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Series involving the Zeta function and multiple Gamma functions

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Abstract

The theory of multiple Gamma functions, which was recently revived in the study of the determinants of the Laplacians, was applied in several earlier works in order to evaluate some families of series involving the Riemann Zeta function as well as to compute the determinants of the Laplacians. Here, in the present paper, the authors address the converse problem and apply various (known or new) formulas for series associated with the Zeta and related functions with a view to developing the corresponding theory of multiple Gamma functions and then using these series to compute the determinants of the Laplacians on the n-dimensional unit sphere $\mathbf{S}^n$ $(n = 5, 6, 7)$ explicitly.

Keywords: Multiple Gamma functions; Riemann's ζ-function; Hurwitz (or generalized) Zeta function; Determinants of the Laplacians; Series involving the Zeta function; Meromorphic continuation; Glaisher–Kinkelin constant; Regularization procedure; Finite part prescription; Weierstrass canonical products

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1. Introduction, definitions and preliminaries

The Riemann Zeta function $\zeta(s)$ is defined by

$$\zeta(s) := \begin{cases} \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & (\Re(s) > 1) \\ (1-2^{1-s})^{-1} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & (\Re(s) > 0; s \neq 1), \end{cases} \tag{1.1}$$

which can, except for a simple pole at $s = 1$ with its residue 1, be continued *meromorphically* to the whole complex s-plane. The Hurwitz (or generalized) Zeta function $\zeta(s,a)$ defined by

$$\zeta(s,a) := \sum_{k=0}^{\infty} \frac{1}{(k+a)^s} \quad (\Re(s) > 1; a \in \mathbb{C} \setminus \mathbb{Z}_0^-; \mathbb{Z}_0^- := \{-1,-2,-3,\ldots\}) \tag{1.2}$$

can, just as $\zeta(s)$, be continued *meromorphically* to the whole complex s-plane except for a simple pole at $s = 1$ with its residue 1 (see, for details, [40, pp. 88–103]).

From the definition (1.2) of $\zeta(s,a)$, it easily follows that

$$\zeta(s,a) = \zeta(s,n+a) + \sum_{k=0}^{n-1} (k+a)^{-s} \quad (n \in \mathbb{N} := \{1,2,3,\ldots\}) \tag{1.3}$$

and

$$\zeta(s) = \zeta(s,n+1) + \sum_{k=1}^{n} k^{-s} \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}). \tag{1.4}$$

The connection between $\zeta(s)$ and the Bernoulli numbers B_n (see [40, pp. 59–61]) is given as follows:

$$\zeta(-n) = \begin{cases} -\frac{1}{2} & (n = 0) \\ -\frac{B_{n+1}}{n+1} & (n \in \mathbb{N}). \end{cases} \tag{1.5}$$

The classical Gamma function $\Gamma(z)$, which was first developed by Euler, has several equivalent forms; the one by Weierstrass is being recalled here:

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} \left\{ \left(1 + \frac{z}{k}\right)^{-1} e^{z/k} \right\} \quad (z \in \mathbb{C} \setminus \mathbb{Z}_0^-), \tag{1.6}$$

where γ denotes the Euler–Mascheroni constant defined by

$$\gamma := \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) \cong 0.577215664901532860606512\cdots. \tag{1.7}$$

The Psi (or Digamma) function $\psi(z)$ defined by

$$\psi(z) := \frac{\mathrm{d}}{\mathrm{d}z}\log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \log\Gamma(z) = \int_1^z \psi(t)\,\mathrm{d}t \tag{1.8}$$

has many useful properties including (for example) the ones recorded below, which will be needed in our investigation:

$$\psi(1) = -\gamma, \quad \psi\left(\frac{1}{2}\right) = -\gamma - 2\log 2 \tag{1.9}$$

and

$$\psi(z+n) = \psi(z) + \sum_{k=1}^{n}\frac{1}{z+k-1} \quad (n \in \mathbb{N}). \tag{1.10}$$

The Stirling numbers $s(n,k)$ of the first kind are defined by the generating functions:

$$z(z-1)\cdots(z-n+1) = \sum_{k=0}^{n} s(n,k)z^k \tag{1.11}$$

and

$$\{\log(1+z)\}^k = k!\sum_{n=k}^{\infty} s(n,k)\frac{z^n}{n!} \quad (|z| < 1). \tag{1.12}$$

The numbers $s(n,k)$ satisfy the following recurrence relation:

$$s(n+1,k) = s(n,k-1) - n\,s(n,k) \quad (n \geqq k \geqq 1) \tag{1.13}$$

and their special values are given by

$$s(0,0) = 1, \quad s(n,0) = 0 \ \ (n \in \mathbb{N}), \quad s(n,n) = 1, \quad \text{and}$$
$$s(n,1) = (-1)^{n+1}(n-1)! \tag{1.14}$$

One of the classical series involving the Riemann Zeta function is the following sum:

$$\sum_{k=2}^{\infty}\{\zeta(k) - 1\} = 1. \tag{1.15}$$

The summation formula (1.15) can indeed be shown to be equivalent to a classical (over two centuries old) theorem of Christian Goldbach (1690–1764), which was stated in a letter dated 1729 from Goldbach to Daniel Bernoulli (1700–1782) and was revived in 1986 by Shallit and Zikan [36] (see also [38,39]) as the following problem:

$$\sum_{\omega\in\mathfrak{S}}(\omega-1)^{-1}=1, \tag{1.16}$$

where $\mathfrak{S}$ denotes the set of all nontrivial integer kth powers, that is,

$$\mathfrak{S}:=\{n^k : n,k\in\mathbb{N}\setminus\{1\}\}. \tag{1.17}$$

Since then many formulas for various families of series involving the Zeta and related functions have been presented by using a variety of techniques (see, e.g., [4,6,19,28,29,38,39]) and, especially, by using the theory of multiple Gamma functions (see [14–21]). Here, in the present paper, we address the converse problem and apply a class of closed-form evaluations of series associated with the Zeta and related functions to give further explicit forms of the multiple Gamma function Γ_n and to evaluate the determinants of the Laplacians on the n-dimensional unit sphere $\mathbf{S}^n$ $(n=5,6,7)$ explicitly. In the course of our investigation, we also consider many other related series and identities involving the multiple Gamma function Γ_n.

2. Multiple Gamma functions

As we noted in Section 1, the subject of evaluations of series involving the Zeta and related functions has a long history which can be traced back to Christian Goldbach (1690–1764) and Leonhard Euler (1707–1783). Many different techniques to evaluate various families of series involving the Zeta and related functions have since then been developed (see, for details, [40, Chapter 3]). Recently, Choi and Srivastava [19] (see also Srivastava and Choi [40, p. 149]) proved, by using the following known identity for $\zeta(s,a)$ [38, p. 18, Eq. (6.13)]:

$$\sum_{k=0}^{\infty}\frac{(s)_k}{k!}\zeta(s+k,a)t^k=\zeta(s,a-t)\quad(|t|<|a|), \tag{2.1}$$

where $(\lambda)_n$ denotes the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n:=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\begin{cases}1 & (n=0;\lambda\neq 0)\\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n\in\mathbb{N};\lambda\in\mathbb{C}),\end{cases} \tag{2.2}$$

an interesting class of series involving the Hurwitz (or generalized) Zeta function as follows:

$$\sum_{k=2}^{\infty} \frac{\zeta(k,a)}{(k)_{n+1}} t^{n+k} = \frac{(-1)^n}{n!} [\zeta'(-n,a-t) - \zeta'(-n,a)] + \sum_{k=1}^{n} \frac{(-1)^{n+k}}{n!} \binom{n}{k} [(H_n - H_{n-k})\zeta(k-n,a) - \zeta'(k-n,a)] t^k + [H_n + \psi(a)] \frac{t^{n+1}}{(n+1)!} \quad (|t| < |a|; n \in \mathbb{N}_0), \tag{2.3}$$

where H_n denotes the harmonic numbers defined by

$$H_n := \sum_{j=1}^{n} \frac{1}{j} \tag{2.4}$$

and it is understood (just as elsewhere in this paper) that an empty sum is nil. The formula (2.3) can be shown to be equivalent to the following result proven *independently* by Kanemitsu et al. (cf., e.g., [28, p. 9, Theorem 5]; see also [29, p. 10, Theorem B]):

$$\sum_{k=2}^{\infty} \frac{\zeta(k,a)}{k+n} t^{k+n} = \sum_{k=0}^{n} \binom{n}{k} \zeta'(-k,a-t) t^{n-k} - \sum_{k=0}^{n-1} \frac{\zeta(-k,a)}{n-k} t^{n-k} + [\psi(a) - H_n] \frac{t^{n+1}}{n+1} - \zeta'(-n,a) \quad (|t| < |a|; n \in \mathbb{N}_0), \tag{2.5}$$

by making use of the elementary identity:

$$\frac{1}{(k)_{n+1}} = \frac{1}{k(k+1)\cdots(k+n)} := \sum_{j=0}^{n} \frac{A_j}{k+j}, \tag{2.6}$$

where

$$A_j = \frac{(-1)^j}{n!} \binom{n}{j} \quad (0 \leqq j \leqq n; j, n \in \mathbb{N}_0);$$

the combinatorial identities:

$$\sum_{j=0}^{n} \frac{(-1)^j}{j+1} \binom{n}{j} = \frac{1}{n+1} \quad (n \in \mathbb{N}_0), \tag{2.7}$$

$$\sum_{j=1}^{n} \frac{(-1)^{j+1}}{j+1} \binom{n}{j} H_j = \frac{H_n}{n+1} \quad (n \in \mathbb{N}_0) \tag{2.8}$$

and

$$\sum_{j=n-k+1}^{n} \frac{(-1)^{j+1}}{j-(n-k)} \binom{n}{j} = (-1)^{n+k} \binom{n}{k} (H_n - H_{n-k}) \quad (0 \leqq k \leqq n; n,k \in \mathbb{N}_0), \tag{2.9}$$

the special case $k = n$ of which is recorded in [25, p. 5, Entry 0.155]; the familiar result:

$$\sum_{j=k}^{n} (-1)^j \binom{j}{k} \binom{n}{j} = 0 \quad (0 \leqq k \leqq n-1; k \in \mathbb{N}_0; n \in \mathbb{N}) \tag{2.10}$$

and some rather simple manipulations using such elementary series identities as the ones involved in

$$\sum_{j=0}^{n} \sum_{k=0}^{j-1} A_{j,k} = \sum_{k=0}^{n-1} \sum_{j=k+1}^{n} A_{j,k} \tag{2.11}$$

and

$$\sum_{j=0}^{n} \sum_{k=0}^{j} A_{j,k} = \sum_{k=0}^{n} \sum_{j=k}^{n} A_{j,k}, \tag{2.12}$$

where $\{A_{j,k}\}(j,k \in \mathbb{N}_0)$ is a suitably bounded double sequence.

Such infinite sums as that occurring in (2.3) can also be evaluated, in a *markedly different* way, in terms of the multiple Gamma function of a form other than the one considered in this section (see, for details, [29, p. 10, Theorem 1]).

Here, in our present investigation, we shall make use of each of the following (easily derivable) consequences of the summation formula (2.5):

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{\zeta(2k,a)}{k+n} t^{2k+2n} &= \sum_{k=0}^{2n} \binom{2n}{k} [\zeta'(-k,a-t) + (-1)^k \zeta'(-k,a+t)] t^{2n-k} \\ &\quad - \sum_{\ell=0}^{n-1} \frac{\zeta(-2\ell,a)}{n-\ell} t^{2n-2\ell} - 2\zeta'(-2n,a) \quad (n \in \mathbb{N}_0; |t| < |a|);\end{aligned} \tag{2.13}$$

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1,a)}{2k+2n+1} t^{2k+2n+1} = \frac{1}{2} \sum_{k=0}^{2n} \binom{2n}{k} [\zeta'(-k,a-t) - (-1)^k \zeta'(-k,a+t)] t^{2n-k} - \sum_{\ell=1}^{n} \frac{\zeta(1-2\ell,a)}{2n-2\ell+1} t^{2n-2\ell+1} - \frac{t^{2n+1}}{2n+1} [\psi(2n+1) - \psi(a) + \gamma] \quad (n \in \mathbb{N}_0; |t| < |a|); \tag{2.14}$$

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1,a)}{k+n+1} t^{2k+2n+2} = \sum_{k=0}^{2n+1} \binom{2n+1}{k} [\zeta'(-k,a-t) - (-1)^k \zeta'(-k,a+t)] t^{2n+1-k} - \sum_{\ell=1}^{n} \frac{\zeta(1-2\ell,a)}{n-\ell+1} t^{2n+2-2\ell} - \frac{t^{2n+2}}{n+1} [\psi(2n+2) - \psi(a) + \gamma] - 2\zeta'(-2n-1,a) \quad (n \in \mathbb{N}_0; |t| < |a|); \tag{2.15}$$

$$\sum_{k=1}^{\infty} \frac{\zeta(2k,a)}{2k+2n+1} t^{2k+2n+1} = \frac{1}{2} \sum_{k=0}^{2n+1} \binom{2n+1}{k} [\zeta'(-k,a-t) + (-1)^k \zeta'(-k,a+t)] t^{2n+1-k} - \sum_{\ell=0}^{n} \frac{\zeta(-2\ell,a)}{2n+1-2\ell} t^{2n+1-2\ell} \quad (n \in \mathbb{N}_0; |t| < |a|). \tag{2.16}$$

Furthermore, in its special case when $a = 1$ and $t = -1$. (2.5) immediately yields

$$\sum_{k=2}^{\infty} \frac{(-1)^k}{k+n} \zeta(k) = \sum_{\ell=0}^{n-1} (-1)^{\ell} \binom{n}{\ell} \zeta'(-\ell) - \sum_{\ell=0}^{n-1} \frac{(-1)^{\ell}}{n-\ell} \zeta(-\ell) + \frac{1}{n+1} [\psi(n+1) + 2\gamma] \quad (n \in \mathbb{N}_0). \tag{2.17}$$

The multiple Gamma functions Γ_n were defined and studied by Barnes ([7,8]) and by others (cf., e.g., [5,26,27,30]) in about 1900 (see also [25, p. 649, Entry 6.441(4); p. 887, Entry 8.333] and [45, p. 264]). Recently, these functions were revived in the study of the determinants of the Laplacians on the n-dimensional unit sphere $\mathbf{S}^n$ (cf. [11,16,17,31,42,44]) and have been investigated in various other ways (*cf.* [40, p. 24]; see also [2,3,10,12,13,32,37]). Barnes [7] gave several explicit Weierstrass canonical product forms of the double Gamma function $\Gamma_2 := 1/G$, one of which is recalled here in the form:

$$\{\Gamma_2(z+1)\}^{-1} = G(z+1) = (2\pi)^{z/2}\exp\left(-\frac{1}{2}z-\frac{1}{2}(\gamma+1)z^2\right)\prod_{k=1}^{\infty}\left\{\left(1+\frac{z}{k}\right)^k \cdot\exp\left(-z+\frac{z^2}{2k}\right)\right\}, \tag{2.18}$$

where γ denotes the Euler–Mascheroni constant given by (1.7).

The double Gamma function $\Gamma_2 := 1/G$ also satisfies the following asymptotic expansion:

$$\log G(z+n+2) = \frac{n+1+z}{2}\log(2\pi) + \left[\frac{n^2}{2}+n+\frac{5}{12}+\frac{z^2}{2}+(n+1)z\right]\log n - \frac{3n^2}{4} - n - nz - \log A + \frac{1}{12} + \mathrm{O}(n^{-1}) \quad (n\to\infty), \tag{2.19}$$

where A denotes the Glaisher–Kinkelin constant defined by

$$\log A = \lim_{n\to\infty}\left[\sum_{k=1}^{n} k\log k - \left(\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}\right)\log n + \frac{n^2}{4}\right] \cong 1.282427130\cdots. \tag{2.20}$$

We introduce here two more mathematical constants B and C (analogous to the Glaisher–Kinkelin constant A) (see [17]), defined by

$$\log B = \lim_{n\to\infty}\left[\sum_{k=1}^{n} k^2\log k - \left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right)\log n + \frac{n^3}{9} - \frac{7}{12}\right] \tag{2.21}$$

and

$$\log C = \lim_{n\to\infty}\left[\sum_{k=1}^{n} k^3\log k - \left(\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}-\frac{1}{120}\right)\log n + \frac{n^4}{16} - \frac{n^2}{12}\right] \tag{2.22}$$

for which the approximate numerical values are given by

$$B \cong 1.03091675\cdots \quad \text{and} \quad C \cong 0.97955746\cdots.$$

The constants A, B, and C are known to be expressible as follows:

$$\log A = \frac{1}{12} - \zeta'(-1), \quad \log B = -\zeta'(-2), \quad \text{and} \quad \log C = -\frac{11}{720} - \zeta'(-3), \tag{2.23}$$

in terms of special values of the derivative $\zeta'(s)$. More generally, Adamchik [1] (see also [9,24] and [40, p. 128]) considered such constants as D_k defined by

$$\log D_k := \lim_{n\to\infty}\left[\sum_{m=1}^{n} m^k \log m - p(n,k)\right] \quad (k \in \mathbb{N}_0) \tag{2.24}$$

with $p(n,k)$ given as follows:

$$p(n,k) := \frac{n^k}{2}\log n + \frac{n^{k+1}}{k+1}\left(\log n - \frac{1}{k+1}\right) + k!\sum_{j=1}^{k}\frac{n^{k-j}B_{j+1}}{(j+1)!(k-j)!}\left[\log n + (1-\delta_{k,j})\sum_{\ell=1}^{j}\frac{1}{k-\ell+1}\right],$$

where $\delta_{k,j}$ is the Kronecker symbol and B_j is the jth Bernoulli number. Adamchik [1] also showed that

$$\log D_k = \frac{B_{k+1}\ H_k}{k+1} - \zeta'(-k) \quad (k \in \mathbb{N}_0), \tag{2.25}$$

where H_n are the harmonic numbers given by (2.4). It is known that (see, e.g., [40, p. 128])

$$D_1 = A, \quad D_2 = B, \quad \text{and} \quad D_3 = C.$$

A recurrence formula of the Weierstrass canonical product forms of the multiple Gamma function Γ_n was given by Vignéras [43] who used the theorem of Dufresnoy and Pisot [23] which provides the existence, uniqueness, and expansion of the series of Weierstrass satisfying a functional equation. By making use of the aforementioned Dufresnoy–Pisot theorem [23] and starting with

$$f_1(x) = -\gamma x + \sum_{n=1}^{\infty}\left[\frac{x}{n} - \log\left(1+\frac{x}{n}\right)\right], \tag{2.26}$$

Vignéras [43] obtained a recurrence formula for Γ_n $(n \in \mathbb{N})$, which is given as follows:

$$\Gamma_n(z) := \{G_n(z)\}^{(-1)^{n-1}} \quad (n \in \mathbb{N}), \tag{2.27}$$

where

$$G_n(z+1) = \exp(f_n(z)) \tag{2.28}$$

and the functions $f_n(z)$ are given by

$$f_n(z) = -zA_n(1) + \sum_{k=1}^{n-1}\frac{p_k(z)}{k!}[f_{n-1}^{(k)}(0) - A_n^{(k)}(1)] + A_n(z) \tag{2.29}$$

with

$$A_n(z) = \sum_{m\in\mathbb{N}_0^{n-1}\times\mathbb{N}} \left[\sum_{k=1}^{n} \frac{(-1)^{n-k}}{k}\left(\frac{z}{L(m)}\right)^k + (-1)^n \log\left(1+\frac{z}{L(m)}\right)\right], \tag{2.30}$$

where

$$L(m) = m_1 + m_2 + \cdots + m_n \quad \text{if } m = (m_1, m_2, \ldots, sm_n) \in \mathbb{N}_0^{n-1}\times\mathbb{N}$$

and the polynomials

$$p_n(z) := 1^n + 2^n + \cdots + (z-1)^n$$

satisfy the following relations:

$$p_n'(z) = B_n(z) \quad \text{and} \quad p_n(0) = 0, \tag{2.31}$$

$B_n(z)$ being the Bernoulli polynomial of degree n in z. Indeed, we have

$$p_n(z) = \frac{1}{n+1}\sum_{k=1}^{n+1}\binom{n+1}{k} B_{n+1-k}\, z^k \quad (n\in\mathbb{N}). \tag{2.32}$$

It is not difficult to verify that $\{\Gamma_n(z)\}^{-1}$ is an entire function with zeros at $z=-k$ $(k\in\mathbb{N}_0)$ with multiplicities

$$\binom{n+k-1}{n-1} \quad (n\in\mathbb{N}; k\in\mathbb{N}_0), \tag{2.33}$$

which is mainly used to derive the explicit forms of $\Gamma_n(z)$ like (2.18). For example, we have

$$\Gamma_3(1+z) = \exp(c_1 z + c_2 z^2 + c_3 z^3)\cdot \prod_{k=1}^{\infty}\left\{\left(1+\frac{z}{k}\right)^{-\frac{1}{2}k(k+1)} \cdot \exp\left[\frac{1}{2}(k+1)z - \frac{1}{4}\left(1+\frac{1}{k}\right)z^2 + \frac{1}{6k}\left(1+\frac{1}{k}\right)z^3\right]\right\}, \tag{2.34}$$

where

$$c_1 = \frac{3}{8} - \frac{1}{4}\log(2\pi) - \log A, \quad c_2 = \frac{1}{4}\left(\gamma + \log(2\pi) + \frac{1}{2}\right),$$
$$c_3 = -\frac{1}{6}\left(\gamma + \frac{\pi^2}{6} + \frac{3}{2}\right);$$

$$\Gamma_4(1+z) = \exp(d_1 z + d_2 z^2 + d_3 z^3 + d_4 z^4) \cdot \prod_{k=1}^{\infty} \left\{ \left(1+\frac{z}{k}\right)^{-\binom{k+2}{3}} \cdot \exp\left[\binom{k+2}{3}\left(\frac{z}{k} - \frac{z^2}{2k^2} + \frac{z^3}{3k^3} - \frac{z^4}{4k^4}\right)\right]\right\}, \tag{2.35}$$

where

$$\begin{aligned}
d_1 &= \frac{7}{24} - \log A - \frac{1}{2}\log B - \frac{1}{6}\log(2\pi),\\
d_2 &= -\frac{1}{144} + \frac{\gamma}{6} + \frac{1}{4}\log(2\pi) + \frac{1}{2}\log A,\\
d_3 &= -\frac{2}{9} - \frac{\gamma}{6} - \frac{1}{12}\log(2\pi) - \frac{\pi^2}{54}, \quad d_4 = \frac{11}{144} + \frac{\gamma}{24} + \frac{\pi^2}{48} + \frac{\zeta(3)}{12};
\end{aligned}$$

$$\Gamma_5(1+z) = \exp(e_1 z + e_2 z^2 + e_3 z^3 + e_4 z^4 + e_5 z^5) \cdot \prod_{k=1}^{\infty} \left\{ \left(1+\frac{z}{k}\right)^{-\binom{k+3}{4}} \cdot \exp\left[\binom{k+3}{4}\left(\frac{z}{k} - \frac{z^2}{2k^2} + \frac{z^3}{3k^3} - \frac{z^4}{4k^4} + \frac{z^5}{5k^5}\right)\right]\right\}, \tag{2.36}$$

where

$$\begin{aligned}
e_1 &= \frac{469}{2^6 \cdot 3^2 \cdot 5} - \frac{1}{8}\log(2\pi) + \frac{11}{12}\zeta'(-1) - \frac{3\zeta(3)}{16\pi^2} + \frac{1}{6}\zeta'(-3)\\
&\quad + \frac{1}{20}\zeta(4) - \frac{1}{20}\zeta(5),\\
e_2 &= \frac{\gamma}{8} + \frac{11}{48}\log(2\pi) - \frac{3}{4}\zeta'(-1) + \frac{\zeta(3)}{16\pi^2},\\
e_3 &= -\frac{161}{2^5 \cdot 3^3} - \frac{11}{2^3 \cdot 3^2}\gamma - \frac{1}{8}\log(2\pi) + \frac{1}{6}\zeta'(-1) - \frac{1}{12}\zeta(2),\\
e_4 &= \frac{7}{64} + \frac{1}{16}\gamma + \frac{1}{48}\log(2\pi) + \frac{11}{96}\zeta(2) + \frac{1}{16}\zeta(3),\\
e_5 &= -\frac{5}{2^5 \cdot 3^2} - \frac{1}{2^3 \cdot 3 \cdot 5}\gamma - \frac{1}{20}\zeta(2) - \frac{11}{120}\zeta(3) - \frac{1}{20}\zeta(4).
\end{aligned}$$

By virtue of (2.29), in order to get the constants c_j's, d_j's, and e_j's involved in (2.34), (2.35) and (2.36), respectively, it is indispensable to evaluate $A_n(1)$ explicitly. To do this, using (2.30) and (2.33), we find that

$$A_n(z) = \sum_{k=1}^{\infty} \binom{n+k-2}{n-1} \left[\sum_{\ell=1}^{n} \frac{(-1)^{n-\ell}}{\ell} \left(\frac{z}{k}\right)^{\ell} + (-1)^n \log\left(1+\frac{z}{k}\right) \right]$$
$$= \sum_{\ell=n+1}^{\infty} \frac{(-1)^{n+\ell-1}}{\ell} z^{\ell} \sum_{k=1}^{\infty} \binom{n+k-2}{n-1} \frac{1}{k^{\ell}},$$

which, by means of the expansion [see Eq. (1.11)]:

$$\binom{n+k-2}{n-1} = \frac{1}{(n-1)!} \sum_{j=0}^{n-1} (-1)^{n+j-1} s(n-1,j) k^j,$$

immediately yields

$$A_n(z) = \frac{1}{(n-1)!} \sum_{j=0}^{n-1} (-1)^j s(n-1,j) \sum_{\ell=n+1}^{\infty} \frac{(-1)^{\ell}}{\ell} \zeta(\ell-j) z^{\ell}. \tag{2.37}$$

Now we can apply (2.5) to (2.37) and we get

$$A_n(z) = \frac{1}{(n-1)!} \sum_{j=0}^{n-1} s(n-1,j) \left[\sum_{k=0}^{j} (-1)^k \binom{j}{k} \zeta'(-k, 1+z) z^{j-k} \right.$$
$$+ (-1)^{j+1} \zeta'(-j) - \sum_{\ell=0}^{j-1} (-1)^{\ell} \frac{\zeta(-\ell)}{j-\ell} z^{j-\ell} + \frac{z^{j+1}}{j+1} (H_j + \gamma)$$
$$\left. - \sum_{k=2}^{n-j} (-1)^k \frac{\zeta(k)}{k+j} z^{k+j} \right], \tag{2.38}$$

the special case $z = 1$ of which is given here for convenience:

$$A_n(1) = \frac{1}{(n-1)!} \sum_{j=0}^{n-1} s(n-1,j) \left[\sum_{k=0}^{j-1} (-1)^k \binom{j}{k} \zeta'(-k) \right.$$
$$\left. - \sum_{k=0}^{j-1} \frac{(-1)^k}{j-k} \zeta(-k) + \frac{1}{j+1} (H_j + 2\gamma) - \sum_{k=2}^{n-j} \frac{(-1)^k}{k+j} \zeta(k) \right]$$
$$(n \in \mathbb{N}). \tag{2.39}$$

By using (1.5), (1.9), (1.10), (1.13), (1.14), (2.23), and a well-known identity:

$$\zeta'(0) = -\frac{1}{2} \log(2\pi), \tag{2.40}$$

some special cases of (2.39) are explicitly obtained here as follows:

$$A_1(1) = \gamma, \quad A_2(1) = 1 + \frac{\gamma}{2} - \frac{1}{2}\log(2\pi), \quad A_3(1) = -\frac{1}{4} - \frac{\gamma}{12} + \frac{\pi^2}{36} + \log A,$$
$$A_4(1) = \frac{5}{36} + \frac{\gamma}{24} + \frac{\pi^2}{432} + \frac{\zeta(3)}{12} - \frac{1}{2}\log A - \frac{1}{2}\log B,$$
$$A_5(1) = -\frac{37}{540} - \frac{19}{720}\gamma + \frac{3}{160}\zeta(2) + \frac{7}{240}\zeta(3) + \frac{1}{20}\zeta(5) - \frac{1}{3}\zeta'(-1) + \frac{\zeta(3)}{8\pi^2} - \frac{1}{6}\zeta'(-3). \tag{2.41}$$

Next, by taking logarithms on each side of (2.18), (2.34), and (2.35), we obtain

$$\log\Gamma_2(1+z) = \frac{1}{2}z - \frac{1}{2}z\log(2\pi) + \frac{1}{2}(\gamma+1)z^2 - \sum_{n=2}^{\infty}\frac{(-1)^n}{n+1}\zeta(n)z^{n+1},$$

$$\log\Gamma_3(1+z) = \left[\frac{3}{8} - \frac{1}{4}\log(2\pi) - \log A\right]z + \left[\frac{\gamma}{4} + \frac{1}{4}\log(2\pi) + \frac{1}{8}\right]z^2 - \left(\frac{1}{4} + \frac{\gamma}{6}\right)z^3 - \frac{1}{2}\sum_{n=2}^{\infty}\frac{(-1)^n}{n+1}\zeta(n)z^{n+1} + \frac{1}{2}\sum_{n=2}^{\infty}\frac{(-1)^n}{n+2}\zeta(n)z^{n+2}$$

and

$$\log\Gamma_4(1+z) = \left[\frac{7}{24} - \log A - \frac{1}{2}\log B - \frac{1}{6}\log(2\pi)\right]z + \frac{1}{2}\left[-\frac{1}{72} + \frac{\gamma}{3} + \frac{1}{2}\log(2\pi) + \log A\right]z^2 - \frac{1}{6}\left[\frac{4}{3} + \gamma + \frac{1}{2}\log(2\pi)\right]z^3 + \frac{1}{24}\left(\frac{11}{6} + \gamma\right)z^4 - \frac{1}{6}\sum_{n=2}^{\infty}\frac{(-1)^n}{n+3}\zeta(n)z^{n+3} + \frac{1}{2}\sum_{n=2}^{\infty}\frac{(-1)^n}{n+2}\zeta(n)z^{n+2} - \frac{1}{3}\sum_{n=2}^{\infty}\frac{(-1)^n}{n+1}\zeta(n)z^{n+1},$$

which, upon employing the special case $a = 1$ of (2.5), can be expressed in terms of the Zeta functions as follows:

$$\log\Gamma_2(1+z) = -\frac{1}{12} + \log A - z\log\Gamma(1+z) + \zeta'(-1, 1+z), \tag{2.42}$$

$$\log \Gamma_3(1+z) = -\frac{1}{24} + \frac{1}{2}\log A + \frac{\zeta(3)}{8\pi^2} + \left(\frac{1}{12} - \log A\right)z + \frac{1}{2}(z^2 - z)\log \Gamma(1+z) + \left(\frac{1}{2} - z\right)\zeta'(-1, 1+z) + \frac{1}{2}\zeta'(-2, 1+z) \tag{2.43}$$

and

$$\log \Gamma_4(1+z) = -\frac{1}{36} + \frac{1}{3}\log A + \frac{\zeta(3)}{8\pi^2} - \frac{1}{6}\zeta'(-3) + \left(\frac{1}{12} - \log A - \frac{\zeta(3)}{8\pi^2}\right)z + \left(-\frac{1}{24} + \frac{1}{2}\log A\right)z^2 - \frac{1}{6}(z^3 - 3z^2 + 2z)\log \Gamma(1+z) + \left(\frac{1}{2}z^2 - z + \frac{1}{3}\right)\zeta'(-1, 1+z) + \frac{1}{6}\zeta'(-3, 1+z), \tag{2.44}$$

where we have also made use of (2.23) and the following known identity (see, e.g., [40, p. 99]):

$$\zeta'(-2n) = (-1)^n \frac{(2n)!}{2(2\pi)^{2n}}\zeta(2n+1) \quad (n \in \mathbb{N}). \tag{2.45}$$

Analogous to the classical result:

$$\Gamma\left(\frac{1}{2}\right) = \pi^{\frac{1}{2}},$$

setting $z = -\frac{1}{2}$ in (2.42), (2.43), and (2.44), we obtain

$$\Gamma_2\left(\frac{1}{2}\right) = 2^{-\frac{1}{24}} \cdot \pi^{\frac{1}{4}} \cdot \mathrm{e}^{-\frac{1}{8}} \cdot A^{\frac{3}{2}}, \tag{2.46}$$

$$\Gamma_3\left(\frac{1}{2}\right) = 2^{-\frac{1}{24}} \cdot \pi^{\frac{3}{16}} \cdot \exp\left[-\frac{1}{8} + \frac{7\zeta(3)}{32\pi^2}\right] \cdot A^{\frac{3}{2}} \tag{2.47}$$

and

$$\Gamma_4\left(\frac{1}{2}\right) = 2^{-\frac{229}{5760}} \cdot \pi^{\frac{5}{32}} \cdot \mathrm{e}^{-\frac{265}{2304}} \cdot A^{\frac{23}{16}} \cdot B^{\frac{3}{4}} \cdot C^{\frac{5}{16}}. \tag{2.48}$$

By integrating the very three equations before (2.42) from 0 to z and using the special case $a = 1$ of (2.5), we get

$$\int_0^z \log \Gamma_2(1+t)\,\mathrm{d}t = \frac{\zeta(3)}{4\pi^2} - \left[\frac{1}{12} + \zeta'(-1)\right]z - \frac{1}{4}z^2\log(2\pi) + \frac{1}{6}z^3 - z\zeta'(-1, 1+z) + \zeta'(-2, 1+z), \tag{2.49}$$

$$\begin{aligned}\int_0^z \log\Gamma_3(1+t)\,\mathrm{d}t &= \frac{\zeta(3)}{8\pi^2}-\frac{1}{2}\zeta'(-3)+\left[\frac{\zeta(3)}{8\pi^2}-\frac{1}{12}+\frac{1}{2}\log A\right]z\\ &\quad+\left[\frac{1}{24}-\frac{1}{8}\log(2\pi)-\frac{1}{2}\log A\right]z^2+\frac{1}{12}[1+\log(2\pi)]z^3\\ &\quad-\frac{1}{24}z^4-\frac{z}{2}\zeta'(-1,1+z)+\frac{1}{2}z^2\zeta'(-1,1+z)\\ &\quad+\frac{1}{2}\zeta'(-2,1+z)-z\zeta'(-2,1+z)+\frac{1}{2}\zeta'(-3,1+z)\end{aligned} \tag{2.50}$$

and

$$\begin{aligned}\int_0^z \log\Gamma_4(1+t)\,\mathrm{d}t &= \frac{\zeta(3)}{12\pi^2}-\frac{1}{2}\zeta'(-3)-\frac{\zeta(5)}{8\pi^4}-\left[\frac{19}{720}+\frac{1}{3}\zeta'(-1)\right.\\ &\qquad\left.-\frac{\zeta(3)}{8\pi^2}+\frac{1}{6}\zeta'(-3)\right]z+\left[\frac{1}{2}\zeta'(-1)-\frac{\zeta(3)}{16\pi^2}\right.\\ &\qquad\left.-\frac{1}{12}\log(2\pi)\right]z^2+\left[\frac{1}{18}+\frac{1}{12}\log(2\pi)-\frac{1}{6}\zeta'(-1)\right]z^3\\ &\quad-\frac{1}{24}\left[1+\frac{1}{2}\log(2\pi)\right]z^4+\frac{1}{120}z^5\\ &\quad-\frac{1}{6}(z^3-3z^2+2z)\zeta'(-1,1+z)+\left(\frac{1}{2}z^2-z+\frac{1}{3}\right)\\ &\quad\cdot\zeta'(-2,1+z)-\frac{1}{2}(z-1)\zeta'(-3,1+z)\\ &\quad+\frac{1}{6}\zeta'(-4,1+z).\end{aligned} \tag{2.51}$$

Upon setting $z=1$ and $z=\frac{1}{2}$ in (2.49), (2.50), and (2.51), and using some previously recorded identities, we have

$$\int_0^1 \log\Gamma_2(1+t)\,\mathrm{d}t = -\frac{1}{12}-\frac{1}{4}\log(2\pi)+2\log A, \tag{2.52}$$

$$\int_0^1 \log\Gamma_3(1+t)\,\mathrm{d}t = \frac{3\zeta(3)}{8\pi^2}-\frac{1}{24}\log(2\pi), \tag{2.53}$$

$$\int_0^1 \log\Gamma_4(1+t)\,\mathrm{d}t = \frac{13}{2160}+\frac{3\zeta(3)}{16\pi^2}-\frac{1}{48}\log(2\pi)+\frac{2}{3}\log C, \tag{2.54}$$

$$\int_0^{\frac{1}{2}} \log\Gamma_2(1+t)\,\mathrm{d}t = -\frac{1}{24}(1+\log 2)-\frac{1}{16}\log\pi+\frac{1}{4}\log A+\frac{7\zeta(3)}{16\pi^2}, \tag{2.55}$$

$$\int_0^{\frac{1}{2}} \log \Gamma_3(1+t)\,\mathrm{d}t = -\frac{1}{256} - \frac{29}{1920}\log 2 - \frac{1}{48}\log \pi + \frac{1}{16}\log A + \frac{3}{4}\log B + \frac{15}{16}\log C \tag{2.56}$$

and

$$\int_0^{\frac{1}{2}} \log \Gamma_4(1+t)\,\mathrm{d}t = \frac{73}{69\,120} - \frac{17}{1920}\log 2 - \frac{3}{256}\log \pi + \frac{1}{32}\log A + \frac{77}{96}\log C + \frac{47\zeta(3)}{384\pi^2} - \frac{31\zeta(5)}{128\pi^4}. \tag{2.57}$$

Barnes [7, p. 283] expressed $\log \Gamma_2(z+a)$ as an integral of $\log \Gamma(t+a)$ as follows:

$$\int_0^z \log \Gamma(t+a)\,\mathrm{d}t = \frac{1}{2}[\log(2\pi) + 1 - 2a]z - \frac{z^2}{2} + (z+a-1)\log \Gamma(z+a) + \log \Gamma_2(z+a) + (1-a)\log \Gamma(a) - \log \Gamma_2(a). \tag{2.58}$$

Similarly, we have

$$\begin{aligned}\int_0^z \log \Gamma_2(t+a)\,\mathrm{d}t = {} & \left[\frac{1}{4} - \frac{1}{2}(a-1)\log(2\pi) + 2\log A + \frac{a^2}{2} - a\right]z \\ & + \frac{1}{4}[2a - 2 - \log(2\pi)]z^2 + \frac{1}{6}z^3 + (z+a-2) \\ & \cdot \log \Gamma_2(z+a) + 2\log \Gamma_3(z+a) + (2-a) \\ & \cdot \log \Gamma_2(a) - 2\log \Gamma_3(a),\end{aligned} \tag{2.59}$$

$$\begin{aligned}\int_0^z \log \Gamma_3(t+a)\,\mathrm{d}t = {} & \left[\frac{1}{8} - \frac{5}{6}a + \frac{3}{4}a^2 - \frac{1}{6}a^3 + (3-2a)\log A \right. \\ & \left. + \frac{3}{2}\log B + \frac{1}{4}(a-1)(a-2)\log(2\pi)\right]z \\ & + \left[-\frac{1}{4}a^2 + \frac{3}{4}a - \frac{5}{12} + \left(\frac{1}{4}a - \frac{3}{8}\right)\log(2\pi) - \log A\right]z^2 \\ & + \left[\frac{1}{4} - \frac{1}{6}a + \frac{1}{12}\log(2\pi)\right]z^3 \\ & - \frac{1}{24}z^4 + (z+a-3)\log \Gamma_3(z+a) + (3-a)\log \Gamma_3(a) \\ & + 3\log \Gamma_4(z+a) - 3\log \Gamma_4(a)\end{aligned} \tag{2.60}$$

and

$$\int_0^z \log\Gamma_4(t+a)\,\mathrm{d}t = \beta_1 z + \beta_2 z^2 + \beta_3 z^3 + \beta_4 z^4 + \beta_5 z^5 - 4\log\Gamma_5(a) + (z+a-4)\log\Gamma_4(z+a) + (4-a)\log\Gamma_4(a) + 4\log\Gamma_5(z+a), \tag{2.61}$$

where

$$\begin{aligned}\beta_1 &= \frac{1937}{2160} + \frac{19}{90}\gamma - a + \frac{11}{12}a^2 - \frac{1}{3}a^3 + \frac{1}{24}a^4 \\ &\quad - \frac{1}{12}(a^3 - 6a^2 + 11a - 6)\log(2\pi) - \left(a^2 - 4a + 1\right)\zeta'(-1) \\ &\quad + \frac{1}{8}(1-3a)\frac{\zeta(3)}{\pi^2} + \frac{2}{3}\zeta'(-3) - \frac{3}{20}\zeta(2) - \frac{7}{30}\zeta(3) - \frac{1}{5}\zeta(4) - \frac{1}{5}\zeta(5), \\ \beta_2 &= -\frac{1}{2} + \frac{11}{12}a - \frac{1}{2}a^2 + \frac{1}{12}a^3 - \left(\frac{1}{8}a^2 - \frac{1}{2}a + \frac{11}{24}\right)\log(2\pi) + (2-a)\zeta'(-1) \\ &\quad - \frac{3\zeta(3)}{16\pi^2}, \\ \beta_3 &= \frac{11}{36} - \frac{1}{3}a + \frac{1}{12}a^2 + \frac{1}{12}(2-a)\log(2\pi) - \frac{1}{3}\zeta'(-1), \\ \beta_4 &= -\frac{1}{12} + \frac{1}{24}a - \frac{1}{48}\log(2\pi), \quad \text{and} \quad \beta_5 = \frac{1}{120}.\end{aligned}$$

In order to derive (2.61), for example, by setting $z = t + a$ in (1.6), $z = t + a - 1$ in (2.18), (2.34), (2.35), and (2.36), and taking the logarithmic derivatives on each side of the resulting equations, we find that

$$\begin{aligned}\frac{\Gamma'(t+a)}{\Gamma(t+a)} &= -\gamma - \sum_{k=1}^{\infty}\left(\frac{1}{t+a-1+k} - \frac{1}{k}\right), \\ \frac{\Gamma_2'(t+a)}{\Gamma_2(t+a)} &= \frac{1}{2} - \frac{1}{2}\log(2\pi) + (1+\gamma)t + a - 1) \\ &\quad + \sum_{k=1}^{\infty} k\left(-\frac{1}{t+a-1+k} + \frac{1}{k} - \frac{t+a-1}{k^2}\right), \\ \frac{\Gamma_3'(t+a)}{\Gamma_3(t+a)} &= c_1 + 2c_2(t+a-1) + 3c_3(t+a-1)^2 + \sum_{k=1}^{\infty}\frac{1}{2}k(k+1) \\ &\quad \cdot\left(-\frac{1}{t+a-1+k} + \frac{1}{k} - \frac{t+a-1}{k^2} + \frac{(t+a-1)^2}{k^3}\right)\end{aligned}$$

and

$$\frac{\Gamma_4'(t+a)}{\Gamma_4(t+a)} = d_1 + 2d_2(t+a-1) + 3d_3(t+a-1)^2 + 4d_4(t+a-1)^3 + \sum_{k=1}^{\infty}\binom{k+2}{3}\left[-\frac{1}{t+a-1+k} + \frac{1}{k} - \frac{t+a-1}{k^2} + \frac{(t+a-1)^2}{k^3} - \frac{(t+a-1)^3}{k^4}\right],$$

all of which can be applied in conjunction with

$$\frac{\Gamma_5'(t+a)}{\Gamma_5(t+a)} = e_1 + 2e_2(t+a-1) + 3e_3(t+a-1)^2 + 4e_4(t+a-1)^3 + 5e_5(t+a-1)^4 + \sum_{k=1}^{\infty}\binom{k+3}{4}\left[-\frac{1}{t+a-1+k} + \frac{1}{k} - \frac{t+a-1}{k^2} + \frac{(t+a-1)^2}{k^3} - \frac{(t+a-1)^3}{k^4} + \frac{(t+a-1)^4}{k^5}\right]$$

to obtain

$$\frac{\Gamma_5'(t+a)}{\Gamma_5(t+a)} = e_1 + \omega_1(t+a-1) + \omega_2(t+a-1)^2 + \omega_3(t+a-1)^3 + \omega_4(t+a-1)^4 - \frac{1}{4}(t+a-1)\frac{\Gamma'(t+a)}{\Gamma(t+a)} - \frac{1}{4}(t+a-1) \cdot \frac{\Gamma_2'(t+a)}{\Gamma_2(t+a)} - \frac{1}{4}(t+a-1)\frac{\Gamma_3'(t+a)}{\Gamma_3(t+a)} - \frac{1}{4}(t+a-1) \cdot \frac{\Gamma_4'(t+a)}{\Gamma_4(t+a)}, \tag{2.62}$$

where

$$\omega_1 = \frac{1}{4} + \frac{11}{48}\log(2\pi) - \zeta'(-1) + \frac{3\zeta(3)}{32\pi^2},$$
$$\omega_2 = -\frac{11}{48} - \frac{1}{8}\log(2\pi) + \frac{1}{4}\zeta'(-1),$$
$$\omega_3 = \frac{1}{12} + \frac{1}{48}\log(2\pi), \quad \text{and} \quad \omega_4 = -\frac{1}{96}.$$

Finally, upon integrating both sides of (2.62) from $t = 0$ to $t = z$ and using (2.58) to (2.60), we are led to the desired identity (2.61).

The foregoing methods can also be applied with a view to deriving explicit expressions for the multiple Gamma functions Γ_n of higher order and various other related identities.

3. Determinants of the Laplacians

During the last two decades, the problem of evaluating the determinants of the Laplacians on Riemann manifolds has received considerable attention by many authors including D'Hoker and Phong [22], Sarnak [35], and Voros [44], who computed the determinants of the Laplacian on compact Riemann surfaces of constant curvature in terms of special values of the Serberg Zeta function. Here we are particularly concerned with the evaluation of the functional determinant for the n-dimensional unit sphere $\mathbf{S}^n$ with standard metric. The theory of multiple Gamma functions given in Section 2 played an important rôle in computations of the determinants of the Laplacians on $\mathbf{S}^n$ (see [11,31,42,44]), which could also be computed by using the closed-form evaluations of a certain family of series involving the Zeta and related functions (see [16,17]). Here, in this section, we will evaluate the determinants of the Laplacians on the n-dimensional unit sphere $\mathbf{S}^n$ $(n = 5, 6, 7)$ explicitly, by using some closed-form evaluations of series associated with the Zeta and related functions. We note that the cases $n = 2$, 3, and 4 were already treated in [16,17] in the same way as in this section (see also [31,33,34,42]).

Let $\{\lambda_n\}$ be a sequence such that

$$0 = \lambda_0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n \leqq \cdots; \quad \lambda_n \uparrow \infty \quad (n \to \infty); \tag{3.1}$$

henceforth we consider only such nonnegative increasing sequences. Then we can show that

$$Z(s) := \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s}, \tag{3.2}$$

which is known to converge absolutely in a half-plane $\Re(s) > \sigma$ for some $\sigma \in \mathbb{R}$. The determinant of the Laplacian Δ on the compact manifold M is defined to be

$$\det{}'\Delta := \prod_{\lambda_k \neq 0} \lambda_k, \tag{3.3}$$

where $\{\lambda_k\}$ is the sequence of eigenvalues of the Laplacian Δ on M (cf. [33]). The sequence $\{\lambda_k\}$ is known to satisfy the condition as in (3.1), but the product in (3.3) is always divergent; so, in order for the expression (3.3) to make sense, some sort of regularization procedure must be used (see [34]). It is easily seen that, formally, $\mathrm{e}^{-Z'(0)}$ is the product of nonzero eigenvalues of Δ. This product does not converge, but $Z(s)$ can be continued analytically to a neighborhood of $s = 0$. Therefore, we can give a meaningful definition:

$$\det{}'\Delta := \mathrm{e}^{-Z'(0)}, \tag{3.4}$$

which is called the functional determinant of the Laplacian Δ on M. The order μ of the sequence $\{\lambda_k\}$ is defined by

$$\mu := \inf\left\{\alpha : \alpha > 0 \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{\alpha}} < \infty\right\}. \tag{3.5}$$

The analogous and shifted analogous Weierstrass canonical products $E(\lambda)$ and $E(\lambda, a)$ of the sequence $\{\lambda_k\}$ are defined, respectively, by

$$E(\lambda) := \prod_{k=1}^{\infty}\left\{\left(1-\frac{\lambda}{\lambda_k}\right)\exp\left(\frac{\lambda}{\lambda_k}+\frac{\lambda^2}{2\lambda_k^2}+\cdots+\frac{\lambda^{[\mu]}}{[\mu]\lambda_k^{[\mu]}}\right)\right\} \tag{3.6}$$

and

$$E(\lambda, a) := \prod_{k=1}^{\infty}\left\{\left(1-\frac{\lambda}{\lambda_k+a}\right)\exp\left(\frac{\lambda}{\lambda_k+a}+\cdots+\frac{\lambda^{[\mu]}}{[\mu](\lambda_k+a)^{[\mu]}}\right)\right\}, \tag{3.7}$$

where $[\mu]$ denotes the greatest integer part of the order μ of the sequence $\{\lambda_k\}$. There exists the following relationship between $E(\lambda)$ and $E(\lambda, a)$ (see [44]):

$$E(\lambda, a) = \exp\left(\sum_{m=1}^{[\mu]} R_{m-1}(-a)\frac{\lambda^m}{m!}\right)\frac{E(\lambda-a)}{E(-a)}, \tag{3.8}$$

where, for convenience,

$$R_{[\mu]}(\lambda-a) := \frac{\mathrm{d}^{[\mu]+1}}{\mathrm{d}\lambda^{[\mu]+1}}\{-\log E(\lambda, a)\}. \tag{3.9}$$

The shifted series $Z(s, a)$ of $Z(s)$ in (3.2) by a is given by

$$Z(s, a) := \sum_{k=1}^{\infty}\frac{1}{(\lambda_k+a)^s}. \tag{3.10}$$

Finally, indeed, we have

$$Z'(0, -\lambda) = -\sum_{k=1}^{\infty}\log(\lambda_k-\lambda),$$

which, under the definition:

$$D(\lambda) := \exp[-Z'(0, -\lambda)], \tag{3.11}$$

immediately implies that

$$D(\lambda) = \prod_{k=1}^{\infty}(\lambda_k-\lambda).$$

In fact, Voros [44] gave the relationship between $D(\lambda)$ and $E(\lambda)$ as follows:

$$D(\lambda) = \exp[-Z'(0)] \exp\left[-\sum_{m=1}^{[\mu]} \mathrm{FP}\, Z(m) \frac{\lambda^m}{m} \right] \cdot \exp\left[-\sum_{m=2}^{[\mu]} C_{-m} \left(\sum_{k=1}^{m-1} \frac{1}{k} \right) \frac{\lambda^m}{m!} \right] E(\lambda), \tag{3.12}$$

where an empty sum is interpreted to be nil and the *finite part* prescription is applied (as usual) as follows (cf. [44, p. 446]):

$$\mathrm{FP} f(s) := \begin{cases} f(s), & \text{if } s \text{ is not a pole,} \\ \lim_{\epsilon \to 0} \left(f(s+\epsilon) - \frac{\text{Residue}}{\epsilon} \right), & \text{if } s \text{ is a simple pole} \end{cases} \tag{3.13}$$

and

$$Z(-m) = (-1)^m m! C_{-m}. \tag{3.14}$$

Now we consider the sequence of eigenvalues on the standard Laplacian Δ_n on $\mathbf{S}^n$. It is known from the work of Vardi [42] (see also [41]) that the standard Laplacian Δ_n $(n \in \mathbb{N})$ has eigenvalues

$$\mu_k := k(k+n-1) \tag{3.15}$$

with multiplicity

$$\binom{k+n}{n} - \binom{k+n-2}{n} = \frac{(2k+n-1)(k+n-2)!}{k!(n-1)!} \quad (k \in \mathbb{N}_0). \tag{3.16}$$

From now on we consider the shifted sequence $\{\lambda_k\}$ of $\{\mu_k\}$ in (3.15) by $\left(\frac{n-1}{2}\right)^2$ as a fundamental sequence. Then the sequence $\{\lambda_k\}$ is written in the following simple and tractable form:

$$\lambda_k = \mu_k + \left(\frac{n-1}{2} \right)^2 = \left(k + \frac{n-1}{2} \right)^2 \tag{3.17}$$

with the same multiplicity as in (3.16). We will exclude the zero mode, that is, we start the sequence at $k=1$ for later use. Furthermore, with a view to emphasizing n on $\mathbf{S}^n$, we choose the notations $Z_n(s)$, $Z_n(s,a)$, $E_n(\lambda)$, $E_n(\lambda,a)$, and $D_n(\lambda)$ instead of $Z(s)$, $Z(s,a)$, $E(\lambda)$, $E(\lambda,a)$, and $D(\lambda)$, respectively.

We readily observe from (3.11) that

$$D_n\left(\left(\frac{n-1}{2} \right)^2 \right) = \det{}' \Delta_n, \tag{3.18}$$

where $\det' \Delta_n$ denotes the *determinants of the Laplacians* on $\mathbf{S}^n$ $(n \in \mathbb{N})$.

In order to evaluate $\det' \Delta_5$, we begin by setting $a=3$, $n=1$, $n=2$, and $t=2$ in (2.13), and then use (1.4), (2.40) and (2.45). We thus obtain

$$\sum_{n=1}^{\infty} \frac{\zeta(2n,3)}{n+1} 2^{2n+2} = 10 + \log(3 \cdot \pi^{-4}) \tag{3.19}$$

and

$$\sum_{n=1}^{\infty} \frac{\zeta(2n,3)}{n+2} 2^{2n+4} = 20 - \frac{13\zeta(3)}{\pi^2} + \log(2^{288} \cdot 3 \cdot \pi^{-16}). \tag{3.20}$$

By setting $n = 5$ in (3.17), we find that the shifted sequence of eigenvalues of Δ^5 of $\mathbf{S}^5$ is given as follows:

$$(k+2)^2 \quad \text{with multiplicity} \quad \frac{1}{12}(k+1)(k+2)^2(k+3) \quad (k \in \mathbb{N}). \tag{3.21}$$

It is seen that the sequence in (3.21) has the order $\mu = \frac{5}{2}$. We also have

$$\begin{aligned} Z_5(s) &= \frac{1}{12} \sum_{k=1}^{\infty} \frac{(k+1)(k+2)^2(k+3)}{(k+2)^{2s}} \\ &= \frac{1}{12}[\zeta(2s-4) - \zeta(2s-2)] + \frac{1}{3}\left(\frac{1}{2^{2s}} - \frac{1}{2^{2s-2}}\right). \end{aligned} \tag{3.22}$$

It is observed that $Z_5(s)$ has simple poles at $s = \frac{3}{2}$ and $s = \frac{5}{2}$ We, therefore, have

$$\mathrm{FP}\, Z_5(1) = Z_5(1) = -\frac{5}{24} \quad \text{and} \quad \mathrm{FP}\, Z_5(2) = Z_5(2) = -\frac{1}{12}\zeta(2) - \frac{5}{48}.$$

We also have

$$C_{-2} = \frac{1}{2} Z_5(-2) = -8 \quad \text{and} \quad Z_5'(0) = \frac{\zeta(5)}{8\pi^4} + \frac{\zeta(3)}{24\pi^2} + 2\log 2.$$

We thus find that

$$\begin{aligned} E_5(\lambda) = \prod_{k=1}^{\infty} &\left(1 - \frac{\lambda}{(k+2)^2}\right)^{\frac{1}{12}(k+1)(k+2)^2(k+3)} \\ &\cdot \exp\left[\frac{1}{12}(k+1)(k+2)^2(k+3)\left(\frac{\lambda}{(k+2)^2} + \frac{\lambda^2}{2(k+2)^4}\right)\right], \end{aligned}$$

which, upon setting $\lambda = 4$ and taking logarithms on each side of the resulting equation, and using (3.19) and (3.20), yields

$$\begin{aligned}
\log E_5(4) &= -\frac{1}{12}\sum_{n=3}^{\infty}\frac{2^{2n}}{n}\left[\zeta(2n-4)-\zeta(2n-2)+\frac{1}{2^{2n-2}}-\frac{1}{2^{2n-4}}\right] \\
&= -\frac{1}{12}\left[\sum_{n=1}^{\infty}\frac{2^{2n+4}}{n+2}\zeta(2n,3)-\sum_{n=2}^{\infty}\frac{2^{2n+2}}{n+1}\zeta(2n,3)\right] \\
&= -\frac{\pi^2}{9}+\frac{13\zeta(3)}{12\pi^2}+\log(2^{-24}\cdot\pi).
\end{aligned}$$

If we set $n = 5$ in (3.18) and use (3.12), we finally have

$$\begin{aligned}
\det{}'\Delta_5 &= D_5(4) \\
&= \exp[-Z_5'(0)-4\mathrm{FP}Z_5(1)-8\mathrm{FP}Z_5(2)-8C_{-2}]E_5(4) \\
&= \frac{\pi}{2^{26}}\exp\left[\frac{197}{3}-\frac{\zeta(3)}{24\pi^2}-\frac{\zeta(5)}{8\pi^4}\right]. \qquad (3.23)
\end{aligned}$$

Next, by setting $n = 6$ in (3.17), we obtain the shifted sequence of eigenvalues of Δ_6 on $\mathbf{S}^6$ as follows:

$$\left(k+\frac{5}{2}\right)^2 \quad \text{with multiplicity} \quad \frac{1}{120}(2k+5)(k+4)(k+3)(k+2)(k+1). \qquad (3.24)$$

We see that

$$\begin{aligned}
Z_6(s) &= \frac{1}{120}\sum_{k=1}^{\infty}\frac{(2k+5)(k+4)(k+3)(k+2)(k+1)}{(k+\frac{5}{2})^{2s}} \\
&= \frac{1}{1920}[(2^{2s}-32)\zeta(2s-5)-10(2^{2s}-8)\zeta(2s-3)+9(2^{2s}-2)\zeta(2s-1)] \\
&\quad -\left(\frac{2}{5}\right)^{2s}. \qquad (3.25)
\end{aligned}$$

It is observed that $Z_6(s)$ has simple poles at $s = 1$, 2, and 3 with its residues $\frac{3}{640}$, $-\frac{1}{48}$, and $\frac{1}{120}$, respectively. It is also seen that the sequence in (3.24) has the order $\mu = 3$. Now we can find that

$$\begin{aligned}
Z_6'(0) &= -\frac{484\,051}{2^8\cdot 3^3\cdot 5\cdot 7}\log 2+2\log 5-\frac{31}{960}\zeta'(-5)+\frac{7}{96}\zeta'(-3) \\
&\quad -\frac{3}{320}\zeta'(-1),
\end{aligned}$$

$$\mathrm{FP}Z_6(1) = \lim_{\epsilon\to 0}\left[Z_6(1+\epsilon)-\frac{3}{640\epsilon}\right] = -\frac{9323}{2^8\cdot 3^2\cdot 5^2}+\frac{3}{320}\gamma+\frac{3}{160}\log 2,$$

$$\mathrm{FP}Z_6(2) = -\frac{4483}{2^5\cdot 3^2\cdot 5^4}+\frac{21}{320}\zeta(3)-\frac{1}{24}(\gamma+2\log 2),$$

$$\mathrm{FP}\, Z_6(3) = -\frac{2^6}{5^6} + \frac{\gamma}{60} + \frac{1}{30}\log 2 - \frac{7}{24}\zeta(3) + \frac{93}{320}\zeta(5),$$

$$C_{-2} = \frac{1}{2} Z_6(-2) = -\frac{2\,217\,581\,021}{2^{15}\cdot 3^2\cdot 5\cdot 7\cdot 11},$$

and

$$C_{-3} = -\frac{1}{6} Z_6(-3) = \frac{62\,451\,523}{2^{18}\cdot 3^4\cdot 5^2\cdot 7\cdot 11\cdot 13} + \frac{5^6}{2^7\cdot 3}.$$

We also see that

$$\log E_6(\lambda) = -\frac{1}{60}\sum_{n=4}^{\infty}\frac{\lambda^n}{n}\left[\zeta\left(2n-5,\frac{7}{2}\right) - \frac{5}{2}\zeta\left(2n-3,\frac{7}{2}\right) + \frac{9}{16}\zeta\left(2n-1,\frac{7}{2}\right)\right], \tag{3.26}$$

which, for $\lambda = \frac{25}{4}$, yields

$$\log E_6\left(\frac{25}{4}\right) = -\frac{1}{60}\left[\sum_{n=1}^{\infty}\frac{\zeta\left(2n+1,\frac{7}{2}\right)}{n+3}\left(\frac{5}{2}\right)^{2n+6} - \frac{5}{2}\sum_{n=2}^{\infty}\frac{\zeta\left(2n+1,\frac{7}{2}\right)}{n+2}\left(\frac{5}{2}\right)^{2n+4} + \frac{9}{16}\sum_{n=3}^{\infty}\frac{\zeta\left(2n+1,\frac{7}{2}\right)}{n+1}\left(\frac{5}{2}\right)^{2n+2}\right]. \tag{3.27}$$

By setting $n = 0, 1, 2$; $a = \frac{7}{2}$, and $t = \frac{5}{2}$ in (2.15), and using some identities recorded in Sections 1 and 2, we have

$$\sum_{n=1}^{\infty}\frac{\zeta\left(2n+1,\frac{7}{2}\right)}{n+1}\left(\frac{5}{2}\right)^{2n+2} = \frac{155}{12} - \frac{25}{4}\gamma - \frac{11}{12}\log 2 - \frac{5}{2}\log 3 - \frac{5}{2}\log 5 + 3\zeta'(-1); \tag{3.28}$$

$$\sum_{n=1}^{\infty}\frac{\zeta\left(2n+1,\frac{7}{2}\right)}{n+2}\left(\frac{5}{2}\right)^{2n+4} = \frac{3325}{2^6} - \frac{625}{2^5}\gamma + \frac{3\,561\,553}{2^5\cdot 5\cdot 3831}\log 2 - \frac{53}{8}\log 3 - \frac{125}{8}\log 5 + 6\zeta'(-1) + \frac{15}{8}\zeta'(-3); \tag{3.29}$$

and

$$\sum_{n=1}^{\infty} \frac{\zeta\left(2n+1,\frac{7}{2}\right)}{n+3}\left(\frac{5}{2}\right)^{2n+6} = \frac{623\,005}{2^8\cdot 3^2} - \frac{15\,625}{2^6\cdot 3}\gamma + \frac{253\,849}{2^6\cdot 3^2\cdot 7}\log 2 - \frac{485}{32}\log 3 - \frac{238\,627}{32}\log 5 + \frac{3133}{8}\zeta'(-1) + \frac{3}{2}\zeta'(-2) + \frac{507}{4}\zeta'(-3) + \frac{15}{8}\zeta'(-4) + \frac{63}{16}\zeta'(-5). \tag{3.30}$$

Applying (3.28) to (3.30) in (3.27), we obtain

$$\log E_6\left(\frac{25}{4}\right) = -\frac{4639}{2^{10}\cdot 3^3} + \frac{1385}{2^8\cdot 3^2}\gamma - \frac{246\,717\,677}{2^8\cdot 3^2\cdot 5\cdot 7\cdot 3831}\log 2 + \frac{118\,711}{2^6\cdot 3\cdot 5}\log 5 - \frac{6053}{2^6\cdot 3\cdot 5}\zeta'(-1) - \frac{1}{40}\zeta'(-2) - \frac{651}{2^6\cdot 5}\zeta'(-3) - \frac{1}{2^5}\zeta'(-4) - \frac{21}{2^6\cdot 5}\zeta'(-5) - \frac{413\,875}{2^{11}\cdot 3^2}\zeta(3) + \frac{96\,875}{2^{12}}\zeta(5). \tag{3.31}$$

If we set $n = 6$ in (3.18) and use (3.12), we get

$$\begin{aligned}\det{}'\Delta_6 &= D_6\left(\frac{25}{4}\right)\\ &= \exp\left[-Z_6'(0) - \frac{25}{4}\mathrm{FP}\,Z_6(1) - \frac{625}{32}\mathrm{FP}\,Z_6(2) - \frac{15\,625}{192}\mathrm{FP}\,Z_6(3) - \frac{625}{32}\left(C_{-2} + \frac{25}{8}C_{-3}\right)\right]\cdot E_6\left(\frac{25}{4}\right),\end{aligned}$$

which, upon using the above computations, yields

$$\det{}'\Delta_6 = 2^{-\frac{3\,990\,625}{735\,552}}\cdot 5^{\frac{116\,791}{960}}\cdot \exp\left[-\frac{38\,441\,354\,615\,245\,651}{5\,441\,253\,801\,984} - \frac{1511}{240}\zeta'(-1) - \frac{1}{40}\zeta'(-2) - \frac{2023}{960}\zeta'(-3) - \frac{1}{32}\zeta'(-4) - \frac{11}{480}\zeta'(-5)\right]. \tag{3.32}$$

By setting $n = 7$ in (3.17), we see that the shifted sequence of eigenvalues of Δ_7 on $\mathbf{S}^7$ is given as follows:

$$\begin{aligned}\lambda_k &= \mu_k + 9\\ &= (k+3)^2 \quad \text{with multiplicity} \quad \frac{1}{360}(k+1)(k+2)(k+3)^2(k+4)(k+5).\end{aligned} \tag{3.33}$$

It is found that

$$Z_7(s) = \frac{1}{360}\sum_{k=1}^{\infty}\frac{(k+1)(k+2)(k+3)^2(k+4)(k+5)}{(k+3)^{2s}}$$
$$= \frac{1}{360}[\zeta(2s-6) - 5\zeta(2s-4) + 4\zeta(2s-2)] - 3^{-2s}, \tag{3.34}$$

which shows us that $Z_7(s)$ has simple poles at $s = \frac{3}{2}, \frac{5}{2}$, and $\frac{7}{2}$ with their residues $\frac{1}{180}$, $-\frac{1}{144}$, and $\frac{1}{720}$, respectively. It is also observed that the sequence in (3.33) has the order $\mu = \frac{7}{2}$.

If we set $n = 7$ in (3.18) and use (3.12), we get

$$\det{}' \Delta_7 = D_7(9)$$
$$= \exp\left[-Z_7'(0) - 9\mathrm{FP}\,Z_7(1) - \frac{81}{2}\mathrm{FP}\,Z_7(2) - 243\mathrm{FP}\,Z_7(3) - \frac{81}{2}\left(C_{-2} + \frac{9}{2}C_{-3}\right)\right] \cdot E_7(9). \tag{3.35}$$

We can also easily verify each of the following evaluations:

$$C_{-2} = -\frac{81}{2}, \quad C_{-3} = \frac{243}{2}, \quad \mathrm{FP}\,Z_7(1) = -\frac{7}{60},$$
$$\mathrm{FP}\,Z_7(2) = -\frac{7}{2^4\cdot 3^4} + \frac{\pi^2}{540},$$
$$\mathrm{FP}\,Z_7(3) = -\frac{161}{2^4\cdot 3^6\cdot 5} - \frac{\pi^2}{2^4\cdot 3^3} + \frac{\pi^4}{2^2\cdot 3^4\cdot 5^2}$$

and

$$Z_7'(0) = -\frac{\zeta(7)}{32\pi^6} - \frac{\zeta(5)}{48\pi^4} - \frac{\zeta(3)}{180\pi^2} + 2\log 3.$$

It is observed that

$$E_7(\lambda) = \prod_{k=1}^{\infty}\left\{\left(1 - \frac{\lambda}{(k+3)^2}\right)^{\frac{1}{360}(k+1)(k+2)(k+3)^2(k+4)(k+5)} \cdot \exp\left[\frac{1}{360}(k+1)(k+2)(k+3)^2(k+4)(k+5) \cdot \left(\frac{\lambda}{(k+3)^2} + \frac{\lambda^2}{2(k+3)^4} + \frac{\lambda^3}{3(k+3)^6}\right)\right]\right\},$$

which, upon setting $\lambda = 9$ and taking logarithms on each side of the resulting equation, and considering the Taylor–Maclaurin expansion of $\log(1-x)$, yields

$$\log E_7(9) = -\frac{1}{360}\sum_{n=1}^{\infty}\frac{\zeta(2n,4)}{n+3}\cdot 3^{2n+6} + \frac{1}{72}\sum_{n=1}^{\infty}\frac{\zeta(2n,4)}{n+2}\cdot 3^{2n+4} - \frac{1}{90}\sum_{n=1}^{\infty}\frac{\zeta(2n,4)}{n+1}\cdot 3^{2n+2} - \frac{39}{80}\pi^2 + \frac{3}{100}\pi^4 + \frac{259}{240}. \tag{3.36}$$

By setting $a = 4$; $n = 1, 2, 3$, and $t = 3$ in (2.13), and using some identities already recorded in this presentation, we have

$$\sum_{n=1}^{\infty}\frac{\zeta(2n,4)}{n+1}\cdot 3^{2n+2} = \frac{63}{2} - 5\log 2 + 9\log 3 + 4\log 5 - 9\log\pi, \tag{3.37}$$

$$\sum_{n=1}^{\infty}\frac{\zeta(2n,4)}{n+2}\cdot 3^{2n+4} = \frac{1071}{4} - 29\log 2 + 243\log 3 + 16\log 5 - 81\log\pi - \frac{27\zeta(3)}{\pi^2}, \tag{3.38}$$

and

$$\sum_{n=1}^{\infty}\frac{\zeta(2n,4)}{n+3}\cdot 3^{2n+6} = \frac{32211}{10} - 125\log 2 + 3645\log 3 - 503\,936 \cdot \log 5 - 729\log\pi - \frac{1215\zeta(3)}{2\pi^2} + \frac{405\zeta(5)}{2\pi^4}. \tag{3.39}$$

If we apply (3.37) to (3.39) in (3.36), we get

$$\log E_7(9) = -\frac{789}{2^5\cdot 5} - \frac{137}{20}\log 3 + 1400\log 5 + \log\pi + \frac{21\zeta(3)}{16\pi^2} - \frac{9\zeta(5)}{16\pi^4} - \frac{39}{80}\pi^2 + \frac{3}{100}\pi^4. \tag{3.40}$$

Finally, from (3.36) and (3.40), and other previously recorded results, we obtain

$$\det{}'\Delta_7 = 3^{-\frac{177}{20}}\cdot 5^{1400}\cdot\pi\cdot\exp\left[-\frac{1\,230\,367}{60} + \frac{949\zeta(3)}{720\pi^2} - \frac{13\zeta(5)}{24\pi^4} + \frac{\zeta(7)}{32\pi^6}\right]. \tag{3.41}$$

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