Image Space Analysis and Scalarization of Multivalued Optimization

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IMAGE SPACE ANALYSIS AND SCALARIZATION OF MULTIVALUED OPTIMIZATION [1]

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Abstract. Using a new method based on generalized sections of feasible sets, we obtain optimality conditions for vector optimization of objective multifunctions with multivalued constraints.
Key Words. C-multifunctions. Scalarization of vector optimization. Superefficiency. Image space analysis.

1. Introduction and Preliminaries

In recent years, there has been a growing interest in vector optimization problems, in particular problems with multivalued constraints and objective multifunctions. The Image Space approach (IS) was initiated in [1] and carried on in some articles; see [2] and references therein. The IS approach has been proved to be a fruitful method in many topics of optimization theory (e.g., optimality condition, existence of solution, duality, and stability); see [1 - 6]. Song [7] investigated cone separation between two suitable subsets of the image space and applied his results to study optimality conditions for nonconvex multifunction in locally convex spaces. In this paper, we introduce a method of scalarization of vector constrained problems based on some generalized sections of its feasible sets. In fact, we extend to multifunctions the method applied for scalarization of vector-valued functions in [1-6, 8],which relies on the level sets of objective mappings.
The paper is organized as follows: In Section 1, we present some basic concepts and different types of solutions of a vector optimization problem. We also apply the directional derivatives technique used by Yang [9] to characterize optimality conditions for vector optimization of multifunctions. In Section 2, by introducing an assumption on the objective multifunctions, we extend a method for scalarization of vector valued functions to multifunctions. In Section 3, we continue our work to obtain some other solutions for vector optimization problems.
Let X be a topological vector space, let Y and Z be two normed linear spaces with norm dual spaces Y^* and Z^*, respectively. Let $C \subset Y$ and $D \subset Z$ be pointed closed and convex cones with int $C \neq \emptyset$ and int $D \neq \emptyset$. For a nonempty subset V of Y, the generated cone of V is given by

$$\text{cone } V := \{\lambda v : \lambda \geq 0, v \in V\} = \cup_{\lambda \geq 0} \lambda V.$$

A convex subset Θ of C is a base for C iff $C = \text{cone } \Theta$ and $0 \notin \text{cl } \Theta$. The positive dual cone C^+ of C is defined by

$$C^+ := \{p \in Y^* : p(y) \geq 0,\ \forall y \in C\},$$

and the set of all positive linear functionals in C^+ is

$$C^{+i} := \{p \in Y^* : p(y) > 0,\ \forall y \in C \setminus \{0\}\}.$$

Note that, if C is a convex cone in Y, then int $C^+ \subseteq C^{+i}$ and equality holds if int $C^+ \neq \emptyset$ [10]. A partial order $\leq_C$ in Y is defined by

$$y_1 \leq_C y_2 \ \Leftrightarrow\ y_2 - y_1 \in C,\ \forall y_1, y_2 \in Y.$$

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In the sequel, we suppose that $F : U \rightrightarrows Y$ be a multifunction defined on a nonempty subset U of X with values in Y, which is partially ordered by cone C. The set

$$\text{dom } F := \{x : F(x) \neq \emptyset\}$$

is called the *domain* of F. The set

$$\text{gr } F := \{(x, y) : x \in \text{dom } F,\ y \in F(x)\} = \bigcup_{x \in \text{dom } F} [\{x\} \times F(x)]$$

is called the *graph* of F and the set

$$\text{epi } F = \{(x, y) : x \in \text{dom } F,\ y \in F(x) + C\} = \bigcup_{x \in \text{dom } F} [\{x\} \times (F(x) + C)]$$

is called the *epigraph* of F.

For simplicity, throughout this paper, we denote $\overset{\circ}{C} := \text{int } C$ and $C_0 := C \setminus \{0\}$.

Definition 1.1 [11 - 12]. Let U be a convex subset of X. A multifunction $F : U \rightrightarrows Y$ is said to be:

(a) C-multifunction on U iff, for all $x_1, x_2 \in U$ and $t \in [0, 1]$, we have

$$tF(x_1) + (1 - t)F(x_2) \subseteq F(tx_1 + (1 - t)x_2) + C,$$

or iff epi F is convex;

(b) quasi C-multifunction on U iff, for all $x_1, x_2 \in U$ and $t \in [0, 1]$, we have

$$(F(x_1) + C) \cap (F(x_2) + C) \subseteq F(tx_1 + (1 - t)x_2)) + C;$$

(c) strictly quasi C-multifunction on U iff, for all $x_1, x_2 \in U$, $x_1 \neq x_2$ and $t \in]0, 1[$, we have

$$(F(x_1) + C) \cap (F(x_2) + C) \subseteq F(tx_1 + (1 - t)x_2) + \overset{\circ}{C}.$$

Remark 1.1. When $C = \mathbb{R}^m_+$, then (a) characterizes convex multifunctions; when $C = \mathbb{R}^m_-$, then (a) characterizes concave multifunctions; when $C \supseteq \mathbb{R}^m_+$, then (a) characterizes epiconvex C-multifunctions; finally, when $C \supseteq \mathbb{R}^m_-$, then (a) characterizes epiconcave C-multifunctions. We can do similar remarks for parts (b) and (c) of Definition 1.1.

The following lemma gives a characterization of a quasi C-multifunction in terms of its generalized level sets.

Lemma 1.1 [12]. *Let $F : U \rightrightarrows Y$ be a multifunction. Then, F is a quasi C-multifunction on U if and only if, for each $y \in Y$, the following generalized level set is convex:*

$$\text{lev}_{\leqslant_C} y - F(.) := \{x \in U : y \in F(x) + C\}.$$ □

Definition 1.2 Let V be a nonempty subset of Y. The set of all vector minimum points of V with respect to C is defined as follows:

$$S(V, C) := \{y \in V : (V \setminus \{y\}) \cap (y - C) = \emptyset\}. \tag{1}$$

Let $F : U \rightrightarrows Y$ and $G : U \rightrightarrows Z$ be two multifunctions with nonempty values. We consider the following vector optimization problem:

$$\min_C\ F(x) \quad \text{s.t.} \quad x \in R := \{x \in U : G(x) \cap (-D) \neq \emptyset\}, \tag{2}$$

where R is called the feasible region of problem (2).

Remark 1.2 It is obvious that, if G is a quasi D-multifunction on a convex subset U of X, then R is a convex set.

Definition 1.3 Let $\bar{x} \in R$ and set $\bar{X} := \{x \in R : F(x) = F(\bar{x})\}$. $\bar{x}$ is called a feeble multifunction minimum point (f.m.m.p.) of problem (2) iff

$$\exists \bar{y} \in F(\bar{x}), \quad \text{s.t.} \quad (F(R \backslash \{\bar{x}\})) \cap (\bar{y} - C_0) = \emptyset; \tag{3}$$

$\bar{x}$ is called a multifunction minimum point (m.m.p.) of problem (2) iff

$$(F(R \setminus \{\bar{x}\})) \cap (y - C_0) = \emptyset, \ \forall y \in F(\bar{x}). \tag{4}$$

The set of all $\bar{x} \in R$ which fulfill (3) or (4) is denoted by $\hat{S}(F, C)$ and $S(F, C)$, respectively. When $R \setminus \{\bar{x}\}$ is replaced by $(R \setminus \{\bar{x}\}) \cap N(\bar{x})$ in (3) and in (4), $N(\bar{x})$ being a neighborhood of $\bar{x}$, then of course we have a local f.m.m.p. and a local m.m.p., respectively.

Definition 1.4 A point $\bar{x} \in R$ is called a feeble multifunction weak minimum point (f.m.w.m.p.) of problem (2) iff

$$\exists \bar{y} \in F(\bar{x}) \quad \text{s.t.} \quad (F(R \backslash \{\bar{x}\})) \cap (\bar{y} - \mathring{C}) = \emptyset; \tag{5}$$

$\bar{x}$ is called a multifunction weak minimum point (m.w.m.p.) of problem (2) iff

$$(F(R \setminus \{\bar{x}\})) \cap (y - \mathring{C}) = \emptyset, \ \forall y \in F(\bar{x}). \tag{6}$$

The set of all $\bar{x} \in R$, which fulfill (5) or (6) is denoted by $\hat{WS}(F, C)$ and $WS(F, C)$, respectively. When $R \setminus \{\bar{x}\}$ is replaced by $(R \setminus \{\bar{x}\}) \cap N(\bar{x})$ in (5)and (6), $N(\bar{x})$ being a neighborhood of $\bar{x}$, then of course we have a local multifunction weak minimum of problem (2).

If either $(\bar{x}, \bar{y})$ or $(\bar{x}, y) \in \text{gr } F$ satisfies one of the above definitions, then it is called a minimizer of F.

The following lemma presents a necessary and sufficient condition for a vector being a f. m. m. p. or m. m .p. of problem (2).

Lemma 1.2 [13]. *Let $\bar{x} \in R$ and $(\bar{x}, \bar{y}) \in \text{gr } F$. Then*
(a) $(\bar{x}, \bar{y})$ is a feeble minimizer of problem (2), if and only if

$$(\bar{y} - C_0, -D) \cap (F(x), G(x)) = \emptyset \quad \forall x \in U \setminus \{\bar{x}\}.$$

(b) *$(\bar{x}, y)$ is a minimizer of problem (2), if and only if*

$$(y - C_0, -D) \cap (F(x), G(x)) = \emptyset \quad \forall x \in U \setminus \{\bar{x}\}, y \in F(\bar{x}). \qquad \square$$

Remark 1.3 By a similar way, we can deduce other types of minima such as weak or local solution of problem (2).

Lemma 1.3 *Let $F : U \rightrightarrows Y$ be strictly quasi C-multifunction and $G : U \rightrightarrows Z$ be quasi D-multifunction. Then $\hat{S}(F, C) = \hat{WS}(F, C)$.*

Proof. Let $x_0 \in \hat{WC}(F,C)$ and suppose that $x_0 \notin \hat{S}(F,C)$. Then there exist $\hat{x} \neq x_0$, in R , $\hat{y} \in F(\hat{x})$ and $(x_0, y_0) \in \text{gr } F$ such that $y_0 - \hat{y} \in C$. Since F is strictly quasi C-multifunction, then for all $t \in]0,1[$, we have:

$$y_0 \in (F(x_0) + C) \cap (F(\hat{x}) + C) \subset F(tx_0 + (1-t)\hat{x})) + \mathring{C}$$

Hence,

$$\forall t \in]0,1[, \quad \exists y_t \in F(tx_0 + (1-t)\hat{x}) \quad \text{s.t.} \quad (y_0 - y_t) \in \mathring{C}.$$

Now by convexity of R, we deduce that $y_t \in F(R \setminus \{x_0\}) \cap (y_0 - \mathring{C})$, which contradicts our assumption. The other inclusion is trivial. □

Definition 1.5 Let V be a nonempty subset in Y. A point $y_0 \in V$ is said to be properly positive solution of V, iff there exists some $p \in C^{+i}$, such that:

$$p(y_0) \leq p(y), \ \forall \ y \in V.$$

A point $x_0 \in U$ is a properly positive solution of problem (2), iff there exists $y_0 \in F(x_0)$, such that y_0 is a properly positive solution of $F(R)$.

The next two results are based on the following definition of Yang [9].

Definition 1.6 Let $S(x_0, R)$ be the cone of feasible directions; i.e.

$$S(x_0, R) := \{v \in X : \exists \delta > 0, x_0 + tv \in R, 0 < t < \delta\}.$$

Then, the limit set of F at x_0 on the direction $v \in S(x_0, F)$ is

$$Y_F^{y_0}(x_0, v) := \{z : z = \lim_{(t_n, u_n) \to (0^+, v)} \frac{f(x_0 + t_n u_n) - y_0}{t_n} \quad u_n \in S(x_0, R)\},$$

where f is any continuous selection of F with $f(x_0) = y_0$. The union of all limit set of F at x_0 on all directions $v \in S(x_0, R)$ is denoted by $Y_F^{y_0}(x_0, S(x_0, R))$.

In the following theorem we obtain a local version of part (i) of Theorem 3.1 in [9].

Theorem 1.1 *Let $x_0 \in R$ and $y_0 \in F(x_0)$ be a local weak solution of the problem (2). Then*

$$Y_F^{y_0}(x_0, S(x_0, R)) \cap (-\mathring{C}) = \emptyset.$$

Proof. Suppose that $x_0 \in R$ is a local weak solution of problem (2) and f is any continuous selection of F such that $y_0 = f(x_0)$. Let $N(x_0)$ be a neighborhood of x_0 such that

$$(F(R \setminus \{x_0\}) \cap N(x_0)) \cap (y_0 - \mathring{C}) = \emptyset.$$

If $z \in Y_F^{y_0}(x_0, S(x_0, R))$, then there exist $v, u_n \in S(x_0, R), u_n \to v, t_n \to 0^+$ such that

$$z = \lim_{(t_n, u_n) \to (0^+, v)} \frac{f(x_0 + t_n u_n) - f(x_0)}{t_n}$$

for some continuous selection f of F. Let $x_n = x_0 + t_n u_n$; then, there exists $n_0 \in N$ such that $x_n \in (R \setminus \{x_0\}) \cap N(x_0)$ for $n \geq n_0$, and hence

$$(F(x_n) - y_0) \cap (-\mathring{C}) = \emptyset \quad , \forall n \geqslant n_0.$$

Furthermore, since f is a continuous selection of F and $y_0 = f(x_0)$, therefore

$$(f(x_n) - f(x_0)) \cap (-\mathring{C}) = \emptyset, \quad \forall n \geqslant n_0.$$

Hence

$$\frac{f(x_n) - f(x_0)}{t_n} \cap (-\overset{\circ}{C}) = \emptyset \quad \forall n \geqslant n_0.$$

Since $-\overset{\circ}{C}$ is an open set, it follows that $z \notin -\overset{\circ}{C}$. □

For the converse of Theorem 1.1, we have the following refinement of part(ii) of Theorem 3.1 in [9].

Theorem 1.2 *Let $x_0 \in R$, $y_0 \in F(x_0)$, $F : U \rightrightarrows Y$ be pseudo C-multifunction at (x_0, y_0), $G : U \rightrightarrows Z$ be quasi D-multifunction and f be a continuous selection of F such that $y_0 = f(x_0)$ with*

$$T(\text{epi } F, (x_0, y_0)) = T(\text{epi } f, (x_0, y_0)).$$

If $Y_F^{y_0}(x_0, S(x_0, R)) \cap (-C) = \emptyset$, then (x_0, y_0) is a feeble minimizer of problem (2).

Proof. As G is a quasi D-multifunction, then R is a convex set and $x - x_0 \in S(x_0, R)$. Since F is pseudo C-multifunction at (x_0, y_0), we have

$$\text{epi } F \subseteq \{(x_0, y_0)\} + T(\text{epi } F, (x_0, y_0)).$$

Hence, for all $x \in R$ and $y \in F(x)$, there exist $t_n \to 0^+, (h_n, k_n) \to (x - x_0, y - y_0)$ such that

$$(x_0 + t_n h_n, y_0 + t_n k_n) \in \text{epi } f, \quad \forall n \in N.$$

Therefore,

$$k_n \in \frac{f(x_0 + t_n h_n) - f(x_0)}{t_n} + C.$$

By taking the limit when $n \to \infty$, we have,

$$y - y_0 \in \lim_{n\to\infty} \frac{f(x_0 + t_n h_n) - f(x_0)}{t_n} + C.$$

Hence, $y - y_0 \in Y_F^{y_0}(x_0, x - x_0) + C$. Now if (x_0, y_0) is not a feeble minimizer of problem (2), then there exist $x \in R \setminus \{x_0\}, y \in F(x)$ such that

$$y_0 - y \in C.$$

On the other hand since, $y - y_0 \in Y_F^{y_0}(x_0, x - x_0) + C$, thus there exists $z \in Y_F^{y_0}(x_0, x - x_0)$ such that $y - y_0 \in z + C$ and therefore

$$z \in y - y_0 - C \subset -C - C = -C.$$

Consequently, $z \in Y_F^{y_0}(x_0, x - x_0) \cap (-C)$, which is a contradiction. □

2. Scalarization of Problem (2)

The theory and methods of scalarization have always been of utmost importance for solving a vector optimization problem. In this section we will propose a method of scalarization of problem (2) which is relied on some level sets of F. This method was initiated first in [4], for single-valued functions.

Let $p \in C^{+i}$ be fixed. For any arbitrary $y \in Y$ as a parameter, we associate (2) with the following multivalued optimization :

$$(\text{SP})_p \qquad \min p(\xi) \quad \text{s.t.} \quad \xi \in F(R \cap (\text{lev}_{\leqslant C} y - F(.))).$$

Let

$$\bar{x} \in R \cap (\text{lev}_{\leqslant C} y - F(.))$$

and $\bar{y} \in F(\bar{x})$, then $(\bar{x}, \bar{y})$ is said to be a feeble minimizer of $(\text{SP})_p$, iff

$$p(\bar{y}) \leq p(\xi), \qquad \forall \xi \in F(R \cap (\text{lev}_{\leqslant C} y - F(.))).$$

It should be noted that the set $R \cap (\mathrm{lev}_{\leqslant_C} y - F(.))$ is not a section of the feasible set R, so we can't deduce each solution or weak solution of problem $(\mathrm{SP})_p$ is also a solution of problem (2). Hence, in order to have a similar result in our case, we have to impose some assumptions on F. Consequently in the presence of these assumptions, we will be able to obtain optimality results for problem (2) and extend the method of [1-6] to vector optimization of objective multifunctions with multivalued constraints.

Definition 2.1 Let

$$\bar{x} \in R \cap (\mathrm{lev}_{\leqslant_C} y - F(.))$$

and $\bar{y} \in F(\bar{x})$, then $(\bar{x}, \bar{y})$ is said to be:
(a) a local feeble minimizer of $(\mathrm{SP})_p$, iff there exist a neighborhood $N(\bar{x})$ such that:

$$p(\xi) \geq p(\bar{y}),\ \forall \xi \in F(x),\ \forall x \in R \cap N(\bar{x}) \cap (\mathrm{lev}_{\leqslant_C} y - F(.));$$

(b) a strict local feeble minimizer of $(\mathrm{SP})_p$, iff there exist a neighborhood $N(\bar{x})$ such that:

$$p(\xi) > p(\bar{y}),\ \forall \xi \in F(x),\ \forall x \in R \cap (N(\bar{x}) \setminus \{\bar{x}\}) \cap (\mathrm{lev}_{\leqslant_C} y - F(.)).$$

Similar to single-valued functions [5], in the next theorem we will show that each solution of problem (2) is equivalent to the impossibility of a system.

Theorem 2.1 *Let $p \in C^{+i}$ be fixed. Then, $(x_0, y_0) \in \mathrm{gr}\ F$ is a feeble minimizer of problem (2), if and only if the system (in the unknown x):*

$$p(y_0 - \xi) > 0, \quad \xi \in F(x), \quad x \in (R \setminus \{x_0\}) \cap (\mathrm{lev}_{\leqslant_{C_0}} y_0 - F(.)) \tag{7}$$

is impossible.

Proof. Suppose on the contrary that the system (7) is possible. Then, there exist

$$\hat{x} \in (R \setminus \{x_0\}) \cap (\mathrm{lev}_{\leqslant_{C_0}} y_0 - F(.))$$

and $\hat{\xi} \in F(\hat{x})$ such that $p(y_0 - \hat{\xi}) > 0$. Since $p \in C^{+i}$, then we have $(y_0 - \hat{\xi}) \neq 0$ and $(y_0 - \hat{\xi}) \in C_0$. Therefore, there exists $\hat{x} \in R \setminus \{x_0\}$ such that $(y_0 - C_0) \cap F(\hat{x}) \neq \emptyset$ and hence (x_0, y_0) is not a feeble minimizer of problem (2). Conversely, if (x_0, y_0) is not a feeble minimizer of problem (2), then there exists $\hat{x} \in R \setminus \{x_0\}$ and $\hat{\xi} \in F(\hat{x})$ such that $(y_0 - \hat{\xi}) \in C_0$ and then,

$$\hat{x} \in (R \setminus \{x_0\}) \cap (\mathrm{lev}_{\leqslant_{C_0}} y_0 - F(.)).$$

But since $p \in C^{+i}$, we have $p(y_0 - \hat{\xi}) > 0$. Hence the system (7) is possible. □

Since the assertion of Theorem 2.1 is equivalent to the fact that $(x_0, y_0) \in \mathrm{gr}\ F$ is a minimum solution of problem (2), thus the impossibility of (7) is necessary and sufficient condition for a point(x_0, y_0) to be a minimizer of $(\mathrm{SP})_p$ at $y = y_0$. Now, in order to obtain a feeble minimizer of problem (2), we have to impose some assumption on F.

Assumption A Let $F : U \rightrightarrows Y$ be a multifunction. We say that F satisfies Assumption A at (x_0, y_0) in gr F with respect to cone C, if when $y_0 \in F(\hat{x}) + C$ for some $\hat{x} \in U$ and $y \in F(x_0) + C$, then $y \in F(\hat{x}) + C$.

It can be shown that all single-valued maps satisfy Assumption A at each point of its graph.

The following example shows that in general the multifunction F may not satisfy Assumption A at some points of its graph.

Example 2.1 Let $U = Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$ and $F : U \rightrightarrows Y$ be a multifunction defined as

$$F(x_1, x_2) := \begin{cases} (y_1, y_2), \quad y_1 = x_1, y_2 = x_2 & \text{if } x_2 < 0, \\ \{(y_1, y_2) : y_2 = x_2, \mid y_1 - x_1 \mid \leq 1\} & \text{if } x_2 \geq 0. \end{cases}$$

Suppose that $x_0 = y_0 = (0,0) \in F((0,0))$. Let $\hat{x} = (0,-1), y = (-.5,1)$, then $y_0 \in F(\hat{x}) + C$ and $y \in F(x_0) + C$ but, $y \notin F(\hat{x}) + C$. Moreover, if we set $\bar{x} = (1,1)$ and $\bar{y} = (0,1)$, then $\bar{y} \in F(\bar{x})$ and one can show that F satisfies Assumption A at $(\bar{x}, \bar{y})$. Furthermore, it is trivial that F satisfies Assumption A at each point $x = (x_1, x_2) \in U$ such that $x_2 < 0$.

We now give another example for the case when F satisfies Assumption A.

Example 2.2 Assume that $f, g : U \longrightarrow Y$ are given functions. Then, the multifunction $F : U \rightrightarrows Y$ defined as

$$F(x) = \{y \in Y : \quad f(x) \leq y \leq g(x)\}$$

satisfies Assumption A at each points of the graph of f.

In the following lemmas we obtain some more examples of those multifunctions satisfying Assumption A in some points of its graph.

Definition 2.2 Let $(x_0, y_0) \in \text{gr } F$. The contingent epiderivative [10, 14] $DF(x_0, y_0)$ of F at (x_0, y_0) is the multifunction from X to Y defined by

$$\text{gr } DF(x_0, y_0) = T(\text{epi } F, (x_0, y_0)).$$

Hence, $y \in DF(x_0, y_0)(x)$, iff there exist a sequence $\{(x_n, y_n)\}$ in epi F and a sequence $\{t_n\}$ of positive real numbers such that

$$\lim_{n\to\infty} (x_n, y_n) = (x_0, y_0) \text{ and } \lim_{n\to\infty} t_n(x_n - x_0, y_n - y_0) = (x, y).$$

Lemma 2.1 *Let* $(x_0, y_0) \in \text{gr } F$ *and* $F : U \rightrightarrows Y$ *be a* C*- multifunction on a convex set* U*. If*

$$\text{epi } F = \{(x_0, y_0)\} + T(\text{epi } (F), (x_0, y_0)),$$

then F *satisfies Assumption* A *at* (x_0, y_0) *with respect to* C*.*

Proof. Suppose that$(x_0, y_0) \in \text{gr } F$, $y_0 \in F(\hat{x}) + C$ and $y \in F(x_0) + C$. We will show that $y \in F(\hat{x}) + C$. Since for all $(x, y) \in \text{epi } F$, we have

$$y - y_0 \in DF(x_0, y_0)(x - x_0).$$

Hence, for $(\hat{x}, y_0)$ and $(x_0, y) \in \text{epi } F$, we obtain

$$y - y_0 \in DF(x_0, y_0)(x_0 - x_0) = DF(x_0, y_0)(0)$$

and

$$0 = y_0 - y_0 \in DF(x_0, y_0)(\hat{x} - x_0).$$

Therefore, $(0, y - y_0)$ and $(\hat{x} - x_0, 0)$ belong to gr $DF(x_0, y_0)$. Now convexity of epi F implies that $T(\text{epi } F, (x_0, y_0))$ is a convex cone and therefore

$$(\hat{x} - x_0, y - y_0) \in \text{gr } DF(x_0, y_0) = T(\text{epi } F, (x_0, y_0)).$$

Hence, $(\hat{x}, y) \in \text{epi } F$, i.e. $y \in F(\hat{x}) + C$. □

Recently, Hu and Fang [15] introduced a class of set-valued increasing-along-rays maps. They showed that this class is related to the corresponding set-valued star-shaped optimization. In the following we consider a class of multifunctions decreasing-along-rays by a similar way and give another example of the multifunction with Assumption A.

Definition 2.3 Let U be a star-shaped set at $x_0 \in U$. A multifunction $F : U \rightrightarrows Y$ is said to be decreasing-along-rays starting at x_0 iff for any $x \in U$ and $0 \leqslant t_1 \leqslant t_2$ with $t_i x + (1 - t_i)x_0 \in U, i = 1, 2$, one has:

$$F(t_1 x + (1 - t_1)x_0) \subseteq F(t_2 x + (1 - t_2)x_0) + C.$$

Lemma 2.2 *Let U be a star-shaped at $x_0 \in U$, $F : U \rightrightarrows Y$ and $(x_0, y_0) \in \text{gr } F$. If F is decreasing-along-rays starting at x_0, then F satisfies Assumption* A *at (x_0, y_0) with respect to cone C.*

Proof. Let $y_0 \in F(\hat{x}) + C$ and $y \in F(x_0) + C$. Since U is star-shaped at x_0, then $x(t) = x_0 + t(\hat{x} - x_0) \in U$ for all $t \in [0, 1]$. Now since F is decreasing-along-rays starting at x_0, then

$$F(x_0) = F(x(0)) \subseteq F(x(1)) + C = F(\hat{x}) + C,$$

and therefore, $F(x_0) + C \subseteq F(\hat{x}) + C$. That is, F satisfies Assumption A at (x_0, y_0).□

Lemma 2.3 *Let $F : U \rightrightarrows Y$ be a multifunction and $(x_0, y_0) \in \text{gr } F$. If $F(x_0)$ is bounded from below by y_0, then F satisfies Assumption* A *at (x_0, y_0) with respect to C.*

Proof. Let $y_0 \in F(\hat{x}) + C$ for some $\hat{x} \in U$ and $y \in F(x_0) + C$. Hence, there exist $\hat{\xi} \in F(\hat{x})$ and $\xi_0 \in F(x_0)$ such that

$$\hat{\xi} \leq y_0, \quad \xi_0 \leq y.$$

Since $F(x_0)$ is bounded from below by y_0, we have $y_0 \leq \xi_0$ and therefore, $\hat{\xi} \leq y$. Hence, $y \in F(\hat{x}) + C$. Thus, F satisfies Assumption A at (x_0, y_0). □

The next result gives a scalarization method of problem (2) in the presence of Assumption A.

Theorem 2.2 *Let $F : U \rightrightarrows Y$, $G : U \rightrightarrows Z$ be multifunctions. Let*

$$x_0 \in R \cap (\text{lev}_{\leqslant C} y - F(.))$$

for some arbitrary $y \in Y$ and F satisfying Assumption A *at $(x_0, y_0) \in \text{gr } F$ with respect to C. If (x_0, y_0) is a feeble minimizer of* $(\text{SP})_p$, *then (x_0, y_0) is also a feeble minimizer of (2).*

Proof. Let

$$x_0 \in R \cap (\text{lev}_{\leqslant C} y - F(.))$$

and (x_0, y_0) be a feeble minimizer of $(\text{SP})_p$. Then, for all

$$x \in R \cap (\text{lev}_{\leqslant C} y - F(.))$$

and $\xi \in F(x)$, we have

$$p(y_0) \leq p(\xi).$$

By Theorem 2.1, it is enough to show that system (7) with respect to unknown x is impossible. Assume on the contrary, i.e. system (7) is possible, hence there exist

$$\hat{x} \in R \cap (\text{lev}_{\leqslant C_0} y_0 - F(.))$$

and $\hat{\xi} \in F(\hat{x})$ such that

$$p(y_0 - \hat{\xi}) > 0.$$

By the above two inequalities, it suffices to show that

$$\hat{x} \in R \cap (\text{lev}_{\leqslant C} y - F(.)),$$

which is a contradiction. As

$$R \cap (\text{lev}_{\leqslant C_0} y_0 - F(.)) \subseteq R \cap (\text{lev}_{\leqslant C} y_0 - F(.),$$

therefore,

$$\hat{x} \in R \cap (\text{lev}_{\leqslant_C} y_0 - F(.)).$$

Thus, $y_0 \in F(\hat{x}) + C$. Now since F satisfies Assumption A at (x_0, y_0) and $y \in F(x_0) + C$, then $y \in F(\hat{x}) + C$. That is

$$\hat{x} \in R \ \cap (\ \text{lev}_{\leqslant_C} y - F(.)). \qquad \square$$

3. Some Other Solutions of Problem (2)

In this section in the presence of Assumption A, we will obtain some sufficient conditions for a local weak solution of problem (2) being feeble. Then, we will obtain some optimality results for Borwein's super efficient and strongly properly solutions of problem (2) by the scalar problem $(\text{SP})_p$.

The next theorem is due to Jabarootian and Zafarani [16].

Theorem 3.1 *Let F be a strictly quasi C-multifunction and G be a quasi D-mutifunction. Then, any local feeble weak solution of problem (2) is a feeble weak solution of problem (2).*

The following theorem gives sufficient conditions for a local solution of $(\text{SP})_p$ to be a weak solution of problem (2).

Theorem 3.2 *Let F be a strictly quasi C-multifunction, G be a quasi D-multifunction and $p \in C^{+i}$. If $(x_0, y_0) \in \text{gr } F$ is a local feeble minimizer of $(\text{SP})_p$ and F satisfies Assumption* A *at (x_0, y_0) with respect to cone C. Then, (x_0, y_0) is a feeble weak minimizer of problem (2).*

Proof. Let

$$x_0 \in R \cap (\text{lev}_{\leqslant_C} y - F(.))$$

and $(x_0, y_0) \in \text{gr } F$ be a local feeble minimizer of $(\text{SP})_p$. Then $y \in F(x_0) + C$. The point (x_0, y_0) is also a local feeble weak minimizer of (2), for otherwise there exist $\hat{x} \in R \setminus \{x_0\} \cap N(x_0)$ and $\hat{y} \in F(\hat{x})$ such that $(y_0 - \hat{y}) \in \mathring{C} \subset C$. Now, since F satisfies Assumption A at (x_0, y_0) with respect to cone C, we have $y \in F(\hat{x}) + C$. So

$$\hat{x} \in R \cap N(x_0) \cap (\text{lev}_{\leqslant_C} y - F(.))$$

and

$$p(\hat{y}) < p(y_0),$$

which is a contradiction. Hence, Theorem 3.1 implies that (x_0, y_0) is a feeble weak minimizer of problem (2). $\square$

Theorem 3.3 *Let F be a quasi C-multifunction, G be a quasi D-multifunction and $p \in C^+ \setminus \{0_{Y^*}\}$ and*

$$x_0 \in R \cap (\text{lev}_{\leqslant_C} y - F(.)).$$

If $(x_0, y_0) \in \text{gr } F$, is a strict local minimizer of $(\text{SP})_p$, and F satisfies Assumption A *at (x_0, y_0) with respect to cone C. Then, (x_0, y_0) is a feeble minimizer of (2).*

Proof. Since (x_0, y_0) is a strict local minimizer of $(\text{SP})_p$, then there exist a neighborhood $N(x_0)$ and $y_0 \in F(x_0)$ such that for each

$$x \in R \cap (N(x_0) \setminus \{x_0\}) \cap (\text{lev}_{\leqslant_C} y - F(.))$$

and each $\xi \in F(x)$, we have:

$$p(\xi) > p(y_0).$$

If (x_0, y_0) is not a feeble minimizer of problem (2), then

$$(F(R \setminus \{x_0\}) - y_0) \cap (-C_0) \neq \emptyset.$$

Therefore, there exist $\hat{x} \in R \setminus \{x_0\}$ and $\hat{y} \in F(\hat{x})$ such that $y_0 - \hat{y} \in C$. Thus $y_0 \in F(\hat{x}) + C$ and as F is a quasi C-multifunction, then for each $t \in [0, 1]$, we have

$$y_0 \in (F(x_0) + C) \cap (F(\hat{x}) + C) \subset F(x_0 + t(\hat{x} - x_0)) + C.$$

Let $x_t = x_0 + t(\hat{x} - x_0)$. Then for each $t \in [0, 1]$, $y_0 \in F(x_t) + C$ and consequently there exists $y_t \in F(x_t)$ such that $y_0 \in y_t + C$. Hence, for such t, we have

$$p(y_0) \geq p(y_t).$$

Now by the above two inequalities it suffices to show that

$$x_t \in R \setminus \{x_0\} \cap N(x_0) \cap (\mathrm{lev}_{\leqslant_C} y - F(.)),$$

which is a contradiction. In fact, since $y \in F(x_0) + C$, $y_0 \in F(x_t) + C$ and F satisfies Assumption A at (x_0, y_0) with respect to cone C, we have $y \in F(x_t) + C$. Thus for small enough $t > 0$, we obtain

$$x_t \in R \cap (N(x_0) \setminus \{x_0\}) \cap (\mathrm{lev}_{\leqslant_C} y - F(.)). \qquad \square$$

In the following we will extend the method of [1 - 6] for strongly properly solution of an objective multifunction with a multivalued constraint.

Definition 3.1 Let V be a nonempty subset of normed space Y and $C \subset Y$ be a closed, convex and pointed cone. A point $y_0 \in V$ is called strongly properly solution iff there exists a closed convex cone $K \neq Y$, $\mathring{K} \neq \varnothing$, $C_0 \subset \mathring{K}$ such that for each 0-neighborhood W there exists a 0-neighborhood H

$$(C \setminus W) + H \subset K,$$

and $y_0 \in \hat{S}(V, K)$. The set of all strongly properly solution of V with respect to C is denoted by $SPS(V, C)$.

Definition 3.2 A point $x_0 \in R$ is called a strongly properly solution of problem (2) iff there exists $y_0 \in F(x_0)$ such that $y_0 \in SPS(F(R), C)$.

In the next theorem, we will show that existence of a strongly properly solution of problem (2) is equivalent to the impossibility of a system.

Theorem 3.4 *Let Y be a reflexive Banach space and C be a closed, convex and pointed cone with a bounded base Θ in Y. Let $p \in C^{+i}$ be fixed and $F(R)$ be weakly compact. Then $(x_0, y_0) \in \mathrm{gr}\ F$ is a strongly properly solution of problem (2) if and only if the system (in the unknown x):*

$$p(y_0 - \xi) > 0, \quad \xi \in F(x), \quad x \in R \cap (\mathrm{lev}_{\leqslant_{C_0}} y_0 - F(.)) \qquad (8)$$

is impossible.

Proof. Let $(x_0, y_0) \in \mathrm{gr}\ F$ be a strongly properly solution of problem (2). Then (x_0, y_0) is a feeble minimizer of problem (2). Hence, by Theorem 2.1, system (8) is impossible. Conversely, if (x_0, y_0) is not a strongly properly solution of problem (2), then by Proposition 2.1 in [17] for any $\epsilon > 0$,

$$(F(R) - y_0) \bigcap [-\mathrm{cone}(\Theta + \epsilon B)] \neq \varnothing.$$

Hence, for each positive integer n, there exist $\xi_n \in F(R)$, $\lambda_n > 0$, , $\theta_n \in \Theta$ and $b_n \in B$ such that

$$y_0 - \xi_n = \lambda_n(\theta_n + \frac{1}{n} b_n).$$

Without loss of generality we can assume that λ_n has a bounded subsequence. For, if $\lambda_n \longrightarrow \infty$, then we must have $\theta_n \longrightarrow 0$, which is impossible since zero is not in cl Θ. Thus $\lambda_n \longrightarrow \lambda$. It is clear that $\lambda > 0$ since Θ is bounded. As $F(R)$ is weakly compact, then ξ_n converges weakly to some $\hat{\xi} \in F(\hat{x})$ and θ_n also converges weakly to some θ_0 in Θ. Hence $p(y_0 - \hat{\xi}) > 0$ and

$$\hat{x} \in R \cap (\mathrm{lev}_{\leqslant_{C_0}} y_0 - F(.)).$$

That is system (8) is possible, which is a contradiction. □

As a consequence of Theorems 2.1, 2.2 and 3.4, we obtain the following equivalence between the strongly properly solution of problem (2) and the scalar minimum solution of $(\mathrm{SP})_p$.

Corollary 3.1 *Under the conditions of Theorem 3.4, let*

$$x_0 \in R \cap (\mathrm{lev}_{\leqslant_C} y - F(.))$$

and F satisfies Assumption A *at $(x_0, y_0) \in$ gr F with respect to cone C. Then, (x_0, y_0) is a minimizer of $(\mathrm{SP})_p$ at arbitrary y, if and only if, (x_0, y_0) is a strongly properly solution of problem (2).* □

Definition 3.3 Let V be a nonempty subset of a normed space X. Then the set of all Borwein's supersolution of V with respect to convex cone C is defined by

$$SS(V, C) := \{y_0 \in V : \exists m > 0 \ s.t. \ (B_Y - C) \cap \overline{\mathrm{cone}(V - y_0)} \subset mB_Y\},$$

where B_Y is the closed unit ball of Y. A point $x_0 \in R$ is said to be supersolution of problem (2), iff there exists $y_0 \in F(x_0)$ such that $y_0 \in SS(F(R), C)$.

It was proved by Borwein and Zhuang [18] that $SS(V, C) \subset \hat{S}(V, C)$.

The following lemma was proved by Rong and Wu [19].

Lemma 3.1 *If the convex pointed ordering cone C has a closed bounded base Θ and if V is a nonempty subset in Y, then $SS(V, C) = SS(V + C, C)$.* □

The next lemma is due to Borwein and Zhuang [18].

Lemma 3.2 *If the convex pointed ordering cone C has a closed bounded base Θ and if V is a convex subset in Y, then $y_0 \in SS(V, C)$ if and only if there exists $p \in \overset{\circ}{C}{}^{+}$ such that $p(y - y_0) \geq 0$ for all $y \in V$.* □

Corollary 3.2 *Suppose that U is a convex set, C is a convex pointed cone which has a closed bounded base and $p \in \overset{\circ}{C}{}^{+}$. Let F be a C-multifunction and G be a quasi D-multifunction on U. Then $(x_0, y_0) \in$ gr F is a superminimizer of problem (2) on $R \cap (\mathrm{lev}_{\leqslant_C} y_0 - F(.))$, if and only if the system (in the unknown x):*

$$p(y_0 - \xi) > 0, \quad \xi \in F(x), \quad x \in R \cap (\mathrm{lev}_{\leqslant_C} y_0 - F(.))$$

is impossible.

Proof. As F is a C-multifunction and G is a quasi D-multifunction, then one can show that

$$F(R \cap \mathrm{lev}_{\leqslant_C} y_0 - F(.)) + C$$

is a convex set. Hence from Lemmas 3.1 and 3.2 we deduce the result.

Theorem 3.5 *Suppose that U is a convex set and C is a closed, convex and pointed cone which has a closed bounded base. Let F be a C-multifunction and G be a quasi D-multifunction on U and F be increasing-along-rays-starting at*

$$x_0 \in R \cap (\text{lev}_{\leqslant_C} y - F(.)).$$

If $(x_0, y_0) \in \text{gr } F$ is a feeble minimizer of $(\text{SP})_p$ for a fixed $p \in \overset{\circ}{C}{}^+$ and $F(x_0)$ is bounded from below by y_0, then (x_0, y_0) is a superminimizer of problem (2).

Proof. Let $p \in \overset{\circ}{C}{}^+$ and (x_0, y_0) be a minimizer of $(\text{SP})_p$. Then

$$x_0 \in R \cap (\text{lev}_{\leqslant_C} y - F(.))$$

and by definition, for each

$$x \in R \cap (\text{lev}_{\leqslant_C} y - F(.))$$

and each $\xi \in F(x)$, we have

$$p(\xi) \geq p(y_0).$$

Let $\hat{x} \in R$ and $\hat{y} \in F(\hat{x})$ be arbitrary. As F is a C-multifunction, then for each $t \in [0, 1]$, we have

$$t\hat{y} + (1 - t)y_0 \in tF(\hat{x}) + (1 - t)F(x_0) \subseteq F(x_0 + t(\hat{x} - x_0)) + C.$$

Since F is increasing-along-rays-starting at x_0, then for each $t \in [0, 1]$, we have

$$F(x_0 + t(\hat{x} - x_0)) + C \subseteq F(x_0) + C.$$

Thus, there exists $\xi_0 \in F(x_0)$ such that $t\hat{y} + (1 - t)y_0 \in \xi_0 + C$. On the other hand, since $F(x_0)$ is bounded from below by y_0, hence,

$$p(\hat{y}) - p(y_0) \geq 0.$$

Since $\hat{x} \in R$ is arbitrary, hence (x_0, y_0) is a minimizer of $\min_{\xi \in F(R)} p(\xi)$. Let $z \in F(R) + C$, then there exist $\xi \in F(R)$ and $c \in C$ such that $z = \xi + c$. Hence,

$$p(z) = p(\xi) + p(c) \geq p(y_0),$$

and as $F(R) + C$ is convex subset of Y, therefore by Lemma 3.2, $y_0 \in SS(F(R) + C, C)$ and by Lemma 3.1, $y_0 \in SS(F(R), C)$, which completes the proof. □

References

1. Castellani, G., Giannessi, F.: Decomposition of mathematical programs by means of theorems of alternative for linear and nonlinear systems. Survey of mathematical programming (Proc. Ninth Internat. Math. Programming Sympos., Budapest), Vol. 2, pp. 423–439, North-Holland, 1979.

2. Giannessi, F.: Constrained Optimization and Image Space Analysis, Volume 1: Separation of Sets and Optimality Conditions, Springer, New York (2005).

3. Dien, P. H., Mastroeni, G., Pappalardo, M., Quang, P. H.: Regularity condition for constrained extreme problems via image space. J. Optim. Theory Appl. **80**, 19-37 (1994).

4. Giannessi, F.: Theorems of the alternative and optimality conditions. J. Optim. Theory Appl. **42**, 331-365 (1984).

5. Giannessi, F., Mastroeni, G., Pellegrini, L.: On the theory of vector optimization and variational inequalities.Image space analysis and separation.Vector variational inequalities and vector equilibria,Mathematical theories. Edited by F. Giannessi, Kluwer Academic Publishers.Dordrecht. London (1999).

6. Giannessi, F., Pellegrini, L.: Image space analysis for vector optimization and variational inequalities. Scalarization. Combinatorial and global optimization. 97 -110, Ser. Appl. Math., 14, World Sci. Publ., River Edge, NJ, (2002).

7. Song, W.: Duality for vector optimization of set-valued functions. J. Math. Anal. Appl. **201**,212-225 (1995).

8. Giannessi, F., Maugeri, A.: Variational Analysis and Applications, Non Convex Optimization and Its Applications, Vol. 79. Springer, New York (2005).

9. Yang, X. Q.: Directional derivatives for set-valued mappings and applications. Math. Methods Oper. Res. **48**, 273-285 (1998).

10. Jahn, J.: Vector Optimization. Theory, Applications, and Extensions. Springer-Verlag, Berlin, (2004).

11. Benoist, J., Borwein, J. M., Popovici, N. A.: Characterization of quasiconvex vector-valued functions. Proc. Amer. Math. Soc. **131**, 1109-1113 (2001).

12. Benoist, J., Popovici, N.: Characterizations of convex and quasiconvex set-valued maps. Mat. Meth. Oper. Res. **57**, 427-435 (2003).

13. Luc, D. T.: Theory of Vector Optimization. Springer-Verlag, Berlin (1989).

14. Aubin, J. P., Ekeland, I.: Applied Nonlinear Analysis. John Wiely & Sons, New York, (1984).

15. Hu, R., Fang, Y. P.: Set-valued increasing-along-rays maps and well-posed set-valued star-shaped optimization. J. Math. Anal. Appl. **331**, 1371-1383 (2007).

16. Jabarootian, T., Zafarani, J.: Characterizations and application of preinvex and prequasi-invex set-valued maps. to appear in Taiwan. J. Math.

17. Bednarczuk, E. M.: A note on lower semicontinuity of minimal points. Nonlinear Analysis **50**, 285-297 (2002).

18. Borwein, J. M., Zhuang, D. M.: Super efficiency in convex vector optimization. ZOR **35**, 175-184 (1991).

19. Rong, W. D., Wu, Y. N.: Characterization of super efficiency in cone-convexlike vector optimization with set-valued maps. Mat. Meth. Oper. Res. **48**, 247-258 (1998).