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Inclusion properties of certain classes of meromorphic functions associated with the generalized hypergeometric function

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Dedicated to Professor H.M. Srivastava on the occasion of his 65th birthday

Abstract

The purpose of the present paper is to introduce several new classes of meromorphic functions defined by using a meromorphic analogue of the Choi–Saigo–Srivastava operator for the generalized hypergeometric function and investigate various inclusion properties of these classes. Some interesting applications involving these and other classes of integral operators are also considered.

Keywords: Meromorphic functions; Differential subordination; Meromorphic starlike functions; Meromorphic convex functions; Meromorphic close-to-convex functions; Generalized hypergeometric function; Choi–Saigo–Srivastava operator; Integral operator

1. Introduction

Let $\mathscr{M}$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k,$$

which are analytic in the punctured open unit disk $\mathbb{D} = \{z \in \mathbb{C} : 0 < |z| < 1\}$. If f and g are analytic in $\mathbb{U} = \mathbb{D} \cup \{0\}$, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function w in $\mathbb{U}$ such that $f(z) = g(w(z))$. For $0 \leqslant \eta, \beta < 1$, we denote by $\mathscr{MS}(\eta)$, $\mathscr{MK}(\eta)$ and $\mathscr{MC}(\eta, \beta)$ the subclasses of $\mathscr{M}$ consisting of all meromorphic functions which are, respectively, starlike of order η, convex of order η and close-to-convex of order β and type η in $\mathbb{U}$ (cf. e.g., [8,9,16]).

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Let $\mathcal{N}$ be the class of all functions ϕ which are analytic and univalent in $\mathbb{U}$ and for which $\phi(\mathbb{U})$ is convex with $\phi(0)=1$ and $\mathrm{Re}\{\phi(z)\}>0$ $(z\in\mathbb{U})$.

Making use of the principle of subordination between analytic functions, we introduce the subclasses $\mathcal{MS}(\eta,\phi)$, $\mathcal{MK}(\eta,\phi)$ and $\mathcal{MC}(\eta,\beta;\phi,\psi)$ of the class $\mathcal{M}$ for $0\leqslant\eta,\beta<1$ and $\phi,\psi\in\mathcal{N}$, which are defined by

$$\mathcal{MS}(\eta;\phi):=\left\{f\in\mathcal{M}:\frac{1}{1-\eta}\left(-\frac{zf'(z)}{f(z)}-\eta\right)\prec\phi(z)\quad\text{in }\mathbb{U}\right\},$$
$$\mathcal{MK}(\eta;\phi):=\left\{f\in\mathcal{M}:\frac{1}{1-\eta}\left(-\left\{1+\frac{zf''(z)}{f'(z)}\right\}-\eta\right)\prec\phi(z)\quad\text{in }\mathbb{U}\right\}$$

and

$$\mathcal{MC}(\eta,\beta;\phi,\psi):=\left\{f\in\mathcal{M}:\exists g\in\mathcal{MS}(\eta;\phi)\quad\text{s.t. }\frac{1}{1-\beta}\left(-\frac{zf'(z)}{g(z)}-\beta\right)\prec\psi(z)\quad\text{in }\mathbb{U}\right\}.$$

We note that the classes mentioned above is the familiar classes which have been used widely on the space of analytic and univalent functions in $\mathbb{U}$ [2,14] and for special choices for the functions ϕ and ψ involved in these definitions, we can obtain the well-known subclasses of $\mathcal{M}$. For examples, we have

$$\mathcal{MS}\left(\eta;\frac{1+z}{1-z}\right)=\mathcal{MS}(\eta),\quad\mathcal{MK}\left(\eta;\frac{1+z}{1-z}\right)=\mathcal{MK}(\eta)$$

and

$$\mathcal{MC}\left(\eta,\beta;\frac{1+z}{1-z},\frac{1+z}{1-z}\right)=\mathcal{MC}(\eta,\beta).$$

For complex parameters

$$\alpha_1,\ldots,\alpha_q\quad\text{and}\quad\beta_1,\ldots,\beta_s$$
$$(\beta_j\in\mathbb{C}\setminus\mathbb{Z}_0^-;\mathbb{Z}_0^-:=\{0,-1,-2,\ldots\};j=1,\ldots,s),$$

we now define the generalized hypergeometric function ${}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_q;z)$ [19,20] as follows:

$${}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_q;z):=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdots(\alpha_q)_k}{(\beta_1)_k\cdots(\beta_s)_k}\frac{z^k}{k!},$$
$$(q\leqslant s+1;q,s\in\mathbb{N}_0:=\mathbb{N}\cup\{0\};\mathbb{N}:=\{1,2,\ldots\};z\in\mathbb{U}),$$

where $(v)_k$ is the Pochhammer symbol (or the shifted factorial) defined (in terms of the Gamma function) by

$$(v)_k:=\frac{\Gamma(v+k)}{\Gamma(v)}=\begin{cases}1 & \text{if } k=0 \text{ and } v\in\mathbb{C}\setminus\{0\},\\ v(v+1)\cdots(v+k-1) & \text{if } k\in\mathbb{N}\text{ and } v\in\mathbb{C}.\end{cases}$$

Corresponding to a function $\mathcal{F}(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)$ defined by

$$\mathcal{F}(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z):=z^{-1}{}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z).\tag{1.1}$$

Liu and Srivastava [13] considered a linear operator $H(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s):\mathcal{M}\to\mathcal{M}$ defined by the following Hadamard product (or convolution):

$$H(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f(z):=\mathcal{F}(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)*f(z).\tag{1.2}$$

We note that the linear operator $H(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)$ was motivated essentially by Dziok and Srivastava [3]. Some interesting developments associated with the generalized hypergeometric function were considered recently by Dziok and Srivastava [4,5] and Liu and Srivastava [11,12].

Corresponding to the function $\mathcal{F}(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)$ defined by (1.1), we introduce a function $\mathcal{F}_\lambda(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)$ given by

$$\mathcal{F}(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)*\mathcal{F}_\lambda(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)=\frac{1}{z(1-z)^\lambda}\quad(\lambda>0).\tag{1.3}$$

Analogous to $H(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)$ defined by (1.2), we now define the linear operator $H_\lambda(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)$ on Σ as follows:

$$H_\lambda(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f(z)=\mathscr{F}_\lambda(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)*f(z) \quad (1.4)$$
$$(\alpha_i,\beta_j\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ i=1,\ldots,q;\ j=1,\ldots,s;\ \lambda>0;\ z\in\mathbb{D};\ f\in\mathscr{M}).$$

For convenience, we write

$$H_{\lambda,q,s}(\alpha_1):=H_\lambda(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s).$$

It is easily verified from the definition (1.3) and (1.4) that

$$z(H_{\lambda,q,s}(\alpha_1+1)f(z))'=\alpha_1H_{\lambda,q,s}(\alpha_1)f(z)-(\alpha_1+1)H_{\lambda,q,s}(\alpha_1+1)f(z) \quad (1.5)$$

and

$$z(H_{\lambda,q,s}(\alpha_1)f(z))'=\lambda H_{\lambda+1,q,s}(\alpha_1)f(z)-(\lambda+1)H_{\lambda,q,s}(\alpha_1)f(z). \quad (1.6)$$

We note that the operator $H_{\lambda,q,s}(\alpha_1)$ is closely related to the Choi–Saigo–Srivastava operator [2] for analytic functions, which includes the integral operator studied by Liu [10] and Noor et al. [17,18].

Next, by using the operator $H_{\lambda,q,s}(\alpha_1)$, we introduce the following classes of meromprphic functions for $\phi,\psi\in\mathscr{N}$, $\lambda>0$ and $0\leqslant\eta,\beta<1$:

$$\mathscr{MS}_{\lambda,\alpha_1}(q,s;\eta;\phi):=\{f\in\mathscr{M}:H_{\lambda,q,s}(\alpha_1)f\in\mathscr{MS}(\eta;\phi)\},$$
$$\mathscr{MK}_{\lambda,\alpha_1}(q,s;\eta;\phi):=\{f\in\mathscr{M}:H_{\lambda,q,s}(\alpha_1)f\in\mathscr{MK}(\eta;\phi)\}$$

and

$$\mathscr{MC}_{\lambda,\alpha_1}(q,s;\eta,\beta;\phi,\psi):=\{f\in\mathscr{M}:H_{\lambda,q,s}(\alpha_1)f\in\mathscr{MC}(\eta,\beta;\phi,\psi)\}.$$

We also note that

$$f(z)\in\mathscr{MK}_{\lambda,\alpha_1}(q,s;\eta;\phi)\iff -zf'(z)\in\mathscr{MS}_{\lambda,\alpha_1}(q,s;\eta;\phi). \quad (1.7)$$

In particular, we set

$$\mathscr{MS}_{\lambda,\alpha_1}\left(q,s;\eta;\frac{1+Az}{1+Bz}\right)=:\mathscr{MS}_{\lambda,\alpha_1}(q,s;\eta;A,B)\quad(-1<B<A\leqslant1)$$

and

$$\mathscr{MK}_{\lambda,\alpha_1}\left(q,s;\eta;\frac{1+Az}{1+Bz}\right)=:\mathscr{MK}_{\lambda,\alpha_1}(q,s;\eta;A,B)\quad(-1<B<A\leqslant1).$$

In this paper, we investgate several inclusion properties of the classes $\mathscr{MS}_{\lambda,\alpha_1}(q,s;\eta;\phi)$, $\mathscr{MK}_{\lambda,\alpha_1}(q,s;\eta;\phi)$ and $\mathscr{MC}_{\lambda,\alpha_1}(q,s;\eta,\beta;\phi,\psi)$ associated with the operator $H_{\lambda,q,s}(\alpha_1)$. Some applications involving integral operators are also considered.

2. Inclusion properties involving the operator $H_{\lambda,q,s}(\alpha_1)$

The following results will be required in our investigation.

Lemma 1 [6]. *Let ϕ be convex univalent in $\mathbb{U}$ with $\phi(0)=1$ and $\mathrm{Re}\{\kappa\phi(z)+\nu\}>0$ $(\kappa,\nu\in\mathbb{C})$. If p is analytic in $\mathbb{U}$ with $p(0)=1$, then*

$$p(z)+\frac{zp'(z)}{\kappa p(z)+\nu}\prec\phi(z)\quad(z\in\mathbb{U})$$

implies

$$p(z)\prec\phi(z)\quad(z\in\mathbb{U}).$$

Lemma 2 [15]. *Let ϕ be convex univalent in $\mathbb{U}$ and ω be analytic in $\mathbb{U}$ with* $\mathrm{Re}\{\omega(z)\} \geqslant 0$. *If p is analytic in $\mathbb{U}$ and $p(0) = \phi(0)$, then*

$$p(z) + \omega(z)zp'(z) \prec \phi(z) \quad (z \in \mathbb{U})$$

implies

$$p(z) \prec \phi(z) \quad (z \in \mathbb{U}).$$

At first, with the help of Lemma 1, we obtain the following:

Theorem 1. *Let $\phi \in \mathcal{N}$ with* $\max_{z\in\mathbb{U}}\mathrm{Re}\{\phi(z)\} < \min\{(\lambda+1-\eta)/(1-\eta), (\alpha_1+1-\eta)/(1-\eta)\}$ $(\lambda, \alpha_1 > 0;\ 0 \leqslant \eta < 1)$. *Then*

$$\mathcal{MS}_{\lambda+1,\alpha_1}(q,s;\eta;\phi) \subset \mathcal{MS}_{\lambda,\alpha_1}(q,s;\eta;\phi) \subset \mathcal{MS}_{\lambda,\alpha_1+1}(q,s;\eta;\phi).$$

Proof. To prove the first part of Theorem 1, let $f \in \mathcal{MS}_{\lambda+1,\alpha_1}(q,s;\eta;\phi)$ and set

$$p(z) = \frac{1}{1-\eta}\left(-\frac{z(H_{\lambda,q,s}(\alpha_1)f(z))'}{H_{\lambda,q,s}(\alpha_1)f(z)} - \eta\right), \tag{2.1}$$

where p is analytic in $\mathbb{U}$ with $p(0) = 1$. Applying (1.6) and (2.1), we obtain

$$\frac{1}{1-\eta}\left(-\frac{z(H_{\lambda+1,q,s}(\alpha_1)f(z))'}{H_{\lambda+1,q,s}(\alpha_1)f(z)} - \eta\right) = p(z) + \frac{zp'(z)}{-(1-\eta)p(z)+\lambda+1-\eta} \quad (z \in \mathbb{U}). \tag{2.2}$$

Since $\max_{z\in\mathbb{U}}\mathrm{Re}\{\phi(z)\} < (\lambda+1-\eta)/(1-\eta)$, we see that

$$\mathrm{Re}\{-(1-\eta)\phi(z)+\lambda+1-\eta\} > 0 (z \in \mathbb{U}).$$

Applying Lemma 1 to (2.3), it follows that $p \prec \phi$, that is, $f \in \mathcal{MS}_{\lambda,\alpha_1}(q,s;\eta;\phi)$. Moreover, by using the arguments similar to those detailed above with (1.5), we can prove the second part of Theorem 1. Therefore we complete the proof of Theorem 1. □

Theorem 2. *Let $\phi \in \mathcal{N}$ with* $\max_{z\in\mathbb{U}}\mathrm{Re}\{\phi(z)\} < \min\{(\lambda+1-\eta)/(1-\eta), (\alpha_1+1-\eta)/(1-\eta)\}$ $(\lambda, \alpha_1 > 0;\ 0 \leqslant \eta < 1)$. *Then*

$$\mathcal{MK}_{\lambda+1,\alpha_1}(q,s;\eta;\phi) \subset \mathcal{MK}_{\lambda,\alpha_1}(q,s;\eta;\phi) \subset \mathcal{MK}_{\lambda,\alpha_1+1}(q,s;\eta;\phi).$$

Proof. Applying (1.7) and Theorem 1, we observe that

$$\begin{aligned} f(z) \in \mathcal{MK}_{\lambda+1,\alpha_1}(q,s;\eta;\phi) &\iff -zf'(z) \in \mathcal{MS}_{\lambda+1,\alpha_1}(q,s;\eta;\phi) \Rightarrow -zf'(z) \in \mathcal{MS}_{\lambda,\alpha_1}(q,s;\eta;\phi) \\ &\iff f(z) \in \mathcal{MK}_{\lambda,\alpha_1}(q,s;\eta;\phi), \end{aligned}$$

and

$$\begin{aligned} f(z) \in \mathcal{MK}_{\lambda,\alpha_1}(q,s;\eta;\phi) &\iff -zf'(z) \in \mathcal{MS}_{\lambda,\alpha_1}(q,s;\eta;\phi) \Rightarrow -zf'(z) \in \mathcal{MS}_{\lambda,\alpha_1+1}(q,s;\eta;\phi) \\ &\iff f(z) \in \mathcal{MK}_{\lambda,\alpha_1+1}(q,s;\eta;\phi), \end{aligned}$$

which evidently proves Theorem 2. □

Taking

$$\phi(z) = \frac{1+Az}{1+Bz} \quad (-1 < B < A \leqslant 1; z \in \mathbb{U})$$

in Theorems 1 and 2, we have

Corollary 1. *Let* $(1+A)/(1+B) < \min\{(\lambda+1-\eta)/(1-\eta), (\alpha_1+1-\eta)/(1-\eta)\}$ $(\lambda, \alpha_1 > 0;\ 0 \leqslant \eta < 1;\ -1 < B < A \leqslant 1)$. *Then*

$$\mathcal{MS}_{\lambda+1,\alpha_1}(q,s;\eta;A,B) \subset \mathcal{MS}_{\lambda,\alpha_1}(q,s;\eta;A,B) \subset \mathcal{MS}_{\lambda,\alpha_1+1}(q,s;\eta;A,B)$$

and

$$\mathcal{MK}_{\lambda+1,\alpha_1}(q,s;\eta;A,B) \subset \mathcal{MK}_{\lambda,\alpha_1}(q,s;\eta;A,B) \subset \mathcal{MK}_{\lambda,\alpha_1+1}(q,s;\eta;A,B).$$

Next, by using Lemma 2, we obtain the following inclusion relation for the class $\mathcal{MC}_{\lambda,\alpha_1}(q,s;\eta,\beta;\phi,\psi)$.

Theorem 3. *Let* $\phi,\psi \in \mathcal{N}$ *with* $\max_{z\in\mathbb{U}}\text{Re}\{\phi(z)\} < \min\{(\lambda+1-\eta)/(1-\eta),(\alpha_1+1-\eta)/(1-\eta)\}$ $(\lambda,\alpha_1 > 0;$ $0 \leqslant \eta < 1)$. *Then*

$$\mathcal{MC}_{\lambda+1,\alpha_1}(q,s;\eta,\beta;\phi,\psi) \subset \mathcal{MC}_{\lambda,\alpha_1}(q,s;\eta,\beta;\phi,\psi) \subset \mathcal{MC}_{\lambda,\alpha_1+1}(q,s;\eta,\beta;\phi,\psi).$$

Proof. To prove the first inclusion of Theorem 3, let $f \in \mathcal{MC}_{\lambda+1,\alpha_1}(q,s;\eta,\beta;\phi,\psi)$. Then, from the definition of $\mathcal{MC}_{\lambda+1,\alpha_1}(q,s;\eta,\beta;\phi,\psi)$, there exists a function $g \in \mathcal{MS}_{\lambda+1,\alpha_1}(q,s;\eta;\phi)$ such that

$$\frac{1}{1-\beta}\left(-\frac{z(H_{\lambda+1,q,s}(\alpha_1)f(z))'}{H_{\lambda+1,q,s}(\alpha_1)g(z)}-\beta\right) \prec \psi(z) \quad (z \in \mathbb{U}).$$

Now let

$$p(z) = \frac{1}{1-\beta}\left(-\frac{z(H_{\lambda,q,s}(\alpha_1)f(z))'}{H_{\lambda,q,s}(\alpha_1)g(z)}-\beta\right), \tag{2.3}$$

where p is analytic in $\mathbb{U}$ with $p(0) = 1$. Using (1.6), we obtain

$$\frac{1}{1-\beta}\left(-\frac{z(H_{\lambda+1,q,s}(\alpha_1)f(z))'}{H_{\lambda+1,q,s}(\alpha_1)g(z)}-\beta\right) = \frac{1}{1-\beta}\left(\frac{\frac{z(H_{\lambda,q,s}(\alpha_1)(-zf'(z)))'}{H_{\lambda,q,s}(\alpha_1)g(z)}+(\lambda+1)\frac{H_{\lambda,q,s}(\alpha_1)(-zf'(z))}{H_{\lambda,q,s}(\alpha_1)g(z)}}{\frac{z(H_{\lambda,q,s}(\alpha_1)g(z))'}{H_{\lambda,q,s}(\alpha_1)g(z)}+\lambda+1}-\beta\right). \tag{2.4}$$

Since $g \in \mathcal{MS}_{\lambda+1,\alpha_1}(q,s;\eta;\phi) \subset \mathcal{MS}_{\lambda,\alpha_1}(q,s;\eta;\phi)$, by Theorem 1, we set

$$q(z) = \frac{1}{1-\eta}\left(-\frac{z(H_{\lambda,q,s}(\alpha_1)g(z))'}{H_{\lambda,q,s}(\alpha_1)g(z)}-\eta\right), \tag{2.5}$$

where $q \prec \phi$ in $\mathbb{U}$ with the assumption for $\phi \in \mathcal{N}$. Then, by virtue of (2.3), (2.4) and (2.5), we observe that

$$\frac{1}{1-\beta}\left(-\frac{z(H_{\lambda+1,q,s}(\alpha_1)f(z))'}{H_{\lambda+1,q,s}(\alpha_1)g(z)}-\beta\right) = p(z) + \frac{zp'(z)}{-(1-\eta)q(z)+\lambda+1-\eta} \prec \psi(z) \quad (z \in \mathbb{U}). \tag{2.6}$$

Since $\lambda > 0$ and $q \prec \phi$ in $\mathbb{U}$ with $\max_{z\in\mathbb{U}}\text{Re}\{\phi(z)\} < (\lambda+1-\eta)/(1-\eta)$,

$$\text{Re}\{-(1-\eta)q(z)+\lambda+1-\eta\} > 0 \quad (z \in \mathbb{U}).$$

Hence, by taking

$$\omega(z) = \frac{1}{-(1-\eta)q(z)+\lambda+1-\eta}$$

in (2.6), and applying Lemma 2, we can show that $p \prec \psi$ in $\mathbb{U}$, so that $f \in \mathcal{MC}_{\lambda,\alpha_1}(q,s;\eta,\beta;\phi,\psi)$. Moreover, we have the second inclusion by using arguments similar to those detailed above with (1.5). Therefore we complete the proof of Theorem 3. $\square$

3. Inclusion properties involving the integral operator F_μ

In this section, we consider the integral operator F_μ (see, e.g., [8]) defined by

$$F_\mu(f) := F_\mu(f)(z) = \frac{\mu}{z^{\mu+1}}\int_0^z t^\mu f(t)\,dt \quad (f \in \mathcal{M};\ \mu > 0). \tag{3.1}$$

From the definition of F_μ defined by (3.1), we observe that

$$z(H_{\lambda,q,s}(\alpha_1)F_\mu(f)(z))' = \mu H_{\lambda,q,s}(\alpha_1)f(z) - (\mu+1)H_{\lambda,q,s}(\alpha_1)F_\mu(f)(z).$$

We first state Theorem 4 below, the proof of which is much akin to that of Theorem 1.

Theorem 4. *Let* $\phi \in \mathcal{N}$ *with* $\max_{z\in\mathbb{U}}\mathrm{Re}\{\phi(z)\} < (\mu+1-\eta)/(1-\eta)$ $(\mu > 0; 0 \leqslant \eta < 1)$. *If* $f \in \mathcal{MS}_{\lambda,\alpha_1}(q,s;\eta;\phi)$, *then* $F_\mu(f) \in \mathcal{MS}_{\lambda,\alpha_1}(q,s;\eta;\phi)$.

Next, we derive an inclusion property involving F_μ, which is obtained by applying (1.7) and Theorem 4.

Theorem 5. *Let* $\phi \in \mathcal{N}$ *with* $\max_{z\in\mathbb{U}}\mathrm{Re}\{\phi(z)\} < (\mu+1-\eta)/(1-\eta)$ $(\mu > 0; 0 \leqslant \eta < 1)$. *If* $f \in \mathcal{MK}_{\lambda,\alpha_1}(q,s;\eta;\phi)$, *then* $F_\mu(f) \in \mathcal{MK}_{\lambda,\alpha_1}(q,s;\eta;\phi)$.

From Theorems 4 and 5, we have

Corollary 2. *Let* $(1 + A)/(1 + B) < (\mu + 1 - \eta)/(1 - \eta)$ $(\mu > 0; -1 < B < A \leqslant 1; 0 \leqslant \eta < 1)$. *Then if* $f \in \mathcal{MS}_{\lambda,\alpha_1}(q,s;\eta;A,B)$ *(or* $\mathcal{MK}_{\lambda,\alpha_1}(q,s;\eta;A,B)$*), then* $F_c(f) \in \mathcal{MS}_{\lambda,\alpha_1}(q,s;\eta;A,B)$ *(or* $\mathcal{MK}_{\lambda,\alpha_1}(q,s;\eta;A,B)$*).*

Finally, we obtain Theorem 6 below by using the same techniques as in the proof of Theorem 3.

Theorem 6. *Let* $\phi,\psi \in \mathcal{N}$ *with* $\max_{z\in\mathbb{U}}\mathrm{Re}\{\phi(z)\} < (\mu+1-\eta)/(1-\eta)$ $(\mu > 0; 0 \leqslant \eta < 1)$. *If* $f \in \mathcal{MC}_{\lambda,\alpha_1}(q,s;\eta,\beta;\phi,\psi)$, *then* $F_c(f) \in \mathcal{MC}_{\lambda,\alpha_1}(q,s;\eta,\beta;\phi,\psi)$.

Remark 1. If we take $\lambda = \alpha_1 = 2$, $\beta_1 = 1$, $\alpha_i = \beta_i$ $(i = 2,\ldots,s)$ and $\alpha_{s+1} = 1$ in all theorems of this section, then we extend the results by Goel and Sohi [7], which reduce the results earlier obtained by Bajpai [1].

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