21] also used this function to prove the classical Kronecker limit formula. Its p-adic analytic extension appeared in a formula of Cassou-Noguès [6] for the p-adic L-functions at the point 0. More recently, Choi *et al.* ([9], [10]) used this function to evaluate sums of series involving the Riemann zeta function. Matsumoto [18], on the other hand, proved asymptotic expansions of the Barnes double zeta function and the double gamma function, and showed an application to the Hecke L-functions of real quadratic fields. Before Barnes, these functions had been introduced in a different form by (for example) Hölder [14], Alexeiewsky [2], and Kinkelin [16].

In this paper we aim at proving some definite integral formulas in terms of quotients of double gamma functions whose special cases become known results. We also systematically evaluate various series involving the zeta function by using the theory of the double gamma function and introducing a (presumably new) mathematical constant.

Barnes [3] gave the following explicit Weierstrass canonical product form of the double gamma function $\Gamma_2 = 1/G$:

$$\begin{aligned}\{\Gamma_2(z+1)\}^{-1} &= G(z+1)\\ &= (2\pi)^{z/2} e^{-(1/2)[(1+\gamma)z^2+z]} \prod_{k=1}^{\infty}\left[\left(1+\frac{z}{k}\right)^k e^{-z+z^2/2k}\right],\end{aligned} \tag{1.1}$$

where γ is the Euler-Mascheroni constant defined by

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) \cong 0.577215664\ldots. \tag{1.2}$$

The double gamma function and the gamma function satisfy the following relations:

$$\begin{gathered}\Gamma(1) = G(1) = 1,\\ \Gamma(z+1) = z\Gamma(z) \qquad \text{and} \qquad G(z+1) = \Gamma(z)G(z) \qquad (z \in \mathbf{C}),\end{gathered} \tag{1.3}$$

where Γ is the familiar gamma function whose Weierstrass canonical product form is

$$\{\Gamma(z)\}^{-1} = ze^{\gamma z}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)e^{-(z/k)}. \tag{1.4}$$

For sufficiently large real x and $a \in \mathbf{C}$, we have the Stirling formula for the G-function:

$$\begin{aligned}\log G(x+a+1) &= \frac{x+a}{2}\log(2\pi) - \log A + \frac{1}{12} - \frac{3x^2}{4} - ax\\ &\quad + \left(\frac{x^2}{2} - \frac{1}{12} + \frac{a^2}{2} + ax\right)\log x + O(1/x) \qquad (x \to \infty),\end{aligned} \tag{1.5}$$

where A is Glaisher's (or Kinkelin's) constant defined by

$$\log A = \lim_{n\to\infty} \log(1^1 \cdot 2^2 \cdots n^n) - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\log n + \frac{n^2}{4}, \qquad (1.6)$$

the numerical value of A being 1.282427130.... It is also known that [3]

$$\Gamma\left(\frac{1}{2}\right) = \pi^{1/2} \text{ and } G\left(\frac{1}{2}\right) = 2^{1/24} \cdot \pi^{-1/4} \cdot e^{1/8} \cdot A^{-(3/2)}. \qquad (1.7)$$

The Riemann zeta function $\zeta(s)$ and the generalized (or Hurwitz's) zeta function $\zeta(s, a)$ defined, when $\mathrm{Re}(s) > 1$, by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} \qquad (\mathrm{Re}(s) > 1) \qquad (1.8)$$

and

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \qquad (\mathrm{Re}(s) > 1; \qquad a \neq 0, -1, -2, \ldots), \qquad (1.9)$$

respectively, can indeed be continued meromorphically to the whole complex s-plane with a simple pole at $s = 1$ (with residue 1) (see Titchmarsh [24] and Ivić [15]; also see Whittaker and Watson [27, pp. 265–280]).

The digamma (or psi) function $\psi(z)$ defined by

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \qquad \text{or} \qquad \log\Gamma(z) = \int_1^z \psi(t)\, dt \qquad (1.10)$$

is meromorphic with simple poles at $z = 0, -1, -2, \ldots$ (with residue -1).

In view of the integrals given in Sect. 2, it may be important to know the special values of the G-function. Here we give two special values of the G-function as an illustration. The following special value of the Γ-function is known (see [22, p. 1]):

$$\Gamma(1/4) \cong 3.62560\,99082\,21908 \ldots. \qquad (1.11)$$

The Catalan constant $\mathbf{G}$ is defined by

$$\mathbf{G} = \frac{1}{2}\int_0^1 \mathbf{K}(k)dk = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} \cong 0.915965\ldots, \qquad (1.12)$$

where $\mathbf{K}$ is the complete elliptic integral of the first kind, given by

$$\mathbf{K}(k) := \int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2\sin^2 t}}. \qquad (1.13)$$

The following integral is known (see [13, p. 526, Entry 4.224]):

$$\int_0^{\pi/4} \log\sin t\, dt = -\frac{\pi}{4}\log 2 - \frac{1}{2}\mathbf{G}. \qquad (1.14)$$

Considering (1.3) and (1.14), we obtain

$$G(3/4) = 2^{-(1/8)} \cdot \pi^{-(1/4)} \cdot e^{\mathbf{G}/2\pi} \cdot \Gamma(1/4) G(1/4). \tag{1.15}$$

We also recall a duplication formula for the G-function (cf. [8, p. 290]; see also Barnes [3, p. 291] for the general case):

$$G(a)\left\{G\left(a+\frac{1}{2}\right)\right\}^2 G(a+1) = e^{1/4} \cdot A^{-3} \cdot 2^{-2a^2+3a-11/12} \cdot \pi^{a-1/2} \cdot G(2a). \tag{1.16}$$

Setting $a = 1/4$ in (1.16), and using (1.3) and (1.7), we obtain

$$\Gamma(1/4)\left\{G\left(\frac{1}{4}\right)G\left(\frac{3}{4}\right)\right\}^2 = 2^{-(1/4)} \cdot \pi^{-(1/2)} \cdot e^{3/8} \cdot A^{-(9/2)}. \tag{1.17}$$

Combining (1.15) and (1.17), we obtain

$$G(1/4) = e^{(3/32)-(\mathbf{G}/4\pi)} \cdot A^{-(9/8)}\left\{\Gamma\left(\frac{1}{4}\right)\right\}^{-(3/4)} \cong 0.293756\ldots \tag{1.18}$$

or, equivalently,

$$G(3/4) = 2^{-(1/8)} \cdot \pi^{-(1/4)} \cdot e^{(3/32)+(\mathbf{G}/4\pi)} \cdot A^{-(9/8)}\left\{\Gamma\left(\frac{1}{4}\right)\right\}^{1/4} \cong 0.848718\ldots. \tag{1.19}$$

Finally, making use of (1.3), (1.18), and (1.19), we get

$$\frac{G(5/4)}{G(3/4)} = 2^{1/8} \cdot \pi^{1/4} \cdot e^{-(\mathbf{G}/2\pi)}. \tag{1.20}$$

2. INTEGRAL FORMULAS INVOLVING THE DOUBLE GAMMA FUNCTION

We begin by recalling the following integral formula:

$$\int_0^z \pi t \cot \pi t \, dt = z \log(2\pi) + \log \frac{G(1-z)}{G(1+z)}, \tag{2.1}$$

which is due originally to Kinkelin [16]. Indeed, in view of (1.1), if we set

$$\Phi(z) := \frac{G(1+z)}{G(1-z)} = (2\pi)^z e^{-z} \prod_{k=1}^{\infty}\left[\left(\frac{1+(z/k)}{1-(z/k)}\right)^k e^{-2z}\right]$$

and differentiate logarithmically with respect to z, applying the partial fraction expansion of $\pi z \cot(\pi z)$ (see [1, p. 189, Eq. (11)]), we shall readily obtain

$$\frac{d}{dz}\log\Phi(z) = \log(2\pi) - \pi z\cot(\pi z) \quad (z \neq 0, \pm 1, \pm 2, \ldots),$$

which, upon integration, yields (2.1), since $\Phi(0) = 1$.

Some simple consequences of Kinkelin's formula (2.1) are worthy of note here (see also Barnes [3, p. 279]). First of all, by using integration by parts in (2.1), we have

$$\int_0^z \log\sin\pi t\, dt = z\log\left(\frac{\sin\pi z}{2\pi}\right) + \log\frac{G(1+z)}{G(1-z)}, \tag{2.2}$$

which, upon setting $t = 1/2 - u$ and replacing z by $1/2 - z$, yields

$$\int_0^z \log\cos\pi t\, dt = \left(z - \frac{1}{2}\right)\log\left(\frac{\cos\pi z}{2\pi}\right) - \frac{1}{2}\log 2 - \log\Gamma\left(\frac{1}{2} - z\right) + \log\frac{G(1/2+z)}{G(1/2-z)}. \tag{2.3}$$

Making use of (2.3), we obtain the following analog of (2.1):

$$\int_0^z \pi t\tan\pi t\, dt = -\frac{1}{2}\log\left(\frac{\cos\pi z}{\pi}\right) - z\log(2\pi) - \log\Gamma\left(\frac{1}{2} - z\right) + \log\frac{G(1/2+z)}{G(1/2-z)}, \tag{2.4}$$

which would follow also from (2.1) by setting $t = 1/2 - u$ and replacing z by $1/2 - z$.

Combining (2.3) and (2.4), we readily have the integral formula:

$$\int_0^z \log\tan\pi t\, dt = z\log\tan\pi z + \frac{1}{2}\log\frac{\cos\pi z}{\pi} + \log\Gamma\left(\frac{1}{2} - z\right) + \log\frac{G(1+z)}{G(1-z)} - \log\frac{G(1/2+z)}{G(1/2-z)}. \tag{2.5}$$

Similarly, by using various trigonometric identities, we can obtain the following integral formulas:

$$\int_0^z \left(\frac{\pi t}{\cos\pi t}\right)^2 dt = \pi z^2\tan\pi z + \log\frac{\cos\pi z}{\pi} + 2z\log(2\pi) + 2\log\Gamma\left(\frac{1}{2} - z\right) - 2\log\frac{G(1/2+z)}{G(1/2-z)}; \tag{2.6}$$

$$\int_0^z \left(\frac{\pi t}{\sin \pi t}\right)^2 dt = -\pi z^2 \cot \pi z + 2z \log(2\pi) + 2\log \frac{G(1-z)}{G(1+z)}, \tag{2.7}$$

$$\int_0^z \pi t \tan \pi t \, dt = \log \frac{G(1-z)}{G(1+z)} - \frac{1}{2} \log \frac{G(1-2z)}{G(1+2z)}, \tag{2.8}$$

which, in view of the known duplication formula (1.16) for the G-function, is the same as (2.4):

$$\int_0^z \frac{\pi t}{\sin \pi t} \, dt = z \log(2\pi) + 4 \log \frac{G(1-z/2)}{G(1+z/2)} - \log \frac{G(1-z)}{G(1+z)}; \tag{2.9}$$

$$\begin{aligned}\int_0^z (\pi t \tan \pi t)^2 \, dt = -\frac{\pi^2 z^3}{3} + \pi z^2 \tan \pi z + \log \frac{\cos \pi z}{\pi} \\ + 2z \log(2\pi) + 2 \log \frac{G(3/2-z)}{G(1/2+z)};\end{aligned} \tag{2.10}$$

$$\begin{aligned}\int_0^z (\pi t \cot^2 \pi t)^2 \, dt = -\frac{\pi^2 z^3}{3} - \pi z^2 \cot \pi z \\ + 2z \log(2\pi) + 2 \log \frac{G(1-z)}{G(1+z)};\end{aligned} \tag{2.11}$$

$$\begin{aligned}&\int_0^z \left(\frac{\pi t}{\sin \pi t}\right)^2 \cos \pi t \, dt \\ &= 2\int_0^{z/2} \left(\frac{\pi t}{\sin \pi t}\right)^2 dt - 2\int_0^{z/2} \left(\frac{\pi t}{\cos \pi t}\right)^2 dt \\ &= -\frac{\pi z^2}{\sin \pi z} - 2 \log \frac{\cos(\pi z/2)}{\pi} \\ &\quad + 4 \log \frac{G(1-z/2)}{G(1+z/2)} + 4 \log \frac{G(1/2+z/2)}{G(3/2-z/2)};\end{aligned} \tag{2.12}$$

$$\begin{aligned}\int_0^z \frac{\pi t}{\sin \pi t \cos \pi t} \, dt = -\frac{1}{2} \log \frac{\cos \pi z}{\pi} + \log \frac{G(1-z)}{G(1+z)} \\ + \log \frac{G(1/2+z)}{G(3/2-z)}.\end{aligned} \tag{2.13}$$

Replacing t by at/π in (2.2), we obtain

$$\int_0^z \log \sin at \, dt = z \log \frac{\sin az}{2\pi} + \frac{\pi}{a} \log \frac{G(1+(a/\pi)z)}{G(1-(a/\pi)z)}, \tag{2.14}$$

which, in view of the trigonometric identity:

$$\cos bt - \cos at = 2\sin\left(\frac{1}{2}(a+b)t\right)\sin\left(\frac{1}{2}(a-b)t\right)$$

yields

$$\begin{aligned}\int_0^z \log(\cos bt - \cos at)\,dt &= z\log\left(\frac{\sin[(1/2)(a+b)z]\sin[(1/2)(a-b)z]}{2\pi^2}\right)\\ &\quad + \frac{2\pi}{a+b}\log\frac{G(1+(a+b/2\pi)z)}{G(1-(a+b/2\pi)z)}\\ &\quad + \frac{2\pi}{a-b}\log\frac{G(1+(a-b/2\pi)z)}{G(1-(a-b/2\pi)z)}. \qquad (2.15)\end{aligned}$$

By differentiating Alexeiewsky's theorem [3, p. 281]:

$$\begin{aligned}\int_0^z \log\Gamma(t+1)\,dt &= \frac{1}{2}[\log(2\pi)-1]z - \frac{z^2}{2}\\ &\quad + z\log\Gamma(z+1) - \log G(z+1) \qquad (2.16)\end{aligned}$$

with respect to z, we obtain

$$\frac{d}{dz}\log G(z) = \frac{1}{2}[\log(2\pi)+1] - z + (z-1)\psi(z), \qquad (2.17)$$

and so we have

$$\begin{aligned}&\frac{\partial}{\partial a}\log G(h(az))\\ &\quad= \left\{\frac{1}{2}[\log(2\pi)+1] - h(az) + [h(az)-1]\psi(h(az))\right\}\frac{\partial}{\partial a}h(az), \qquad (2.18)\end{aligned}$$

where $h(az)$ is a function of z and a is a constant.

Differentiating each side of (2.15) with respect to a, using (2.18) and the relation:

$$\psi(1+z) - \psi(1-z) = \frac{1}{z} - \pi\cot\pi z,$$

we obtain

$$\begin{aligned}\int_0^z \frac{t\sin at}{\cos bt - \cos at}\,dt &= \frac{2az}{a^2-b^2}\log(2\pi)\\ &\quad + \frac{2\pi}{(a+b)^2}\log\frac{G(1-(a+b/2\pi)z)}{G(1+(a+b/2\pi)z)}\\ &\quad + \frac{2\pi}{(a-b)^2}\log\frac{G(1-(a-b/2\pi)z)}{G(1+(a-b/2\pi)z)}. \qquad (2.19)\end{aligned}$$

Replacing t by $(1/\pi)\arctan at$ in (2.5), we obtain

$$\begin{aligned}\int_0^z \frac{\log t}{1+a^2t^2}\,dt &= \left(\frac{\log z}{a}\right)\arctan az + \frac{\pi}{2a}\log\left(\frac{\cos(\arctan az)}{\pi}\right)\\ &\quad + \frac{\pi}{a}\log\frac{G(1+(1/\pi)\arctan az)G(3/2-(1/\pi)\arctan az)}{G(1-(1/\pi)\arctan az)G(1/2+(1/\pi)\arctan az)}. \end{aligned}\tag{2.20}$$

Setting $a=1$ and $z=1$ in (2.20), and using (1.20), we obtain

$$\int_0^1 \frac{\log t}{1+t^2}\,dt = -\mathbf{G}. \tag{2.21}$$

It is easy to verify that

$$\int_0^z \frac{a}{1+a^2+2a\cos t}\,dt = \frac{2a}{1-a^2}\arctan\left(\frac{1-a}{1+a}\tan\frac{1}{2}z\right);$$

$$\int_0^z \frac{\cos t}{1+a^2+2a\cos t}\,dt = \frac{z}{2a} - \frac{1+a^2}{a(1-a^2)}\arctan\left(\frac{1-a}{1+a}\tan\frac{1}{2}z\right).$$

Using these identities and applying the Leibniz rule (see De Lillo [11, p. 665]) to the following integral and again integrating the resulting equation with respect to a from 1 to a, we obtain

$$\begin{aligned}\int_0^z \log\left(\frac{1+a^2+2a\cos t}{2+2\cos t}\right)dt &= z\log a - 2\int_1^a \arctan\left(\frac{1-t}{1+t}\tan\frac{1}{2}z\right)\frac{dt}{t}\\ &= (2z-\pi)\log a + 2\int_1^a \arctan\left(\frac{t+\cos z}{\sin z}\right)\frac{dt}{t}\end{aligned}$$

or, equivalently, from (2.3) and the following integral:

$$\int_0^z \log(2+2\cos t)\,dt = (2z-2\pi)\log\left(\frac{\cos(z/2)}{\pi}\right) + 4\pi\log\frac{G(1/2+z/2\pi)}{G(3/2-z/2\pi)},$$

we have

$$\begin{aligned}\int_0^z \log(1+a^2+2a\cos t)\,dt &= \pi\log a + (2z-2\pi)\log\left(\frac{a\cos(z/2)}{\pi}\right)\\ &\quad + 4\pi\log\frac{G(1/2+z/2\pi)}{G(3/2-z/2\pi)} + 2\int_1^a \arctan\left(\frac{t+\cos z}{\sin z}\right)\frac{dt}{t}.\end{aligned}$$

Applying integration by parts to the last integral gives us the following equivalent form:

$$\begin{aligned}\int_0^z \log(1+a^2+2a\cos t)\,dt &= \pi\log a + (z-\pi)\log\left(\frac{a^2\cot(z/2)}{2\pi^2}\right)\\ &\quad + 2\arctan\left(\frac{a+\cos z}{\sin z}\right)\log\left(\frac{a}{\sin z}\right)\\ &\quad + 4\pi\log\frac{G(1/2+z/2\pi)}{G(3/2-z/2\pi)}\\ &\quad - 2\int_{\cot(1/2)z}^{(a+\cos z)/\sin z} \frac{\log(t-\cot z)}{1+t^2}\,dt. \end{aligned} \tag{2.22}$$

Setting $z=\pi/2$ in (2.22), and using (1.20), (2.20), and (2.21), we obtain

$$\begin{aligned}\int_0^{\pi/2} \log(1+a^2+2a\cos t)\,dt &= -\pi\log\left(\frac{\cos(\arctan a)}{\pi}\right)\\ &\quad -2\pi\log\frac{G(1+(1/\pi)\arctan a)G(3/2-(1/\pi)\arctan a)}{G(1-(1/\pi)\arctan a)G(1/2+(1/\pi)\arctan a)}. \end{aligned} \tag{2.23}$$

Integrating by parts and using (2.20), we obtain

$$\begin{aligned}\int_0^z \frac{\arctan at}{t}\,dt &= -\frac{\pi}{2}\log\left(\frac{\cos(\arctan az)}{\pi}\right)\\ &\quad -\pi\log\frac{G(1+(1/\pi)\arctan az)G(3/2-(1/\pi)\arctan az)}{G(1-(1/\pi)\arctan az)G(1/2+(1/\pi)\arctan az)}. \end{aligned} \tag{2.24}$$

Replacing t by $(1/\pi)\arcsin at$ in (2.2), we obtain

$$\begin{aligned}\int_0^z \frac{\log t\,dt}{\sqrt{1-a^2t^2}} &= \frac{1}{a}\arcsin az\log\left(\frac{z}{2\pi}\right)\\ &\quad + \frac{\pi}{a}\log\frac{G(1+(1/\pi)\arcsin az)}{G(1-(1/\pi)\arcsin az)}. \end{aligned} \tag{2.25}$$

Using (2.25) and considering the following relation:

$$\int_0^z \frac{\arcsin at}{t}\,dt = \arcsin az\log z - a\int_0^z \frac{\log t\,dt}{\sqrt{1-a^2t^2}},$$

we obtain

$$\int_0^z \frac{\arcsin at}{t}\,dt = \arcsin az \log(2\pi) + \pi \log \frac{G(1-(1/\pi)\arcsin az)}{G(1+(1/\pi)\arcsin az)}. \tag{2.26}$$

Integrating by parts and using (2.20), we obtain

$$\begin{aligned}\int_0^z \frac{(\arctan at)^2\,dt}{t^2} &= -\frac{(\arctan az)^2}{z} \\ &\quad - \pi a \log\left(\frac{\cos(\arctan az)}{\pi}\right) - a\log(1+a^2z^2)\arctan az \\ &\quad + 2\pi a \log \frac{G(1-(1/\pi)\arctan az)G(1/2+(1/\pi)\arctan az)}{G(1+(1/\pi)\arctan az)G(3/2-(1/\pi)\arctan az)} \\ &\quad + a^2\int_0^z \frac{\log(1+a^2t^2)}{1+a^2t^2}\,dt.\end{aligned} \tag{2.27}$$

Numerous special cases of the integral formulas (considered in this section) include the following results:

Setting $z = 1/2$ in (2.12), and using (1.20), we obtain

$$\int_0^{\pi/2} \left(\frac{x}{\sin x}\right)^2 \cos x\,dx = -\frac{\pi^4}{4} + 4\mathbf{G}. \tag{2.28}$$

Setting $z = 1$, $1/2$, and $1/4$ in (2.16), and using (1.3), (1.7), and (1.18), we obtain

$$\int_0^{1/2} \log\Gamma(t+1)\,dt = -\frac{1}{2} - \frac{7}{24}\log 2 + \frac{1}{4}\log\pi + \frac{3}{2}\log A; \tag{2.29}$$

$$\int_0^{1/4} \log\Gamma(t+1)\,dt = -\frac{1}{4} - \frac{3}{8}\log 2 + \frac{1}{8}\log\pi + \frac{9}{8}\log A + \frac{\mathbf{G}}{4\pi}. \tag{2.30}$$

Setting $a = 1$ and $z = 1$ in (2.24), and using (1.20), we obtain

$$\int_0^1 \frac{\arctan x}{x}\,dx = \mathbf{G}. \tag{2.31}$$

Setting $a = 1$ and $z = 1/\sqrt{2}$ in (2.26), and using (1.20), we obtain

$$\int_0^{1/\sqrt{2}} \frac{\arcsin x}{x}\,dx = \frac{\pi}{8}\log 2 + \frac{\mathbf{G}}{2}. \tag{2.32}$$

Setting $z = 1/4$ in (2.1), (2.3), (2.4), (2,5), (2.6), (2.7), (2.10), (2.11), and (2.13), and using (1.20), we obtain

$$\int_0^{\pi/4} x \cot x \, dx = \frac{\pi}{8} \log 2 + \frac{\mathbf{G}}{2}; \tag{2.33}$$

$$\int_0^{\pi/4} \log \cos x \, dx = -\frac{\pi}{4} \log 2 + \frac{\mathbf{G}}{2}; \tag{2.34}$$

$$\int_0^{\pi/4} x \tan x \, dx = -\frac{\pi}{8} \log 2 + \frac{\mathbf{G}}{2}; \tag{2.35}$$

$$\int_0^{\pi/4} \log \tan x \, dx = -\mathbf{G}; \tag{2.36}$$

$$\int_0^{\pi/4} \left(\frac{x}{\cos x} \right)^2 dx = \frac{\pi^2}{16} + \frac{\pi}{4} \log 2 - \mathbf{G}; \tag{2.37}$$

$$\int_0^{\pi/4} \left(\frac{x}{\sin x} \right)^2 dx = -\frac{\pi^2}{16} + \frac{\pi}{4} \log 2 + \mathbf{G}; \tag{2.38}$$

$$\int_0^{\pi/4} x^2 \tan^2 x \, dx = -\frac{\pi^3}{192} + \frac{\pi^2}{16} + \frac{\pi}{4} \log 2 - \mathbf{G}; \tag{2.39}$$

$$\int_0^{\pi/4} x^2 \cot^2 x \, dx = -\frac{\pi^3}{192} - \frac{\pi^2}{16} + \frac{\pi}{4} \log 2 + \mathbf{G}. \tag{2.40}$$

Setting $z = 1/2$ in (2.9), and using (1.20), we obtain

$$\int_0^{\pi/2} \frac{x}{\sin x} \, dx = 2\mathbf{G}. \tag{2.41}$$

3. THE EVALUATION OF AN INTEGRAL INVOLVING $\log G(z)$

We first introduce a mathematical constant B defined by (3.4) below, which can easily be deduced from the Maclaurin summation formula [12, p. 117]:

$$\sum u_x = C + \int u_x \, dx - \frac{1}{2} u_x + \frac{B_1}{2!} \frac{d}{dx} u_x - \frac{B_3}{4!} \frac{d^3}{dx^3} u_x + \frac{B_5}{6!} \frac{d^5}{dx^5} u_x - \frac{B_7}{8!} \frac{d^7}{dx^7} u_x + \frac{B_9}{10!} \frac{d^9}{dx^9} u_x - \cdots, \tag{3.1}$$

where C is an arbitrary constant to be determined in each special case,

$$\sum u_x = u_{x-1} + u_{x-2} + \cdots + u_a,$$

u_a is some fixed term of the series, and

$$B_1 = \frac{1}{6}, \qquad B_3 = \frac{1}{30}, \qquad B_5 = \frac{1}{42}, \qquad B_7 = \frac{1}{30}, \qquad B_9 = \frac{5}{66}, \ \ldots$$

are the Bernoulli numbers.

Letting $u_x = x^2 \log x$ in (3.1), and adding $x^2 \log x$ to both sides of the resulting equation, we obtain

$$\begin{aligned} 1^2 \log 1 + 2^2 \log 2 + \cdots + (x-1)^2 \log(x-1) + x^2 \log x \\ = \log B + \left(\frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{6}\right) \log x - \frac{x^3}{9} + \frac{x}{12} \\ - \frac{1}{360x} + \frac{1}{7560x^3} - \frac{1}{25200x^5} + \frac{1}{33264x^7} - \cdots, \end{aligned} \tag{3.2}$$

where B is a constant. Replacing x by a positive integer n in (3.2) yields

$$\sum_{k=1}^{n} k^2 \log k = \log B + \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) \log n - \frac{n^3}{9} + \frac{n}{12} + \frac{c(n)}{n},$$

where

$$c(n) = -\frac{1}{360} + \frac{1}{7560n^2} - \frac{1}{25200n^4} + \frac{1}{33264n^6} - \cdots.$$

Now it is readily seen that

$$\begin{aligned} |c(n)| &\leq 1 + \frac{1}{n^2} + \frac{1}{n^4} + \frac{1}{n^6} + \cdots \\ &= \frac{1}{1-(1/n^2)} = \frac{n^2}{n^2-1} < 2 \qquad (n \in \mathbf{N} \setminus \{1\}). \end{aligned}$$

We thus have

$$\begin{aligned} \sum_{k=1}^{n} k^2 \log k = \log B + \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) \log n \\ - \frac{n^3}{9} + \frac{n}{12} + O\left(\frac{1}{n}\right) \qquad (n \to \infty), \end{aligned} \tag{3.3}$$

which leads us to

$$\log B = \lim_{n\to\infty}\left[\sum_{k=1}^{n} k^2 \log k - \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) \log n + \frac{n^3}{9} - \frac{n}{12}\right]. \tag{3.4}$$

Letting $n = 1000$ in (3.4), we have an approximate numerical value:

$$B = 1.03092\ldots.$$

It is known that (cf. [3, p. 288])

$$\int_0^1 \log G(t+1)\,dt = \frac{1}{12} + \frac{1}{4}\log(2\pi) - 2\log A. \tag{3.5}$$

Replacing z by t in the logarithmic form of (1.1), and integrating both sides of the resulting equation from $t = 0$ to $t = 1/2$, we obtain

$$\begin{aligned}\int_0^{1/2} &\log G(t+1)\,dt \\ &= \frac{1}{16}\log(2\pi) - \frac{\gamma}{48} - \frac{1}{12} \\ &\quad + \frac{1}{4}\sum_{k=1}^{\infty}\Big[(4k^2+2k)\log(2k+1) \\ &\qquad - (4k^2+2k)\log(2k) - 2k - \frac{1}{2} + \frac{1}{12k}\Big].\end{aligned} \tag{3.6}$$

Consider the sum S_n given by

$$\begin{aligned} S_n &:= \sum_{k=1}^{n}\big[(4k^2+2k)\log(2k+1) - (4k^2+2k)\log(2k)\big] \\ &= \sum_{k=1}^{n}\big[(4k^2+2k)\log(2k+1) - (4k^2+2k)\log(2k)\big] \\ &= \sum_{k=1}^{n}\big[(2k+1)^2\log(2k+1) - (2k+1)\log(2k+1) \\ &\qquad - (2k)^2\log(2k) - 2k\log(2k)\big] \\ &= \left[\sum_{k=1}^{2n+1} k^2\log k - 2\sum_{k=1}^{n}(2k)^2\log(2k) - \sum_{k=1}^{2n+1} k\log k\right] \\ &= \left[\sum_{k=1}^{2n+1} k^2\log k - 8\sum_{k=1}^{n} k^2\log k - \sum_{k=1}^{2n+1} k\log k - 8\log 2\sum_{k=1}^{n} k^2\right],\end{aligned}$$

which, in view of (1.6) and (3.4), yields

$$\begin{aligned} S_n = \Big[&\log B + \left\{\frac{(2n+1)^3}{3} + \frac{(2n+1)^2}{2} + \frac{(2n+1)}{6}\right\}\log(2n+1) \\ &- \frac{(2n+1)^3}{9} + \frac{(2n+1)}{12} - 8\log B - 8\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)\log n \end{aligned}$$

$$
\begin{aligned}
&+\frac{8}{9}n^3-\frac{8}{12}n-\log A-\left\{\frac{(2n+1)^2}{2}+\frac{(2n+1)}{2}+\frac{1}{12}\right\}\log(2n+1)\\
&+\frac{(2n+1)^2}{4}-8\log 2\,\frac{n(n+1)(2n+1)}{6}\Bigg]+O\left(\frac{1}{n}\right)\qquad (n\to\infty)\\
=&-\log A-7\log B+\frac{2}{9}\\
&+\lim_{n\to\infty}\Bigg[\left(\frac{8}{3}n^3+4n^2+\frac{4}{3}n-\frac{1}{12}\right)\log(2n+1)\\
&\qquad-\left(\frac{8}{3}n^3+4n^2+\frac{4}{3}n\right)\log n-\frac{1}{3}n^2-\frac{1}{6}n\\
&\qquad-\left(\frac{8}{3}\log 2\right)n^3-(4\log 2)n^2-\left(\frac{4}{3}\log 2\right)n\Bigg]\\
&+O\left(\frac{1}{n}\right)\qquad (n\to\infty).
\end{aligned}
\tag{3.7}
$$

Now, substituting from (3.7), and using (1.2) and the Maclaurin series of $\log(1+1/2n)$, we evaluate the summation part of (3.6) as follows:

$$
\begin{aligned}
T_n :=&-\log A-7\log B+\frac{2}{9}\\
&+\Bigg[\left(\frac{8}{3}n^3+4n^2+\frac{4}{3}n-\frac{1}{12}\right)\log(2n+1)\\
&\qquad-\left(\frac{8}{3}n^3+4n^2+\frac{4}{3}n\right)\log n-\frac{1}{3}n^2-\frac{1}{6}n\\
&\qquad-\left(\frac{8}{3}\log 2\right)n^3-(4\log 2)n^2-\left(\frac{4}{3}\log 2\right)n\\
&\qquad-2\frac{n(n+1)}{2}-\frac{n}{2}+\frac{1}{12}(\gamma+\log n)\Bigg]+O\left(\frac{1}{n}\right)\qquad (n\to\infty)\\
=&\ \frac{\gamma}{12}-\log A-7\log B-\frac{1}{12}\\
&+\Bigg[\left(\frac{8}{3}n^3+4n^2+\frac{4}{3}n-\frac{1}{12}\right)\log 2\\
&\qquad+\left(\frac{8}{3}n^3+4n^2+\frac{4}{3}n-\frac{1}{12}\right)\log\left(1+\frac{1}{2n}\right)\\
&\qquad-\left(\frac{8}{3}\log 2\right)n^3-\left(4\log 2+\frac{4}{3}\right)n^2-\left(\frac{4}{3}\log 2+\frac{5}{3}\right)n\Bigg]\\
&\quad+O\left(\frac{1}{n}\right)\qquad (n\to\infty)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\gamma}{12} - \log A - 7\log B - \frac{1}{12} \\
&\quad + \left[\left(\frac{8}{3}n^3 + 4n^2 + \frac{4}{3}n - \frac{1}{12} \right) \log 2 + \frac{4}{3}n^2 + \frac{5}{3}n + \frac{5}{18} \right. \\
&\qquad \left. - \left(\frac{8}{3}\log 2 \right) n^3 - \left(4\log 2 + \frac{4}{3} \right) n^2 - \left(\frac{4}{3}\log 2 + \frac{5}{3} \right) n \right] \\
&\qquad + O\left(\frac{1}{n} \right) \qquad (n \to \infty),
\end{aligned}
$$

which, upon taking the limit as $n \to \infty$ and considering (3.6), yields our desired result:

$$
\int_0^{1/2} \log G(t+1)\,dt = \frac{1}{24}(\log 2 - 1) + \frac{1}{16}\log \pi - \frac{1}{4}\log A - \frac{7}{4}\log B \tag{3.8}
$$

in terms of the (presumably new) mathematical constant B defined by (3.4).

4. SERIES INVOLVING THE ZETA FUNCTION

This subject can be traced back to an over two-century old theorem of Christian Goldbach (1690–1764) as noted by Srivastava [23]. Recently, Choi *et al.* ([9], [10]) showed that the theory of the double gamma function is useful in evaluating some series involving the zeta function, for which Srivastava [23] has given a comprehensive unified treatment. In this section, we show that many other classes of series involving the zeta function can be evaluated by using the mathematical constant B introduced in Sect. 3 and by applying essentially the same technique as in Choi *et al.* ([9], [10]). Some of our results here can overlap with earlier works. However, we will provide those results together with our new ones to maintain the system of our evaluation.

We first recall that Choi *et al.* [10, p. 385, Eq. (2.4)] obtained the following integral formula:

$$
\begin{aligned}
\int_0^z \log\Gamma(t+a)\,dt &= \left\{ \frac{1}{2} + \frac{1}{2}\log(2\pi) - a \right\} z - \frac{z^2}{2} \\
&\quad + (z+a-1)\log\Gamma(z+a) - \log G(z+a) \\
&\quad + (1-a)\log\Gamma(a) + \log G(a),
\end{aligned} \tag{4.1}
$$

which, for $a = 1$, reduces to the Alexeiewsky's theorem (2.16). Furthermore, it is well-known that (see Whittaker and Watson [27, p. 276]; Gradshteyn and Ryzhik [13, p. 1074, Entry 9.532]):

$$
\sum_{k=2}^{\infty} (-1)^k \zeta(k,a) \frac{t^k}{k} = \log\Gamma(a+t) - \log\Gamma(a) - t\psi(a) \qquad (|t| < |a|), \tag{4.2}
$$

which readily yields the following consequences:

$$\sum_{k=2}^{\infty} \zeta(k,a)\frac{t^k}{k} = \log\Gamma(a-t) - \log\Gamma(a) + t\psi(a) \qquad (|t| < |a|), \tag{4.3}$$

$$\sum_{k=1}^{\infty} \zeta(2k,a)\frac{t^{2k}}{k} = \log\Gamma(a+t) + \log\Gamma(a-t) - 2\log\Gamma(a) \qquad (|t| < |a|) \tag{4.4}$$

and

$$\sum_{k=1}^{\infty} \zeta(2k+1,a)\frac{t^{2k+1}}{2k+1} = \frac{1}{2}\{\log\Gamma(a-t) - \log\Gamma(a+t)\} + t\psi(a) \qquad (|t| < |a|). \tag{4.5}$$

Integrating by parts and using (4.1), we obtain

$$\begin{aligned}\int_0^z t\psi(a+t)\,dt = {}& \left\{a - \frac{1}{2} - \frac{1}{2}\log(2\pi)\right\}z + \frac{z^2}{2} \\ &+ (1-a)\log\Gamma(z+a) + \log G(z+a) \\ &+ (a-1)\log\Gamma(a) - \log G(a)\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}&\int_0^z t^2\psi(a+t)\,dt \\ &\quad = (a-1)\left\{\frac{1}{2} + \frac{1}{2}\log(2\pi) - a\right\}z + \left\{\frac{1}{4} - \frac{1}{4}\log(2\pi)\right\}z^2 \\ &\qquad + \frac{z^3}{3} + (a-1)^2\log\Gamma(z+a) + (z-a+1)\log G(z+a) \\ &\qquad - (a-1)^2\log\Gamma(a) + (a-1)\log G(a) - \int_0^z \log G(t+a)\,dt.\end{aligned} \tag{4.7}$$

In view of (4.1) to (4.7), it is not difficult to derive the following series involving the Hurwitz zeta function:

$$\begin{aligned}&\sum_{k=2}^{\infty}(-1)^k\zeta(k,a)\frac{z^{k+1}}{k+1} \\ &\quad = \left\{a - \frac{1}{2} - \frac{1}{2}\log(2\pi)\right\}z + \frac{1}{2}\{1-\psi(a)\}z^2 \\ &\qquad + (1-a)\log\Gamma(z+a) + \log G(z+a) \\ &\qquad + (a-1)\log\Gamma(a) - \log G(a) \qquad (|z| < |a|);\end{aligned} \tag{4.8}$$

$$\begin{aligned}
&\sum_{k=2}^{\infty} \zeta(k, a) \frac{z^{k+1}}{k+1} \\
&\quad = \left\{ a - \frac{1}{2} - \frac{1}{2} \log(2\pi) \right\} z + \frac{1}{2} \{\psi(a) - 1\} z^2 \\
&\qquad + (a-1) \log \Gamma(a-z) - \log G(a-z) \\
&\qquad + (1-a) \log \Gamma(a) + \log G(a) \qquad (|z| < |a|); \qquad (4.9)
\end{aligned}$$

$$\begin{aligned}
&\sum_{k=1}^{\infty} \zeta(2k, a) \frac{z^{2k+1}}{2k+1} \\
&\quad = \left\{ a - \frac{1}{2} - \frac{1}{2} \log(2\pi) \right\} z \\
&\qquad + \frac{1-a}{2} \{\log \Gamma(a+z) - \log \Gamma(a-z)\} \\
&\qquad + \frac{1}{2} \{\log G(a+z) - \log G(a-z)\} \qquad (|z| < |a|); \qquad (4.10)
\end{aligned}$$

$$\begin{aligned}
&\sum_{k=1}^{\infty} \zeta(2k+1, a) \frac{z^{2k+2}}{k+1} \\
&\quad = \{\psi(a) - 1\} z^2 + (a-1) \{\log \Gamma(a+z) + \log \Gamma(a-z)\} \\
&\qquad - \log G(a+z) - \log G(a-z) \\
&\qquad + 2(1-a) \log \Gamma(a) + 2 \log G(a) \qquad (|z| < |a|); \qquad (4.11)
\end{aligned}$$

$$\begin{aligned}
\sum_{k=2}^{\infty} (-1)^k \zeta(k, a) \frac{z^{k+2}}{k+2} &= (a-1) \left\{ \frac{1}{2} + \frac{1}{2} \log(2\pi) - a \right\} z \\
&\quad + \left\{ \frac{1}{4} - \frac{1}{4} \log(2\pi) \right\} z^2 \\
&\quad + \frac{1}{3} \{1 - \psi(a)\} z^3 + (a-1)^2 \log \Gamma(z+a) \\
&\quad + (z - a + 1) \log G(z+a) \\
&\quad - (a-1)^2 \log \Gamma(a) + (a-1) \log G(a) \\
&\quad - \int_0^z \log G(t+a)\, dt \qquad (|z| < |a|); \qquad (4.12)
\end{aligned}$$

$$\begin{aligned}\sum_{k=2}^{\infty} \zeta(k, a) \frac{z^{k+2}}{k+2} = {} & (1-a)\left\{\frac{1}{2}+\frac{1}{2} \log (2 \pi)-a\right\} z \\ & +\left\{\frac{1}{4}-\frac{1}{4} \log (2 \pi)\right\} z^2 \\ & +\frac{1}{3}\{\psi(a)-1\} z^3+(a-1)^2 \log \Gamma(a-z) \\ & +(1-a-z) \log G(a-z) \\ & -(a-1)^2 \log \Gamma(a)+(a-1) \log G(a) \\ & -\int_0^{-z} \log G(t+a)\, d t \qquad (|z|<|a|);\end{aligned} \tag{4.13}$$

$$\begin{aligned}\sum_{k=1}^{\infty} & \zeta(2 k, a) \frac{z^{2 k+2}}{k+1} \\ = {} & \left\{\frac{1}{2}-\frac{1}{2} \log (2 \pi)\right\} z^2+(a-1)^2\left\{\log \Gamma(a+z)+\log \Gamma(a-z)\right\} \\ & +(z-a+1) \log G(a+z)+(1-a-z) \log G(a-z) \\ & -2(a-1)^2 \log \Gamma(a)+2(a-1) \log G(a) \\ & -\int_0^z \log G(t+a)\, d t-\int_0^{-z} \log G(t+a)\, d t \qquad (|z|<|a|);\end{aligned} \tag{4.14}$$

$$\begin{aligned}\sum_{k=1}^{\infty} & \zeta(2 k+1, a) \frac{z^{2 k+3}}{2 k+3} \\ = {} & \frac{1-a}{2}\{1+\log (2 \pi)-2 a\} z+\frac{1}{3}\{\psi(a)-1\} z^3 \\ & +\frac{(a-1)^2}{2}\left\{\log \Gamma(a-z)-\log \Gamma(a+z)\right\} \\ & \times \frac{1-a-z}{2} \log G(a-z)+\frac{a-z-1}{2} \log G(a+z) \\ & +\frac{1}{2} \int_0^z \log G(t+a)\, d t \\ & -\frac{1}{2} \int_0^{-z} \log G(t+a)\, d t \qquad (|z|<|a|).\end{aligned} \tag{4.15}$$

Now we are ready to evaluate some series involving zeta functions explicitly. Setting $a=1$ in (4.11), and using (1.3) and (1.7), and some known formulas for the ψ-function in [17, p. 15], and then letting $z = 1/2$ in the resulting equation, we obtain

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{(k+1)2^{2k}} = -2 - \frac{1}{3}\log 2 - \gamma + 12\log A. \tag{4.16}$$

Setting $a = 2$ in (4.8), (4.9), (4.10), and (4.11), and using a functional relation for $\zeta(s,a)$ in [17, p. 22] and some known formulas for the ψ-function in [17, p. 15], and then letting $z = 1/2$, 1, 3/2 and $z \to 2$ in the resulting form of (4.8) and letting $z = 1/2$, 1, and 3/2 in resulting forms of (4.9), (4.10), and (4.11), we get the following results:

$$\sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)-1}{(k+1)2^k} = \frac{7}{4} + \log(2^{19/12}\cdot 3^{-2}) + \frac{\gamma}{4} - 3\log A; \tag{4.17}$$

$$\sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)-1}{k+1}\left(\frac{3}{2}\right)^{k+1} = \frac{19}{8} + \log(2^{-(17/24)}\cdot 5^{-1}) + \frac{9}{8}\gamma - \frac{3}{2}\log A; \tag{4.18}$$

$$\sum_{k=2}^{\infty} (-1)^k \{\zeta(k)-1\}\frac{2^{k+1}}{k+1} = 3 - \log 6 - \log \pi + 2\gamma; \tag{4.19}$$

$$\sum_{k=2}^{\infty} \frac{\zeta(k)-1}{(k+1)2^k} = \frac{5}{4} - \frac{31}{12}\log 2 - \frac{\gamma}{4} + 3\log A; \tag{4.20}$$

$$\sum_{k=2}^{\infty} \frac{\zeta(k)-1}{k+1}\left(\frac{3}{2}\right)^{k+1} = \frac{17}{8} - \frac{19}{24}\log 2 - \frac{9}{8}\gamma + \frac{3}{2}\log A; \tag{4.21}$$

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)-1}{(2k+1)2^{2k}} = \frac{3}{2} - \log(2^{1/2}\cdot 3); \tag{4.22}$$

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)-1}{2k+1} = \frac{3}{2} - \log 2 - \frac{1}{2}\log \pi; \tag{4.23}$$

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)-1}{2k+1}\left(\frac{3}{2}\right)^{2k+1} = \frac{9}{4} + \log(2^{-(3/4)}\cdot 5^{-(1/2)}); \tag{4.24}$$

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1)-1}{(k+1)2^{2k}} = -1 + \log(2^{-(25/3)} \cdot 3^4) - \gamma + 12 \log A; \tag{4.25}$$

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1)-1}{k+1} = -\gamma + \log 2; \tag{4.26}$$

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1)-1}{k+1} \left(\frac{3}{2}\right)^{2k+2} = -\frac{1}{4} + \log(2^{-(1/12)} \cdot 5) - \frac{9}{4}\gamma + 3 \log A. \tag{4.27}$$

Using (1.3), (2.16), (2.29), (3.5), and (3.8), we can similarly deduce the following results:

$$\begin{aligned}\int_0^{-(1/2)} \log \Gamma(t+1)\, dt &= -\int_0^1 \log \Gamma(t+1)\, dt + \int_0^{1/2} \log \Gamma(t+1)\, dt + \frac{1}{2}\log 2 - \frac{1}{2} \\ &= -\frac{7}{24}\log 2 - \frac{1}{4}\log \pi + \frac{3}{2}\log A;\end{aligned} \tag{4.28}$$

$$\begin{aligned}&\int_0^{-(1/2)} \log G(t+1)\, dt \\ &= -\int_0^1 \log G(t+1)\, dt + \int_0^{1/2} \log G(t+1)\, dt \\ &\quad + \int_0^1 \log \Gamma(t+1)\, dt - \int_0^{1/2} \log \Gamma(t+1)\, dt + \frac{1}{2} - \frac{1}{2}\log 2 \\ &= -\frac{1}{8} + \frac{1}{12}\log 2 + \frac{1}{16}\log \pi + \frac{1}{4}\log A - \frac{7}{4}\log B;\end{aligned} \tag{4.29}$$

$$\begin{aligned}\int_0^{(3/2)} \log G(t+2)\, dt &= \frac{3}{2}\log\frac{3}{2} - \frac{1}{2} + \int_0^1 \log G(t+1)\, dt \\ &\quad + \int_0^{1/2} \log G(t+1)\, dt \\ &\quad + \int_0^1 \log \Gamma(t+1)\, dt + 2\int_0^{1/2} \log \Gamma(t+1) dt \\ &= -\frac{59}{24} + \log(2^{-(31/24)} 3^{3/2}) + \frac{21}{16}\log \pi \\ &\quad + \frac{3}{4}\log A - \frac{7}{4}\log B;\end{aligned} \tag{4.30}$$

$$\begin{aligned}
&\int_0^{-(3/2)} \log G(t+2)\,dt \\
&\qquad = -\int_0^1 \log G(t+1)\,dt + \int_0^{-(1/2)} \log G(t+1)\,dt \\
&\qquad = -\frac{5}{24} - \frac{1}{6}\log 2 - \frac{3}{16}\log \pi + \frac{9}{4}\log A - \frac{7}{4}\log B. \qquad (4.31)
\end{aligned}$$

Setting $a = 1$ in (4.12), (4.13), (4.14), and (4.15), and using (1.7) and some known formulas for the ψ-function in [17, p. 15], and then letting $z \to 1$ and $z = 1/2$ in the resulting form of (4.12) and letting $z = 1/2$ in resulting forms of (4.13), (4.14), and (4.15), we obtain the following results:

$$\sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{(k+2)2^k} = \frac{5}{6} - \frac{1}{3}\log 2 + \frac{\gamma}{6} - 2\log A + 7\log B; \qquad (4.32)$$

$$\sum_{k=2}^{\infty} \frac{\zeta(k)}{(k+2)2^k} = \frac{1}{3} - \frac{2}{3}\log 2 - \frac{\gamma}{6} + 2\log A + 7\log B; \qquad (4.33)$$

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{(k+1)2^{2k}} = \frac{7}{6} - \log 2 + 14\log B; \qquad (4.34)$$

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{(2k+3)2^{2k}} = -\frac{1}{2} - \frac{1}{3}\log 2 - \frac{1}{3}\gamma + 4\log A. \qquad (4.35)$$

Setting $a = 2$ in (4.12), (4.13), (4.14), and (4.15), and using a functional relation for $\zeta(s, a)$ in [17, p. 22] and some known formulas for the ψ-function in [17, p. 15], and then letting $z = 1/2$, 1, 3/2 and $z \to 2$ in the resulting form of (4.12) and letting $z = 1/2$, 1, and 3/2 in resulting forms of (4.13), (4.14), and (4.15), we can readily evaluate the following series involving the zeta function:

$$\begin{aligned}
\sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)-1}{(k+2)2^k} &= -\frac{5}{6} + \log\left(2^{-(13/3)} \cdot 3^4\right) \\
&\quad + \frac{\gamma}{6} - 2\log A + 7\log B; \qquad (4.36)
\end{aligned}$$

$$\begin{aligned}
\sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)-1}{k+2}\left(\frac{3}{2}\right)^{k+2} &= \frac{5}{6} + \log(2^{-3} \cdot 5) + \frac{9}{8}\gamma \\
&\quad - \frac{3}{2}\log A + \frac{7}{4}\log B; \qquad (4.37)
\end{aligned}$$

$$\sum_{k=2}^{\infty}(-1)^k\{\zeta(k)-1\}\frac{2^{k+2}}{k+2} = \frac{11}{6} + \log(2^{-2}\cdot 3) - 2\log\pi + \frac{8}{3}\gamma + 4\log A; \tag{4.38}$$

$$\sum_{k=2}^{\infty}\frac{\zeta(k)-1}{(k+2)2^k} = 3 - \frac{14}{3}\log 2 - \frac{\gamma}{6} + 2\log A + 7\log B; \tag{4.39}$$

$$\sum_{k=2}^{\infty}\frac{\zeta(k)-1}{k+2} = \frac{11}{6} - \frac{1}{2}\log(2\pi) - \frac{\gamma}{3} - 2\log A; \tag{4.40}$$

$$\sum_{k=2}^{\infty}\frac{\zeta(k)-1}{k+2}\left(\frac{3}{2}\right)^{k+2} = \frac{65}{24} - \frac{5}{4}\log 2 - \frac{9}{8}\gamma + \frac{3}{2}\log A + \frac{7}{4}\log B; \tag{4.41}$$

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)-1}{(k+1)2^{2k}} = \frac{13}{6} + \log(2^{-9}\cdot 3^4) + 14\log B; \tag{4.42}$$

$$\sum_{k=1}^{\infty}\frac{\zeta(2k)-1}{k+1}\left(\frac{3}{2}\right)^{2k+2} = \frac{85}{24} + \log\left(2^{-(17/4)}\cdot 3^{3/2}\cdot 5\right) + \frac{7}{2}\log B; \tag{4.43}$$

$$\sum_{k=1}^{\infty}\frac{\zeta(2k+1)-1}{(2k+3)2^{2k}} = -\frac{13}{6} - \frac{13}{3}\log 2 - \frac{\gamma}{3} + 4\log A; \tag{4.44}$$

$$\sum_{k=1}^{\infty}\frac{\zeta(2k+1)-1}{2k+3} = \frac{13}{12} - \frac{1}{2}\log 2 - \frac{1}{3}\gamma - 2\log A; \tag{4.45}$$

$$\sum_{k=1}^{\infty}\frac{\zeta(2k+1)-1}{2k+3}\left(\frac{3}{2}\right)^{2k+3} = \frac{15}{16} + \log\left(2^{7/8}\cdot 5^{-(1/2)}\right) - \frac{9}{8}\gamma + \frac{3}{2}\log A. \tag{4.46}$$

5. SERIES EVALUATED IN TERMS OF THE CATALAN'S CONSTANT G

In this section we also evaluate series associated with the zeta function in terms of Catalan's constant **G** by using some formulas which we already developed here or elsewhere.

Setting $a = 1$ and $a = 2$ in (4.8) to (4.10) and putting $z = 1/4$ and $z = 3/4$ in the resulting equations, with the aid of (1.18) and (1.19), and a relation between the Gamma function and circular function (see [27, p. 239]), we obtain the following results:

$$\sum_{k=2}^{\infty}(-1)^k 2^{-2k}\frac{\zeta(k)}{k+1} = 1+\frac{\gamma}{8}-\frac{\mathbf{G}}{\pi}-\frac{1}{2}\log\left(2\cdot\pi\cdot A^9\cdot\left\{\Gamma\left(\frac{1}{4}\right)\right\}^{-2}\right); \tag{5.1}$$

$$\sum_{k=2}^{\infty}(-1)^k\left(\frac{3}{4}\right)^{k+1}\frac{\zeta(k)}{k+1} = \frac{3}{4}+\frac{9}{32}\gamma+\frac{\mathbf{G}}{4\pi}+\frac{1}{8}\log\left(\pi^3\cdot A^{-9}\cdot\left\{\Gamma\left(\frac{1}{4}\right)\right\}^{-6}\right); \tag{5.2}$$

$$\sum_{k=2}^{\infty}(-1)^k 2^{-2k}\frac{\zeta(k)-1}{k+1} = \frac{15}{8}+\frac{\gamma}{8}-\frac{\mathbf{G}}{\pi}+\frac{1}{2}\log\left(2^{15}\cdot 5^{-8}\cdot\pi^{-1}\cdot A^{-9}\cdot\left\{\Gamma\left(\frac{1}{4}\right)\right\}^{2}\right); \tag{5.3}$$

$$\sum_{k=2}^{\infty}(-1)^k\left(\frac{3}{4}\right)^{k+1}\frac{\zeta(k)-1}{k+1} = \frac{39}{32}+\frac{9}{32}\gamma+\frac{\mathbf{G}}{4\pi}+\frac{1}{8}\log\left(2^{16}\cdot 7^{-8}\cdot\pi^3\cdot A^{-9}\cdot\left\{\Gamma\left(\frac{1}{4}\right)\right\}^{-6}\right); \tag{5.4}$$

$$\sum_{k=2}^{\infty}2^{-2k}\frac{\zeta(k)}{k+1} = -\frac{\gamma}{8}-\frac{\mathbf{G}}{\pi}+\frac{1}{2}\log\left(\pi\cdot A^9\cdot\left\{\Gamma\left(\frac{1}{4}\right)\right\}^{-2}\right); \tag{5.5}$$

$$\sum_{k=2}^{\infty}\left(\frac{3}{4}\right)^{k+1}\frac{\zeta(k)}{k+1} = -\frac{9}{32}\gamma+\frac{\mathbf{G}}{4\pi}+\frac{1}{8}\log\left(2^{-3}\cdot\pi^{-3}\cdot A^9\cdot\left\{\Gamma\left(\frac{1}{4}\right)\right\}^{6}\right); \tag{5.6}$$

$$\sum_{k=2}^{\infty} 2^{-2k} \frac{\zeta(k)-1}{k+1}$$
$$= \frac{9}{8} - \frac{\gamma}{8} - \frac{\mathbf{G}}{\pi} + \frac{1}{2}\log\left(2^{-16}\cdot 3^{8}\cdot \pi \cdot A^{9}\cdot \left\{\Gamma\left(\frac{1}{4}\right)\right\}^{-2}\right); \qquad (5.7)$$

$$\sum_{k=2}^{\infty} \left(\frac{3}{4}\right)^{k+1} \frac{\zeta(k)-1}{k+1}$$
$$= \frac{33}{32} - \frac{9}{32}\gamma + \frac{\mathbf{G}}{4\pi} + \frac{1}{8}\log\left(2^{-19}\cdot \pi^{-3}\cdot A^{9}\cdot \left\{\Gamma\left(\frac{1}{4}\right)\right\}^{6}\right); \qquad (5.8)$$

$$\sum_{k=1}^{\infty} 2^{-4k} \frac{\zeta(2k)}{2k+1} = \frac{1}{2} - \frac{\mathbf{G}}{\pi} - \frac{1}{4}\log 2; \qquad (5.9)$$

$$\sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^{2k+1} \frac{\zeta(2k)}{2k+1} = \frac{3}{8} + \frac{\mathbf{G}}{4\pi} - \frac{3}{16}\log 2; \qquad (5.10)$$

$$\sum_{k=1}^{\infty} 2^{-4k} \frac{\zeta(2k)-1}{2k+1} = \frac{3}{2} - \frac{\mathbf{G}}{\pi} + \frac{1}{4}\log\left(2^{-1}\cdot 3^{8}\cdot 5^{-8}\right); \qquad (5.11)$$

$$\sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^{2k+1} \frac{\zeta(2k)-1}{2k+1} = \frac{9}{8} + \frac{\mathbf{G}}{4\pi} + \frac{1}{16}\log(2^{-3}\cdot 7^{-8}). \qquad (5.12)$$

Setting $a = 1$ and $a = 2$ in (2.14) and (2.35) of [9] yields

$$\sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k(k+1)} z^{k+1}$$
$$= \{\log(2\pi) - 1\}\frac{z}{2} + (\gamma - 1)\frac{z^2}{2}$$
$$+ z\log\Gamma(1+z) - \log G(1+z) \qquad (|z| < 1) \qquad (5.13)$$

and

$$\sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)-1}{k(k+1)} z^{k+1}$$
$$= \{\log(2\pi) - 3\}\frac{z}{2} + (\gamma - 2)\frac{z^2}{2}$$
$$+ (z+1)\log\Gamma(z+2) - \log G(z+2) \qquad (|z| < 2); \qquad (5.14)$$

$$\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k(2k+1)} z^{2k+1} = \{\log(2\pi) - 1\}z + z \log \Gamma(1+z)\Gamma(1-z) + \log \frac{G(1-z)}{G(1+z)} \qquad (|z| < 1) \tag{5.15}$$

and

$$\sum_{k=1}^{\infty} \frac{\zeta(2k) - 1}{k(2k+1)} z^{2k+1} = \{\log(2\pi) - 3\}z + (z+1) \log \Gamma(2+z) + (z-1) \log \Gamma(2-z) + \log \frac{G(2-z)}{G(2+z)} \qquad (|z| < 2). \tag{5.16}$$

Setting $z = 1/4$ and $z = 3/4$ in (5.13), (5.14), (5.15), and (5.16), and using (1.18) and (1.19), we obtain the following summation formulas:

$$\sum_{k=2}^{\infty} (-1)^k 2^{-2k} \frac{\zeta(k)}{k(k+1)} = -1 + \frac{\gamma}{8} + \frac{\mathbf{G}}{\pi} + \frac{1}{2} \log(2^{-3} \cdot \pi \cdot A^9); \tag{5.17}$$

$$\sum_{k=2}^{\infty} (-1)^k \left(\frac{3}{4}\right)^{k+1} \frac{\zeta(k)}{k(k+1)} = -\frac{3}{4} + \frac{9}{32}\gamma - \frac{\mathbf{G}}{4\pi} + \frac{1}{8} \log(2^{-9} \cdot 3^6 \cdot \pi^3 \cdot A^9); \tag{5.18}$$

$$\sum_{k=2}^{\infty} (-1)^k 2^{-2k} \frac{\zeta(k) - 1}{k(k+1)} = -\frac{17}{8} + \frac{\gamma}{8} + \frac{\mathbf{G}}{\pi} + \frac{1}{2} \log(2^{-23} \cdot 5^{10} \cdot \pi \cdot A^9); \tag{5.19}$$

$$\sum_{k=2}^{\infty} (-1)^k \left(\frac{3}{4}\right)^{k+1} \frac{\zeta(k) - 1}{k(k+1)} = -\frac{57}{32} + \frac{9}{32}\gamma - \frac{\mathbf{G}}{4\pi} + \frac{1}{8} \log(2^{-37} \cdot 3^6 \cdot 7^{14} \cdot \pi^3 \cdot A^9); \tag{5.20}$$

$$\sum_{k=1}^{\infty} 2^{-4k} \frac{\zeta(2k)}{k(2k+1)} = -1 + \frac{2\mathbf{G}}{\pi} + \log\left(\frac{\pi}{2}\right); \tag{5.21}$$

$$\sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^{2k} \frac{\zeta(2k)}{k(2k+1)} = -1 - \frac{2\mathbf{G}}{3\pi} + \log\left(\frac{3\pi}{2}\right); \tag{5.22}$$

$$\sum_{k=1}^{\infty} 2^{-4k} \frac{\zeta(2k)-1}{k(2k+1)} = -3 + \frac{2\mathbf{G}}{\pi} + \log(2^{-5} \cdot 3^{-3} \cdot 5^{5} \cdot \pi); \tag{5.23}$$

$$\sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^{2k+1} \frac{\zeta(2k)-1}{k(2k+1)} = -\frac{9}{4} - \frac{\mathbf{G}}{2\pi} + \frac{1}{4}\log(2^{-15} \cdot 3^{3} \cdot 7^{7} \cdot \pi^{3}). \tag{5.24}$$

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REFERENCES

1. L. V. Ahlfors, "Complex Analysis," Third ed., McGraw-Hill, New York, 1979.
2. W. Alexeiewsky, Über eine Classe von Funktionen, die der Gammafunktion analog sind, *Leipzig Weidmannsche Buchhandluns* **46** (1894), 268–275.
3. E. W. Barnes, The theory of the G-function, *Quart. J. Math.* **31** (1899), 264–314.
4. E. W. Barnes, Genesis of the double gamma function, *Proc. London Math. Soc.* **31** (1900), 358–381.
5. E. W. Barnes, The theory of the double gamma function, *Philos. Trans. Roy. Soc. London Ser. A* **196** (1901), 265–388.
6. P. Cassou-Noguès, Analogues p-adiques des fonctions Γ-multiples, *in* "Journées Arithmétiques de Luminy" (Colloq. Internat. CNRS, Centre Univ. Luminy, Luminy, 1978), pp. 43–55, *Astérisque*, **61**, Soc. Math. France, Paris, 1979.
7. J. Choi, Determinant of Laplacian on S^3, *Math. Japon.* **40** (1994), 155–166.
8. J. Choi, A duplication formula for the double gamma function Γ_2, *Bull. Korean Math. Soc.* **33** (1996), 289–294.
9. J. Choi and H. M. Srivastava, Sums associated with the zeta function, *J. Math. Anal. Appl.* **206** (1997), 103–120.
10. J. Choi, H. M. Srivastava, and J. R. Quine, Some series involving the zeta function, *Bull. Austral. Math. Soc.* **51** (1995), 383–393.
11. N. J. De Lillo, "Advanced Calculus with Applications," Macmillan, New York, 1982.
12. J. Edwards, "A Trestise on the Integral Calculus with Applications, Examples and Problems," Vols. 1 and 2, Chelsea, New York, 1954.
13. I. S. Gradshteyn and I. M. Ryzhik, "Tables of Integrals, Series, and Products," (corrected and enlarged ed., prepared by A. Jeffrey), Academic, New York, 1980.
14. V. O. Hölder, "Über eine Transcendente Funktion," pp. 514–522. Dieterichsche Verlags-Buchhandlung, Göttingen, 1886.
15. A. Ivić, "The Riemann Zeta-Function," Wiley, New York, 1985.
16. V. H. Kinkelin, Über eine mit der Gamma Funktion verwandte Transcendente und deren Anwendung auf die Integralrechnung, *J. Reine Angew. Math.* **57** (1860), 122–158.
17. W. Magnus, F. Oberhettinger, and R. P. Soni, "Formulas and Theorems for the Special Functions of Mathematical Physics," Third ed., Springer-Verlag, New York, 1966.

18. K. Matsumoto, Asymptotic series for double zeta, double gamma, and Hecke L-functions, *Math. Proc. Cambridge Philos. Soc.* **123** (1998), 385–405.
19. B. Osgood, R. Phillips, and P. Sarnak, Extremals of determinants of Laplacians, *J. Funct. Anal.* **80** (1988), 148–211.
20. J. R. Quine and J. Choi, Zeta regularized products and functional determinants on spheres, *Rocky Mountain J. Math.* **26** (1996), 719–729.
21. T. Shintani, A proof of the classical Kronecker limit formula, *Tokyo J. Math.* **3** (1980), 191–199.
22. M. R. Spiegel, "Mathematical Handbook," McGraw-Hill, New York, 1968.
23. H. M. Srivastava, A unified presentation of certain classes of series of the Riemann zeta function, *Riv. Mat. Univ. Parma* (4) **14** (1988), 1–23.
24. E. C. Titchmarsh, "The Theory of the Riemann Zeta-Function," Oxford University (Clarendon), Oxford, 1951; Second ed., (Revised by D. R. Heath-Brown), 1986.
25. I. Vardi, Determinants of Laplacians and multiple gamma functions, *SIAM J. Math. Anal.* **19** (1988), 493–507.
26. A. Voros, Special functions, spectral functions and the Selberg zeta function, *Comm. Math. Phys.* **110** (1987), 439–465.
27. E. T. Whittaker and G. N. Watson, "A Course of Modern Analysis," Fourth ed., Cambridge University, Cambridge, 1963.