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A New Approach to Constrained Optimization via Image Space Analysis

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Abstract In this article, by introducing a class of nonlinear separation functions, the image space analysis is employed to investigate a class of constrained optimization problems. Furthermore, the equivalence between the existence of nonlinear separation function and a saddle point condition for a generalized Lagrangian function associated with the given problem is proved.

Keywords Image space analysis · Scalarization of vector optimization · Linear and nonlinear separation · Saddle point · Generalized Lagrangian

Mathematics Subject Classification (2000) 90C26 · 90C29 · 26B25

1 Introduction

In recent years, there has been a growing interest in vector optimization problems with multifunction constraints and objective multifunctions. The Image space approach (IS) was initiated in [3] and was carried on in some other articles; see e.g. [11] and references therein. The (IS) approach has been proved to be a fruitful method in many topics of optimization theory (e.g., optimality condition, vector variational inequalities and vector equilibrium problems); see [3-12]. Moreover, it has been shown that several theoretical aspects of a constrained extremum problem as duality, penalty methods, regularity and Lagrangian- type optimality can be developed by image space analysis.
In this approach, the optimality condition for a constrained extremum problem is expressed under the form of the impossibility of a parametric system or

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equivalently, as the disjunction of two suitable subsets $\mathcal{K}$ and $\mathcal{H}$ of the image space which is associated with the given problem. In order to separate $\mathcal{K}$ and $\mathcal{H}$, one can show that they lie in two disjoint level sets of a possibly nonlinear suitable separating function.

Here, we first introduce a method of scalarization of vector constrained problems via generalized sections of their feasible sets. In fact, we extend the method applied for scalarization of vector-valued functions in [3,10] to multifunctions, which relies on the level sets of objective mappings. Then, by nonlinear separation function that was introduced by Hiriart-Urruty, [15], we obtain some necessary and sufficient optimality conditions in the vector optimization problems with multifunction constraints and objective multifunctions.

The article is organized as follows: In Section 2, we present some basic concepts and different types of solutions of vector optimization problems. In Section 3, we recall the main concepts concerning the image space analysis and we consider the main properties of the image problem. Then, by introducing an assumption on the objective multifunctions, we extend a method for scalarization of vector valued functions to multifunctions. In Section 4, we show that the existence of a nonlinear separation function between $\mathcal{K}$ and $\mathcal{H}$ is equivalent to a saddle point condition for the generalized Lagrangian function.

2 Preliminaries

Let X be a topological vector space and let Y and Z be two normed linear spaces with normed dual spaces Y^* and Z^*, respectively. Let $C \subset Y$ and $D \subset Z$ be pointed, closed and convex cones with nonempty interiors. The positive dual cone of C is defined by

$$C^+ := \{p \in Y^* : p(y) \geq 0, \ \forall y \in C\},$$

and the set of all positive linear functionals in C^+ is

$$C^{+i} := \{p \in Y^* : p(y) > 0, \ \forall y \in C \setminus \{0\}\}.$$

Note that if C is a convex cone in Y with $intC^+ \neq \emptyset$ then $intC^+ = C^{+i}$ [14]. A partial order $\leq_C$ in Y is defined by

$$y_1 \leq_C y_2 \quad \Leftrightarrow \quad y_2 - y_1 \in C, \quad \forall y_1, y_2 \in Y.$$

In the sequel, we suppose that $F : U \rightrightarrows Y$ is a multifunction defined on a nonempty subset U of X with values in Y. The set

$$\text{dom } F := \{x : F(x) \neq \emptyset\}$$

is called the *domain* of F. The set

$$\text{gr } F := \{(x,y) : x \in \text{dom } F,\ y \in F(x)\} = \bigcup_{x \in \text{dom } F} [\{x\} \times F(x)]$$

is called the *graph* of F and the set

$$\text{epi } F = \{(x,y) : x \in \text{dom } F,\ y \in F(x) + C\} = \bigcup_{x \in \text{dom } F} [\{x\} \times (F(x) + C)]$$

is called the *epigraph* of F.
For simplicity, throughout this article, we denote $\mathring{C} :=$ int C and $C_0 := C \setminus \{0\}$.

Definition 2.1 Let U be a convex subset of X. A multifunction $F : U \rightrightarrows Y$ is said to be:

(a) C-multifunction on U, iff for all $x_1, x_2 \in U$ and $t \in [0,1]$, we have

$$tF(x_1) + (1-t)F(x_2) \subseteq F(tx_1 + (1-t)x_2) + C;$$

(b) quasi C-multifunction on U, iff for all $x_1, x_2 \in U$ and $t \in [0,1]$, we have

$$(F(x_1) + C) \cap (F(x_2) + C) \subseteq F(tx_1 + (1-t)x_2)) + C.$$

The following lemma gives a characterization of a quasi C-multifunction in terms of its generalized level sets.

Lemma 2.1 [2] *Let $F : U \rightrightarrows Y$ be a multifunction. Then F is quasi C-multifunction on U, iff for each $y \in Y$ the following generalized level set*

$$lev_{\leqslant_C} y - F(.) := \{x \in U : y \in F(x) + C\}$$

is convex.

Definition 2.2 [16] Let $F : U \rightrightarrows Y$ be a multifunction. Then, F is said to be C-convexlike on U if for any $x_1, x_2 \in U$, any $y_i \in F(x_i)$, $i = 1,2$ and any $t \in (0,1)$, there exists $x_3 \in U$ satisfying:

$$ty_1 + (1-t)y_2 \in F(x_3) + C,$$

or iff $F(U) + C$ is convex .

Remark 2.2 It is obvious that any C-multifunction on a convex set U is C-convexlike on U.
Let $F : U \rightrightarrows Y$ and $G : U \rightrightarrows Z$ be two multifunctions with nonempty values. We consider the following vector optimization problem:

$$\min_C F(x) \quad s.t. \qquad x \in R := \{x \in U : G(x) \cap (-D) \neq \emptyset\}, \tag{1}$$

where R is called the feasible region of Problem (1), which we suppose nonempty.

Lemma 2.2 *Let $F : U \rightrightarrows Y$ and $G : U \rightrightarrows Z$ be C-multifunction and D-multifunction on U, respectively, then the ordered pair map (F, G) defined by $(F, G)(x) = F(x) \times G(x)$ is a $C \times D$-convexlike multifunction.*

Proof We need to show that the set $(F, G)(U) + C \times D$ is a convex set. Let $x_i \in U$, $y_i \in F(x_i)$, $z_i \in G(x_i)$, $c_i \in C$, $d_i \in D$ for $(i = 1, 2)$. Set $x = tx_1 + (1-t)x_2$, $c = tc_1 + (1-t)c_2$ and $d = td_1 + (1-t)d_2$ and $t \in]0, 1[$. Since U, C and D are convex sets, then $x \in U$, $c \in C$ and $d \in D$. Again, since F and G are C-multifunction and D-multifunction respectively, we have

$$t[(y_1, z_1) + (c_1, d_1)] + (1-t)[(y_2, z_2) + (c_2, d_2)] =$$

$$(ty_1 + (1-t)y_2 + c, tz_1 + (1-t)z_2 + d) \in (F \times G)(x) + C \times D.$$

Thus, (F, G) is $C \times D$-convexlike multifunction. □

Definition 2.3 A point $\bar{x} \in R$ is called a minimum point (in short m.p.) of Problem (1) iff

$$\exists \bar{y} \in F(\bar{x}) \quad s.t. \quad (F(R)) \cap (\bar{y} - C_0) = \emptyset. \tag{2}$$

In this case we say that $(\bar{x}, \bar{y})$ is a minimizer for Problem (1). The set of all $\bar{x} \in R$, which fulfill (2), will be denoted by $S(F, C)$.

Definition 2.4 A point $\bar{x} \in R$ is called a weak minimum point (in short w.m.p.) of Problem (1) iff

$$\exists \bar{y} \in F(\bar{x}) \quad \text{s.t.} \quad (F(R)) \cap (\bar{y} - \mathring{C}) = \emptyset. \tag{3}$$

In this case we say that $(\bar{x}, \bar{y})$ is a weak minimizer for Problem (1). The set of all $\bar{x} \in R$ which fulfill (3), will be denoted by $WS(F, C)$.
The following result presents a necessary and sufficient condition for a vector to be an m.p. or a w.m.p. of Problem (1).

Lemma 2.3 [18] *Let $\bar{x} \in R$ and $(\bar{x}, \bar{y}) \in \text{gr } F$. Then*

(i) $(\bar{x}, \bar{y})$ *is a minimizer of Problem (1) iff*

$$(\bar{y} - C_0, -D) \cap (F(x), G(x)) = \emptyset \quad \forall x \in U.$$

(ii) $(\bar{x}, \bar{y})$ *is a weak minimizer of problem (1) iff*

$$(\bar{y} - \mathring{C}, -D) \cap (F(x), G(x)) = \emptyset \quad \forall x \in U.$$

3 Image Space Analysis

In this section, we develop the image space analysis for vector optimization with multifunction constraints and multifunction objective. Let $\bar{x} \in R$ and $\bar{p} := (\bar{x}, \bar{y}) \in \text{gr } F$. We introduce the multifunction $A_{\bar{p}} : U \rightrightarrows Y \times Z$, defined by

$$A_{\bar{p}}(x) := \{(\bar{y} - y, -z) : \ y \in F(x) \ , \ z \in G(x), \ x \in U\},$$

and we associate the following sets to $\bar{p} \in \text{gr } F$

$$\mathcal{H} = C_0 \times D \quad , \quad \mathcal{K}_{\bar{p}} = A_{\bar{p}}(U).$$

The set $\mathcal{K}_{\bar{p}}$ is called the image space associated with Problem (1). By Lemma 2.3, $\bar{p} = (\bar{x}, \bar{y})$ is a minimizer of Problem (1) iff

$$\mathcal{K}_{\bar{p}} \cap \mathcal{H} = \emptyset. \tag{4}$$

and $\bar{p} = (\bar{x}, \bar{y})$ is a weak minimizer of Problem (1) iff

$$\mathcal{K}_{\bar{p}} \cap \mathcal{H}_{ic} = \emptyset, \tag{5}$$

where, $\mathcal{H}_{ic} = \mathring{\text{C}} \times D$.

Remark 3.1 In general, the image space $\mathcal{K}_{\bar{p}}$ is not convex, even when the two functions F and G are C-multifunction and D-multifunction on the convex set U, respectively. To overcome this defect, we introduce the extended image space $\mathcal{K}_{\bar{p}}$ with respect to the cone cl $\mathcal{H}$ as $\mathcal{E}_{\bar{p}} = \mathcal{K}_{\bar{p}} - \text{cl } \mathcal{H}$. In fact, by imposing some convexity assumptions on F and G, we obtain convexity of the extended image space.

Lemma 3.1 *Let $F : U \rightrightarrows Y$ and $G : U \rightrightarrows Z$ be C-multifunction and D-multifunction on the convex set U, respectively. Then the extended image $\mathcal{E}_{\bar{p}} = \mathcal{K}_{\bar{p}} - cl\, \mathcal{H}$ is convex and*

$$\mathcal{K}_{\bar{p}} \cap \mathcal{H} = \emptyset \Longleftrightarrow \mathcal{E}_{\bar{p}} \cap \mathcal{H} = \emptyset.$$

Proof Since $F : U \rightrightarrows Y$ and $G : U \rightrightarrows Z$ are C-multifunction and D-multifunction on the convex set U respectively, then by Lemma 2.2, the ordered pair (F, G) and so the multifunction $-A_{\bar{p}}(x) = (-\bar{y}, 0) + (F, G)(x)$ are $C \times D$-convexlike multifunctions. Hence, $-\mathcal{E}_{\bar{p}} = -A_{\bar{p}}(U) + \text{cl } \mathcal{H} = -\mathcal{K}_{\bar{p}} + \text{cl } \mathcal{H}$ or equivalently, $\mathcal{E}_{\bar{p}}$ is convex.
Since, $\mathcal{H} + \text{cl } \mathcal{H} = \mathcal{H}$, we deduce that

$$\mathcal{E}_{\bar{p}} - \mathcal{H} = \mathcal{K}_{\bar{p}} - \mathcal{H}.$$

Therefore, we have

$$0 \in \mathcal{E}_{\bar{p}} - \mathcal{H} \Longleftrightarrow 0 \in \mathcal{K}_{\bar{p}} - \mathcal{H}.$$

Hence,

$$\mathcal{K}_{\bar{p}} \cap \mathcal{H} = \emptyset \Longleftrightarrow \mathcal{E}_{\bar{p}} \cap \mathcal{H} = \emptyset. \qquad \square$$

Corollary 3.1 *Let $\bar{x} \in R$. Then $\bar{p} = (\bar{x}, \bar{y}) \in \text{gr } F$ is a minimizer of Problem (1) iff*

$$\mathcal{E}_{\bar{p}} \cap \mathcal{H} = \emptyset.$$

Remark 3.2 Since $\mathcal{H}_{ic} + \text{cl } \mathcal{H} = \mathcal{H}_{ic}$, we deduce that

$$\mathcal{E}_{\bar{p}} - \mathcal{H}_{ic} = \mathcal{K}_{\bar{p}} - \mathcal{H}_{ic},$$

and for $\bar{x} \in R$, we have $\bar{p} = (\bar{x}, \bar{y}) \in \text{gr } F$ is a weak minimizer of Problem (1) iff

$$\mathcal{E}_{\bar{p}} \cap \mathcal{H}_{ic} = \emptyset.$$

Remark 3.3 Let $\mathcal{H}_{C_0}$ be a subset of $\mathcal{H}$, defined by $\mathcal{H}_{C_0} = C_0 \times \{0_z\}$. Then by a similar argument as that of the proof of Proposition 2.1 in [13], we can deduce that (4) is equivalent to

$$\mathcal{E}_{\bar{p}} \cap \mathcal{H}_{C_0} = \emptyset.$$

Remark 3.4 Recall that by Lemma 2.1, F is a quasi C-multifunction on the convex set U iff for each $y \in Y$ the generalized level set $\text{lev}_{\leqslant_C} y - F(.)$ is convex. In next Lemma by imposing some convexity conditions on F and G, we obtain the convexity of some other extension of the image space. We denote $A_{\bar{p}}(\text{lev}_{\leqslant_C} \bar{y} - F(.))$ by $\mathcal{K}_{\bar{p}}^{\text{lev}\bar{y}}$.

Lemma 3.2 *Let $F : U \rightrightarrows Y$ and $G : U \rightrightarrows Z$ be C-multifunction and D-multifunction on the convex set U, respectively. Then the extended image $\mathcal{E}_{\bar{p}}^{lev\bar{y}} = \mathcal{K}_{\bar{p}}^{lev\bar{y}} - \text{cl } \mathcal{H}$ is convex and*

$$\mathcal{E}_{\bar{p}}^{lev\bar{y}} \cap \mathcal{H} = \emptyset \Longleftrightarrow \mathcal{K}_{\bar{p}}^{lev\bar{y}} \cap \mathcal{H} = \emptyset.$$

Proof Since $F : U \rightrightarrows Y$ and $G : U \rightrightarrows Z$ are C-multifunction and D-multifunction on the convex set U, respectively, then by a similar argument as that of the proof of Lemma 3.1, we obtain the extended image $\mathcal{E}_{\bar{p}}^{\text{lev}\bar{y}} = \mathcal{K}_{\bar{p}}^{\text{lev}\bar{y}} - \text{cl } \mathcal{H}$ is convex and from $\mathcal{H} + \text{cl } \mathcal{H} = \mathcal{H}$, we deduce that

$$\mathcal{E}_{\bar{p}}^{\text{lev}\bar{y}} - \mathcal{H} = \mathcal{K}_{\bar{p}}^{\text{lev}\bar{y}} - \mathcal{H}.$$

Hence,

$$\mathcal{E}_{\bar{p}}^{\text{lev}\bar{y}} \cap \mathcal{H} = \emptyset \Longleftrightarrow \mathcal{K}_{\bar{p}}^{\text{lev}\bar{y}} \cap \mathcal{H} = \emptyset. \qquad \square$$

Theorem 3.1 *Let $\bar{x} \in R$. Then $\bar{p} = (\bar{x}, \bar{y}) \in \text{gr } F$ is a minimizer of Problem (1) iff*

$$\mathcal{K}_{\bar{p}}^{lev_{\bar{y}}} \cap \mathcal{H} = \emptyset. \tag{6}$$

Proof Let $\bar{p} = (\bar{x}, \bar{y})$ be a minimizer of Problem (1), then by (4), $\mathcal{K}_{\bar{p}} \cap \mathcal{H} = \emptyset$, hence, $\mathcal{K}_{\bar{p}}^{\text{lev}_{\bar{y}}} \cap \mathcal{H} = \emptyset$.
Conversely, if $\bar{p} = (\bar{x}, \bar{y})$ is not a minimizer of Problem (1), then there exist $\hat{x} \in R$, $\hat{y} \in F(\hat{x})$ such that $\bar{y} - \hat{y} \in C_0$. Consequently, there exist $\hat{z} \in G(\hat{x}) \cap (-D)$ such that $(\bar{y} - \hat{y}, -\hat{z}) \in \mathcal{H}$ and $\bar{y} \in F(\hat{x}) + C_0$. Hence, $\hat{x} \in (\text{lev}_{\leqslant_{C_0}} \bar{y} - F(.))$. Thus, we deduce that $\mathcal{K}_{\bar{p}}^{\text{lev}_{\bar{y}}} \cap \mathcal{H} \neq \emptyset$. □

If we replace $\mathcal{H}$ by $\mathcal{H}_{ic}$ in Theorem 3.1, we obtain the next result.

Theorem 3.2 *Let $\bar{x} \in R$. Then $\bar{p} = (\bar{x}, \bar{y}) \in \text{gr } F$ is a weak minimizer of Problem (1) iff*

$$\mathcal{K}_{\bar{p}}^{lev_{\bar{y}}} \cap \mathcal{H}_{ic} = \emptyset.$$

In Theorems 3.1, and 3.2 if we replace an arbitrary y instead of $\bar{y}$ in the generalized level set $\text{lev}_{\leqslant_{C_0}} \bar{y} - F(.)$, then a similar result does not hold. To overcome this difficulty, we have to impose an assumption on F.

Assumption (A) [5] Let $F : U \rightrightarrows Y$ be a multifunction. We say that F satisfies Assumption(A) at $\bar{p} = (\bar{x}, \bar{y}) \in \text{gr } F$ with respect to the cone C, if $\bar{y} \in F(\hat{x}) + C$ for some $\hat{x} \in U$ and $y \in F(\bar{x}) + C$, then $y \in F(\hat{x}) + C$.

Remark 3.5 It can be shown that all single-valued maps satisfy Assumption (A) at each point of their graph. Some sufficient conditions for a set-valued map F that satisfies Assumption (A) are given in [5]. For instance, by Lemma 2.3 of [5], if $y_0 \in F(x_0)$ is a below bound of $F(x_0)$, then F satisfies Assumption (A) at $(x_0, y_0) \in \text{gr } F$ with respect to cone C.

In the presence of Assumption (A), we show that we can replace an arbitrary y instead of $\bar{y}$ in the generalized level set of Theorems 3.1 and 3.2.

Theorem 3.3 *Let $F : U \rightrightarrows Y$, $G : U \rightrightarrows Z$ be multifunctions. Suppose*

$$\bar{x} \in R \cap lev_{\leqslant_C} y - F(.)$$

for some arbitrary y and F satisfies Assumption(A) at $\bar{p} = (\bar{x}, \bar{y}) \in \text{gr } F$ with respect to the cone C. Then, $\bar{p}$ is a minimizer of Problem (1) iff

$$\mathcal{K}_{\bar{p}}^{lev_y} \cap \mathcal{H} = \emptyset.$$

Proof Let $\bar{p}$ be a minimizer of Problem (1) , then by (4), $\mathcal{K}_{\bar{p}}^{\text{lev}_y} \cap \mathcal{H} = \emptyset$. Conversely, if $\bar{p}$ is not a minimizer of Problem (1), then there exists $\hat{x} \in R$, $\hat{y} \in F(\hat{x})$ and $\hat{z} \in G(\hat{x}) \cap (-D)$ such that $(\bar{y} - \hat{y}, -z) \in \mathcal{H}$ and
$\hat{x} \in R \cap (\text{lev}_{\leqslant C_0} \bar{y} - F(.))$. To prove that $(\bar{y} - \hat{y}, -z) \in \mathcal{K}_{\bar{p}}^{\text{lev}_y}$, it is enough to show that

$$\hat{x} \in R \cap (\text{lev}_{\leqslant C} y - F(.)).$$

From $\bar{y} - \hat{y} \in C_0$, we deduce that $\hat{x} \in R \cap \text{lev}_{\leqslant C} \bar{y} - F(.)$. Thus, $\bar{y} \in F(\hat{x}) + C$. Now since F satisfies Assumption(A) at $(\bar{x}, \bar{y})$ with respect to the cone C and $y \in F(\bar{x}) + C$, then $y \in F(\hat{x}) + C$. Hence, $\hat{x} \in R \cap (\text{lev}_{\leqslant C} y - F(.))$, that is

$$\mathcal{K}_{\bar{p}}^{\text{lev}_y} \cap \mathcal{H} \neq \emptyset. \qquad \square$$

Theorem 3.4 *Let $F : U \rightrightarrows Y$, $G : U \rightrightarrows Z$ be multifunctions. Let*

$$\bar{x} \in R \cap lev_{\leqslant C} y - F(.)$$

for some arbitrary y and F satisfying Assumption(A) at $\bar{p} = (\bar{x}, \bar{y}) \in \text{gr } F$ with respect to the cone $\mathring{C}$. Then, $\bar{p}$ is a weak minimizer of Problem (1) iff

$$\mathcal{K}_{\bar{p}}^{lev_y} \cap \mathcal{H}_{ic} = \emptyset.$$

4 Separation

Let $\bar{x} \in R$ and $\bar{p} = (\bar{x}, \bar{y}) \in \text{gr } F$. To prove disjunction between the two sets $\mathcal{K}_{\bar{p}}$ and $\mathcal{H}$, we will show that $\mathcal{K}_{\bar{p}}$ and $\mathcal{H}$ lie in two disjoint level sets of a linear or nonlinear separation function.
Suppose that Γ is a given set of parameters and $\omega_0 : Z \times \Gamma \mapsto \mathbb{R}$ is a mapping which will be specified in the sequel. We consider the class of functions
$\omega : Y \times Z \times Y^* \times \Gamma \mapsto \mathbb{R}$ defined by

$$\omega(u, v, \theta, \gamma) := \langle \theta, u \rangle + \omega_0(v, \gamma).$$

Definition 4.1 Let $\bar{x} \in R$ and $\bar{p} = (\bar{x}, \bar{y}) \in \text{gr } F$. Then we say that $\mathcal{K}_{\bar{p}}$ and $\mathcal{H}$ admit a separation iff there exists $\bar{\theta} \in Y^*$ and $\bar{\gamma} \in \Gamma$ such that $\omega(u, v, \bar{\theta}, \bar{\gamma}) \not\equiv 0$ and

$$\mathcal{H} \subseteq \text{lev}_{\geq 0} \ \ \omega(., ., \bar{\theta}, \bar{\gamma}); \tag{7}$$

$$\mathcal{K}_{\bar{p}} \subseteq \text{lev}_{\leq 0} \ \ \omega(., ., \bar{\theta}, \bar{\gamma}), \tag{8}$$

in which $\text{lev}_{\leq 0} \ \ \omega(., ., \bar{\theta}, \bar{\gamma}) := \{(u, v) \in Y \times Z : \omega(u, v, \bar{\theta}, \bar{\gamma}) \leq 0\}$ denotes the level set of $\omega(., ., \bar{\theta}, \bar{\gamma})$. If $\bar{\theta} \in C^{+i}$, then the separation is said to be regular, and if $\Gamma \subseteq Z^*$ and $\omega_0(v, \bar{\gamma}) = \langle \bar{\gamma}, v \rangle$, then the separation is said to be a linear separation.

Assumption (B) We say that the nonlinear class of functions

$$\omega(u,v,\theta,\gamma) := \langle \theta, u\rangle + \omega_0(v,\gamma)$$

satisfies Assumption(B) if

$$\inf_{\gamma\in\Gamma}\sup_{v\in D}\omega_0(v,\gamma) = 0,$$

and

$$\inf_{\gamma\in\Gamma}\sup_{v\notin D}\omega_0(v,\gamma) = -\infty.$$

The Assumption (B) is considered in different forms in the literature, see e.g. Theorem 3.1 in [13] and Assumption A in [22].

The following result is a modification of Theorem 3.1 in [13] for set-valued optimization problems.

Theorem 4.1 *Let $\bar{x}\in R$, $\bar{p}=(\bar{x},\bar{y})\in \mathrm{gr}\ F$ and $\theta\in C^*$ be fixed. Then for the class of nonlinear functions ω which satisfies the Assumption (B) and positive, we have*

$$\sup_{(u,v)\in\mathcal{K}_{\bar{p}}}\inf_{\gamma\in\Gamma}\omega(u,v,\theta,\gamma) = \sup_{\{(u,v)\in\mathcal{K}_{\bar{p}}:v\in D\}}\langle\theta,u\rangle.$$

Proof Let $\bar{x}\in R$ and $\omega(u,v,\theta,\gamma)=\langle\theta,u\rangle+\omega_0(v,\gamma)$ be a nonlinear separation for some $\theta\in C^*$. Since

$$\inf_{\gamma}\sup_{v}\omega_0(v,\gamma)\geq\inf_{\gamma}\omega_0(v,\gamma),\quad\forall v,$$

so from Assumption (B), we obtain

$$\inf_{\gamma\in\Gamma}\omega(u,v,\theta,\gamma) = \begin{cases}-\infty, & \text{if } v\notin D;\\ \langle\theta,u\rangle, & \text{if } v\in D.\end{cases} \tag{9}$$

As $\bar{x}\in R$ and $\bar{p}\in \mathrm{gr}\ F$, then $(\bar{y}-\bar{y},-z)=(0,v)\in\mathcal{K}_{\bar{p}}$ for some $v\in D$, therefore, by taking the supremum in (9), the desired result is obtained. □

Lemma 4.1 *Let the separation ω be linear. If condition* (7) *holds for some $\bar{\theta}\in Y^*$ and $\bar{\gamma}\in Z^*$, then $\bar{\theta}\in C^*$ and $\bar{\gamma}\in D^*$.*

Proof Suppose that $\bar{\theta}\notin C^*$, then there exists $\bar{c}\in C_0$ such that $\langle\bar{\theta},\bar{c}\rangle<0$. Since the sequence $\{(n\bar{c},v)\}$ is in $\mathcal{H}$ for an arbitrary $v\in D$, thus we have

$$\lim_{n\to\infty}\omega(n\bar{c},v,\bar{\theta},\bar{\gamma}) = \lim_{n\to\infty}[n\langle\bar{\theta},\bar{c}\rangle+\langle\bar{\gamma},v\rangle] = -\infty,$$

which contradicts (7). For proving $\bar{\gamma}\in D^*$, we can apply a similar approach. For an arbitrary $v\in D$, we consider the sequence $\{(\frac{1}{n}\bar{c},v)\}$ in $\mathcal{H}$, then we have

$$\langle\bar{\gamma},v\rangle = \lim_{n\to\infty}[\frac{1}{n}\langle\bar{\theta},\bar{c}\rangle+\langle\bar{\gamma},v\rangle]\geq 0.$$

That is $\bar{\gamma}\in D^*$. □

Proposition 4.1 *Let $\bar{x} \in R$, $\bar{p} = (\bar{x}, \bar{y}) \in \text{gr } F$ and $\mathcal{K}_{\bar{p}}$ and $\mathcal{H}$ admit a regular linear separation, then $\bar{p}$ is a minimizer of Problem (1).*

Proof Suppose that the two conditions (7) and (8) hold for some $\bar{\theta} \in C^{+i}$ and $\bar{\gamma} \in Z^*$, then by Lemma 4.1, $\bar{\gamma} \in D^*$ and for each $(u, v) \in \mathcal{H}$, we have $\omega(u, v, \bar{\theta}, \bar{\gamma}) = \langle \bar{\theta}, u \rangle + \langle \bar{\gamma}, v \rangle > 0$, so the strict inequality in (7) holds and therefore $\mathcal{K}_{\bar{p}} \cap \mathcal{H} = \emptyset$. □

Definition 4.2 Suppose that $A \subseteq Y$ and $d_A(y) = \inf\{\|a - y\| : a \in A\}$ is the distance function from A. We consider the function $\triangle_A : Y \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\triangle_A(y) = d_A(y) - d_{Y \setminus A}(y),$$

where $d_{\emptyset}(y) = +\infty$.

This function was defined in [15], and some of its main properties are gathered in the following proposition.

Proposition 4.2 [21] *If the set A is nonempty and $A \neq Y$ with nonempty interior, then :*

(i) $\triangle_A$ is real valued and 1-Lipschitzian function;
(ii) $\triangle_A < 0$ for every $y \in intA$, $\triangle_A = 0$ for every $y \in \partial A$, and $\triangle_A > 0$ for every $y \in int(Y \setminus A)$;
(iii) If A is closed, then it holds that $A = \{y : \triangle_A(y) \leq 0\}$;
(iv) If A is convex, then $\triangle_A$ is convex;
(v) If A is a cone, then $\triangle_A$ is positively homogeneous;
(vi) If A is a closed convex cone, then $\triangle_A$ is nonincreasing with respect to the ordering relation induced by C on Y.

In the sequel, we obtain some sufficient and necessary optimality conditions for Problem (1).

Remark 4.1 The set of all functions $\omega : Y \times Z \times Y^* \times Z^* :\mapsto \mathbb{R}$ defined by

$$\omega(u, v, \theta, \gamma) := \langle \theta, u \rangle - \triangle_{\mathcal{R}_+}(\langle \gamma, v \rangle),$$

is a class of linear separation functions ; see Remark 3.1 in [4].

Proposition 4.3 *Let $\bar{x} \in R$ and $\bar{p} = (\bar{x}, \bar{y}) \in \text{gr } F$. Then,*
(a) if there exists $\rho \in \text{cl}\mathcal{H}$ such that for any $x \in U$, $y \in F(x)$ and $z \in G(x)$, we have:

$$\triangle_{\text{cl}\mathcal{H}}((\bar{y} - y, -z) + \rho) > 0.$$

Then, $\bar{p}$ is a minimizer of Problem (1).
(b) if $\bar{p}$ is a weak minimizer of Problem (1) and $\text{int}\mathcal{H} \neq 0$, then there exists $\bar{\rho} \in \text{int}\mathcal{H}$ such that for any $x \in U$, $y \in F(x)$ and $z \in G(x)$, we have:

$$\triangle_{\text{int}\mathcal{H}}((\bar{y} - y, -z) - \bar{\rho}) > 0.$$

Proof (a) Suppose that $\bar{p}$ is not a minimizer of Problem (1), then by (4), $\mathcal{K}_{\bar{p}} \cap \mathcal{H} \neq \emptyset$. Therefore, there exists $\hat{x} \in R$, $\hat{y} \in F(\hat{x})$ and $\hat{z} \in G(\hat{x})$, such that

$$(\bar{y} - \hat{y}, -\hat{z}) \in \mathcal{K}_{\bar{p}} \cap \mathcal{H}.$$

Then, by Proposition 4.2, for each $\rho \in \mathrm{cl}\mathcal{H}$ we deduce

$$0 < \triangle_{\mathrm{cl}\mathcal{H}}((\bar{y} - \hat{y}, -\hat{z}) + \rho) \leq \triangle_{\mathrm{cl}\mathcal{H}}(\bar{y} - \hat{y}, -\hat{z}) \leq 0,$$

which is a contradiction.
(b) Suppose on the contrary that for each $\rho \in \mathrm{int}\mathcal{H}$, there exists $\hat{x} \in R$, $\hat{y} \in F(\hat{x})$ and $\hat{z} \in G(\hat{x})$, such that

$$\triangle_{\mathrm{int}\mathcal{H}}((\bar{y} - \hat{y}, -\hat{z}) - \rho) \leq 0.$$

Hence, $\triangle_{\mathrm{cl}\mathcal{H}}((\bar{y} - \hat{y}, -\hat{z}) - \rho) \leq 0$, since $\triangle_A(y) \leq \triangle_B(y)$ for $B \subseteq A$. By Proposition 4.2, $((\bar{y} - \hat{y}, -\hat{z}) - \rho) \in \mathrm{cl}\mathcal{H}$ for each $\rho \in \mathrm{int}\mathcal{H}$. It follows that

$$(\bar{y} - \hat{y}, -\hat{z}) \in \mathrm{int}\mathcal{H}.$$

By Lemma 2.3, $\bar{p}$ is not a weak minimizer of Problem (1). □

The next result is a nonlinear version of Proposition 4.1.

Proposition 4.4 *Let $\bar{x} \in R$ and $\bar{p} = (\bar{x}, \bar{y}) \in \mathrm{gr}\ F$. If $\mathcal{K}_{\bar{p}}$ and $\mathcal{H}$ admit the following regular nonlinear separation function*

$$\omega(u, v, \bar{\theta}, \bar{\gamma}) := \langle \bar{\theta}, u \rangle + \omega_0(v, \bar{\gamma}),$$

then $\bar{p}$ is a minimizer of Problem (1).

Proof The proof is similar to the proof of Proposition 4.1, so we omit it. □

Proposition 4.5 *Let $F : U \rightrightarrows Y$ and $G : U \rightrightarrows Z$ be two multifunctions, $\bar{p} = (\bar{x}, \bar{y}) \in \mathrm{gr}\ F$ and $\mathring{\mathcal{H}} = \mathring{\mathrm{C}} \times \mathring{\mathrm{D}} \neq \emptyset$. Then,*
(i) if F, G are C-multifunction and D-multifunction on the convex set U, respectively and $\bar{p}$ is a weak minimizer of Problem (1), then $\mathcal{K}_{\bar{p}}$ and $\mathring{\mathcal{H}}$ admit a linear separation.
(ii) if $\mathcal{K}_{\bar{p}}$ and $\mathring{\mathcal{H}}$ admit a regular linear separation, then $\bar{p}$ is a weak minimizer of Problem (1).

Proof (i) Let $\bar{p}$ be a weak minimizer of Problem (1). Since F and G are C-multifunction and D-multifunction on the convex set U respectively, then by Lemma 2.2, the ordered pair (F, G) and consequently the multifunction $(F - \bar{y}, G)$ is a $C \times D$-convexlike multifunction and

$$(F - \bar{y}, G)(U) \cap -\mathring{\mathcal{H}} = \emptyset.$$

Therefore, by separation theorem, we deduce that there exists

$$(\bar{\theta}, \bar{\gamma}) \in (C^* \times D^*) \setminus \{(0, 0)\}$$

such that

$$\langle \bar{\theta}, \bar{y} - y \rangle + \langle \bar{\gamma}, -z \rangle \leq 0, \forall x \in U, \forall y \in F(x), \forall z \in G(x).$$

In the other words, for each $(u, v) \in \mathcal{K}_{\bar{p}}$, we have

$$\omega(u, v, \bar{\theta}, \bar{\gamma}) = \langle \bar{\theta}, u \rangle + \langle \bar{\gamma}, v \rangle \leq 0.$$

That is, $\mathcal{K}_{\bar{p}}$ and $\mathring{\mathcal{H}}$ admit a linear separation.
(ii) It is obvious that if $\mathcal{K}_{\bar{p}}$ and $\mathring{\mathcal{H}}$ admit a regular linear separation, then the two conditions (7) and (8) hold for some $\bar{\theta} \in C^{+i}$ and $\bar{\gamma} \in Z^*$. Now by Lemma 4.1, we deduce that $\bar{\gamma} \in D^*$ and for each $(u, v) \in \mathring{\mathcal{H}}$, we obtain

$$\omega(u, v, \bar{\theta}, \bar{\gamma}) = \langle \bar{\theta}, u \rangle + \langle \bar{\gamma}, v \rangle > 0$$

with strict inequality in (7). Therefore,

$$\mathcal{K}_{\bar{p}} \cap \mathcal{H}_{ic} = \emptyset.$$

Thus, by (5), we have $\bar{p}$ is a weak minimizer of Problem (1). □

Theorem 4.2 *Let $\bar{x} \in R$, $\bar{p} = (\bar{x}, \bar{y}) \in \mathrm{gr}\ F$ and $\theta \in C^{+i}$ be fixed. Let ω be a regular linear separation. If for each $z \in G(x) \cap (-D)$,*

$$\inf_{\gamma \in D^*} \sup_{\{y \in F(x): x \in R\}} \omega(\bar{y} - y, -z, \theta, \gamma) \leq 0,$$

then, $\bar{p}$ is a minimizer of Problem (1).

Proof Suppose on the contrary that $\bar{p}$ is not a minimizer of Problem (1), then by (4), $\mathcal{K}_{\bar{p}} \cap \mathcal{H} \neq \emptyset$. Therefore, there exists $\hat{x} \in R$, $\hat{y} \in F(\hat{x})$ and $\hat{z} \in G(\hat{x})$, such that

$$(\bar{y} - \hat{y}, -\hat{z}) \in \mathcal{K}_{\bar{p}} \cap \mathcal{H}.$$

Hence,

$$\sup_{\{y \in F(x): x \in R\}} \omega(\bar{y} - y, -z, \theta, \gamma) \geq \langle \theta, \bar{y} - \hat{y} \rangle + \langle \gamma, -\hat{z} \rangle.$$

Since $\inf_{\gamma \in D^*} \langle \gamma, -\hat{z} \rangle \geq 0$ and $(\bar{y} - \hat{y}) \in C_0$, then

$$\inf_{\gamma \in D^*} \sup_{\{y \in F(x): x \in R\}} \omega(\bar{y} - y, -z, \theta, \gamma) \geq \langle \theta, \bar{y} - \hat{y} \rangle > 0,$$

which is a contradiction. □

In order to obtain saddle point conditions for the generalized Lagrangian function associated with Problem (1), we consider the generalized Lagrangian function $\mathcal{L} : U \times C^* \times \Gamma \mapsto \mathbb{R}$ defined by

$$\mathcal{L}(x, \theta, \gamma) = \inf_{y \in F(x)} \langle \theta, y \rangle - \sup_{z \in G(x)} \omega_0(-z, \gamma),$$

where F is compact valued.

The generalized Lagrangian function, $\mathcal{L}(x, \theta, \gamma)$, developed in this section is a modified version of similar cases in the literature.

The next result shows that the existence of a nonlinear separation satisfying assumption (B) between $\mathcal{K}_{\bar{p}}$ and $\mathcal{H}$ is equivalent to the existence of a saddle point for the generalized Lagrangian function $\mathcal{L}(x, \theta, \gamma)$.

Theorem 4.3 *Let $\bar{p} = (\bar{x}, \bar{y}) \in \mathrm{gr}\ F$, $\inf_{y \in F(\bar{x})} \langle \bar{\theta}, y \rangle = \langle \bar{\theta}, \bar{y} \rangle$ for a fixed $\bar{\theta} \in C^*$ and $\omega(u, v, \theta, \gamma) = \langle \theta, u \rangle + \omega_0(v, \gamma)$ be the class of functions satisfying Assumption(B) and ω_0 be positive on D. Then $\bar{x} \in R$ and $\mathcal{K}_{\bar{p}}$ and $\mathcal{H}$ admit a nonlinear separation if and only if $(\bar{x}, \bar{\gamma})$ is a saddle point for the generalized Lagrangian function $\mathcal{L}(x, \bar{\theta}, \gamma)$ i.e.*

$$\mathcal{L}(\bar{x}, \bar{\theta}, \gamma) \leq \mathcal{L}(\bar{x}, \bar{\theta}, \bar{\gamma}) \leq \mathcal{L}(x, \bar{\theta}, \bar{\gamma}), \quad \forall x \in U, \ \forall \gamma \in \Gamma.$$

Proof Suppose that $\mathcal{K}_{\bar{p}}$ and $\mathcal{H}$ admit a nonlinear separation and $\bar{x} \in R$. Then for each $x \in U$, $y \in F(x)$ and $z \in G(x)$, we have

$$\langle \bar{\theta}, \bar{y} - y \rangle + \omega_0(-z, \bar{\gamma}) \leq 0;$$

or equivalently

$$\sup_{y \in F(x)} \langle \bar{\theta}, -y \rangle + \sup_{z \in G(x)} \omega_0(-z, \bar{\gamma}) \leq \langle \bar{\theta}, -\bar{y} \rangle.$$

By assumption (B), for any $x \in U$, $\gamma \in \Gamma$, we have

$$\sup_{z \in G(\bar{x}) \cap -D} \omega_0(-z, \gamma) \geq 0.$$

Since $\sup_{y \in F(\bar{x})} \langle \bar{\theta}, -y \rangle = \langle \bar{\theta}, -\bar{y} \rangle$, we deduce

$$\sup_{y \in F(\bar{x})} \langle \bar{\theta}, -y \rangle + \sup_{z \in G(\bar{x}) \cap -D} \omega_0(-z, \gamma) \geq$$

$$\sup_{y \in F(\bar{x})} \langle \bar{\theta}, -y \rangle + \sup_{z \in G(\bar{x}) \cap -D} \omega_0(-z, \bar{\gamma}) \geq \langle \bar{\theta}, -\bar{y} \rangle \geq$$

$$\sup_{y \in F(x)} \langle \bar{\theta}, -y \rangle + \sup_{z \in G(x)} \omega_0(-z, \bar{\gamma}).$$

Therefore,

$$\mathcal{L}(\bar{x},\bar{\theta},\gamma) \leq \mathcal{L}(\bar{x},\bar{\theta},\bar{\gamma}) \leq \mathcal{L}(x,\bar{\theta},\bar{\gamma}), \forall x \in U,\ \forall \gamma \in \Gamma;$$

i.e. $(\bar{x},\bar{\gamma})$ is a saddle point for the generalized Lagrangian function on $z \in G(x) \cap -D$ and for all $x \in U$.
Conversely, suppose that $(\bar{x},\bar{\gamma})$ is a saddle point for the generalized Lagrangian function $\mathcal{L}(x,\bar{\theta},\gamma)$, then

$$\mathcal{L}(\bar{x},\bar{\theta},\gamma) \leq \mathcal{L}(\bar{x},\bar{\theta},\bar{\gamma}) \leq \mathcal{L}(x,\bar{\theta},\bar{\gamma}),\ \forall x \in U,\ \forall \gamma \in \Gamma,$$

or, equivalently for each $\gamma \in \Gamma$, we have

$$\sup_{z\in G(\bar{x})} \omega_0(-z,\gamma) + \sup_{y\in F(\bar{x})} \langle \bar{\theta}, -y\rangle \geq$$

$$\sup_{z\in G(\bar{x})} \omega_0(-z,\bar{\gamma}) + \sup_{y\in F(\bar{x})} \langle \bar{\theta}, -y\rangle \geq \sup_{z\in G(x)} \omega_0(-z,\bar{\gamma}) + \sup_{y\in F(x)} \langle \bar{\theta}, -y\rangle. \tag{10}$$

First, we prove that $\bar{x} \in R$. On the contrary, suppose that $\bar{x} \notin R$. Then $G(\bar{x}) \cap -D = \emptyset$. So, in the first inequality in (10), $-z \notin D$ and by assumption (B), we have

$$\inf_{\gamma\in\Gamma} \sup_{z\in G(\bar{x})} \omega_0(-z,\gamma) = -\infty,$$

which contradicts the first inequality in (10). Therefore, $\bar{x} \in R$ and

$$\sup_{z\in G(\bar{x})} \omega_0(-z,\bar{\gamma}) \leq 0. \tag{11}$$

On the other hand, by assumption (B), we have $\sup_{z\in G(\bar{x})\cap -D} \omega_0(-z,\bar{\gamma}) \geq 0$. Then (11) implies

$$\sup_{z\in G(\bar{x})\cap -D} \omega_0(-z,\bar{\gamma}) = 0.$$

Since $\sup_{y\in F(\bar{x})} \langle \bar{\theta}, -y\rangle = \langle \bar{\theta}, -\bar{y}\rangle$, by the second part of (10), we deduce

$$\sup_{z\in G(x)} \omega_0(-z,\bar{\gamma}) + \sup_{y\in F(x)} \langle \bar{\theta}, -y\rangle + \langle \bar{\theta}, \bar{y}\rangle \leq 0.$$

Or equivalently, for each $x \in U,\ y \in F(x)$ and $z \in G(x)$, we have

$$\langle \bar{\theta}, \bar{y} - y\rangle + \omega_0(-z,\bar{\gamma}) \leq 0.$$

Which shows that $\mathcal{K}_{\bar{p}}$ and $\mathcal{H}$ admit a nonlinear separation. □

Remark 4.2 If $\bar{\theta} \in C^{+i}$ in theorem 4.3, then we obtain a similar result for the regular nonlinear separation.

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