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Bloch constant and Landau's theorem for planar p-harmonic mappings ☆

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ABSTRACT

In this paper, our main aim is to introduce the concept of planar p-harmonic mappings and investigate the properties of these mappings. First, we discuss the p-harmonic Bloch mappings. Two estimates on the Bloch constant are obtained, which are generalizations of the main results in Colonna (1989) [9]. As a consequence of these investigations, we establish a Bloch and Landau's theorem for p-harmonic mappings.

1. Introduction

A $2p$-times continuously differentiable complex-valued function $f = u + iv$ in a domain $D \subseteq \mathbb{C}$ is *p-harmonic* if f satisfies the p-harmonic equation

$$\Delta^p f := \Delta(\Delta^{p-1}) f = 0,$$

where Δ represents the Laplacian operator

$$\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

When $p = 1$ (resp. $p = 2$), the mapping f is called harmonic (resp. biharmonic), and the properties of these two classes of mappings have been investigated extensively by many authors [1–3,8,11,14,16]. If f is harmonic in a simply connected domain $D \subset \mathbb{C}$, then there are two analytic functions g and h on D such that $f = h + \bar{g}$. In the case of a biharmonic mapping F, one has $F = |z|^2 G + H$, where G and H are complex-valued harmonic functions in D. Throughout this paper we consider p-harmonic mappings of the unit disk $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$.

Concerning p-harmonic mappings, we have the following characterization which is crucial in our investigations.

Proposition 1. *A mapping f is p-harmonic in $\mathbb{D}$ if and only if f has the following representation*:

$$f(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z), \tag{1.1}$$

where G_{p-k+1} is harmonic for each $k \in \{1, \ldots, p\}$.

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We remark that the proof of Proposition 1 is straightforward and we omit its proof here. Furthermore, representation (1.1) continues to hold even if f is p-harmonic in a simply connected domain D.

A well-known harmonic version of the classical Schwarz lemma due to Heinz [13] states that if $f:\mathbb{D}\to\mathbb{D}$ is harmonic such that $f(0)=0$, then

$$\left|f(z)\right| \leqslant \frac{4}{\pi}\arctan|z| \leqslant \frac{4}{\pi}|z|, \quad z\in\mathbb{D}. \tag{1.2}$$

This result played an important role in determining bounds for Bloch and Landau constants (see [4,6,7,19]). We refer to works of Minda [21] (see also [20]) and the references therein for estimates concerning these constants for analytic functions.

For a harmonic mapping f, the Jacobian of f is given by $J_f=|f_z|^2-|f_{\bar z}|^2$, and by the inverse function theorem, f is locally univalent (one-to-one) if J_f is nonzero. A result of Lewy shows that the converse is also true for harmonic mappings, i.e. for harmonic f, $J_f\neq 0$ if and only if f is locally univalent. However, if we let

$$\lambda_f=\big||f_z|-|f_{\bar z}|\big| \quad \text{and} \quad \Lambda_f=|f_z|+|f_{\bar z}|,$$

then $J_f=\lambda_f\Lambda_f$ if $J_f\geqslant 0$.

Definition 1. A p-harmonic function f is called a *p-harmonic Bloch function* if

$$B_f=\sup_{z,w\in\mathbb{D},\, z\neq w}\frac{|f(z)-f(w)|}{\rho(z,w)}<\infty,$$

where

$$\rho(z,w)=\frac{1}{2}\log\left(\frac{1+|\frac{z-w}{1-\bar z w}|}{1-|\frac{z-w}{1-\bar z w}|}\right)$$

denotes the hyperbolic distance in $\mathbb{D}$, and B_f is called the Bloch constant of f.

In [9], Colonna discussed the harmonic Bloch functions and established that the Bloch constant B_f of a harmonic mapping $f=h+\overline{g}$ can be expressed in terms of the moduli of the derivatives of g and h, see [9, Theorem 1]. Also she obtained a bound for the Bloch constant for the family of harmonic mappings f of $\mathbb{D}$ into itself and showed that this bound is the best possibility, see [9, Theorem 3]. One of our aims in this paper is to generalize her results for p-harmonic Bloch mappings.

The classical theorem of Landau proves the existence of a $\rho=\rho(M)>0$ such that every function f, analytic in $\mathbb{D}$ with $f(0)=f'(0)-1=0$ and $|f(z)|<M$, is univalent in the disk $\mathbb{D}_\rho=\{z\colon |z|<\rho\}$ and in addition, the range $f(\mathbb{D}_\rho)$ contains a disk of radius $M\rho^2$ (cf. [15]). Recently, many authors considered Landau's theorem for planar harmonic mappings, see, for example, [4,6,7,10,12,18,19,22] and biharmonic mappings, see [1,5,6,17]. In Theorem 2, we derive an analogous result for planar p-harmonic mappings.

2. Preliminaries

We now recall a lemma which is a generalization of [10, Lemma 3] and [22, Theorem 4].

Lemma A. *(See [18, Lemma 2.1].) Suppose that $f=h+\overline{g}$ is a harmonic mapping of $\mathbb{D}$ with $h(z)=\sum_{n=1}^{\infty}a_nz^n$ and $g(z)=\sum_{n=1}^{\infty}b_nz^n$ for $z\in\mathbb{D}$. If $J_f(0)=1$ and $|f(z)|<M$, then*

$$|a_n|,|b_n|\leqslant\sqrt{M^2-1}, \quad n=2,3,\ldots, \tag{2.1}$$

$$|a_n|+|b_n|\leqslant\sqrt{2M^2-2}, \quad n=2,3,\ldots \tag{2.2}$$

and

$$\lambda_f(0)\geqslant\lambda(M)=\begin{cases}\frac{\sqrt{2}}{\sqrt{M^2-1}+\sqrt{M^2+1}} & \text{if } 1\leqslant M\leqslant\frac{\pi}{2\sqrt[4]{2\pi^2-16}},\\ \frac{\pi}{4M} & \text{if } M>\frac{\pi}{2\sqrt[4]{2\pi^2-16}}.\end{cases} \tag{2.3}$$

The following lemma is crucial in the proof of Theorems 2 and 3. This lemma has been proved by the authors in [7] with additional assumption that $f(0)=0$. Without this assumption, the proof is slightly different and we omit its proof here.

Lemma 1. *Let $f = h + \bar{g}$ be a harmonic mapping of $\mathbb{D}$ such that $|f(z)| < M$ with $h(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$. Then $|a_0| \leqslant M$ and for any $n \geqslant 1$,*

$$|a_n| + |b_n| \leqslant \frac{4M}{\pi}. \tag{2.4}$$

The estimate (2.4) is sharp. The extremal functions are $f(z) \equiv M$ or

$$f_n(z) = \frac{2M\alpha}{\pi} \arg\left(\frac{1+\beta z^n}{1-\beta z^n}\right),$$

where $|\alpha| = |\beta| = 1$.

3. Main results and their proofs

In [9, Theorem 1,3], Colonna discussed harmonic Bloch mappings and obtained the following result.

Theorem B. *(See [9, Theorem 1].) Let f be a harmonic mapping in $\mathbb{D}$. Then*

(1) $B_f = \sup_{z\in\mathbb{D}}(1-|z|^2)(|h'(z)| + |g'(z)|)$.
(2) $B_f \leqslant 4/\pi$, *if in addition $f(\mathbb{D}) \subset \mathbb{D}$.*

In the following, we consider the p-harmonic Bloch mappings. Our corresponding result is as follows.

Theorem 1. *Let f be a p-harmonic mapping in $\mathbb{D}$ of the form (1.1) satisfying $B_f < \infty$. Then*

$$\begin{aligned} B_f &= \sup_{z\in\mathbb{D}}\big(1-|z|^2\big)\left\{\left|\sum_{k=1}^{p}|z|^{2(k-1)}(G_{p-k+1})_z(z) + \sum_{k=1}^{p}(k-1)\bar{z}|z|^{2(k-2)}G_{p-k+1}(z)\right|\right.\\ &\quad \left. + \left|\sum_{k=1}^{p}|z|^{2(k-1)}(G_{p-k+1})_{\bar{z}}(z) + \sum_{k=1}^{p}(k-1)z|z|^{2(k-2)}G_{p-k+1}(z)\right|\right\}\\ &\geqslant \sup_{z\in\mathbb{D}}\big(1-|z|^2\big)\left|\left|\sum_{k=1}^{p}|z|^{2(k-1)}(G_{p-k+1})_z(z)\right| - \left|\sum_{k=1}^{p}|z|^{2(k-1)}(G_{p-k+1})_{\bar{z}}(z)\right|\right| \end{aligned} \tag{3.1}$$

and (3.1) is sharp. The equality sign in (3.1) occurs when f is analytic or anti-analytic.

Furthermore, if for each $k \in \{1, 2, \dots, p\}$, the harmonic functions G_{p-k+1} in (1.1) are such that $|G_{p-k+1}(z)| \leqslant M$, then

$$B_f \leqslant 2M\phi_p(y_0). \tag{3.2}$$

Here y_0 is the unique root in $(0, 1)$ of the equation $\phi_p'(y) = 0$, where

$$\phi_p(y) = \frac{2}{\pi}\sum_{k=1}^{p} y^{2(k-1)} + y\big(1-y^2\big)\sum_{k=2}^{p}(k-1)y^{2(k-2)}. \tag{3.3}$$

The bound in (3.2) is sharp when $p = 1$, where M is a positive constant. The extremal functions are $f(z) = \frac{2M\alpha}{\pi}\,\mathrm{Im}(\log\frac{1+S(z)}{1-S(z)})$, where $|\alpha| = 1$ and $S(z)$ is a conformal automorphism of $\mathbb{D}$.

Proof. We first calculate B_f. Let $w = z + re^{i\theta}$ $(0 \leqslant \theta < 2\pi)$. Then, as in [9], we have

$$\begin{aligned} B_f &= \sup_{z\in\mathbb{D}} \max_{0\leqslant\theta<2\pi} \lim_{r\to 0} \frac{|f(z+re^{i\theta}) - f(z)|}{\rho(z+re^{i\theta}, z)}\\ &= \sup_{z\in\mathbb{D}} \max_{0\leqslant\theta<2\pi} \lim_{r\to 0} \frac{|f(z+re^{i\theta}) - f(z)|}{r}\frac{r}{\rho(z+re^{i\theta}, z)}\\ &= \sup_{z\in\mathbb{D}}\big(1-|z|^2\big) \max_{0\leqslant\theta<2\pi}\big|\cos\theta f_x(z) + \sin\theta f_y(z)\big|\\ &= \sup_{z\in\mathbb{D}}\big(1-|z|^2\big)\big(|f_z| + |f_{\bar{z}}|\big)\\ &= \sup_{z\in\mathbb{D}}\big(1-|z|^2\big)\left\{\left|\sum_{k=1}^{p}|z|^{2(k-1)}(G_{p-k+1})_z(z) + \sum_{k=1}^{p}(k-1)\bar{z}|z|^{2(k-2)}G_{p-k+1}(z)\right|\right. \end{aligned}$$

$$+\left|\sum_{k=1}^{p}|z|^{2(k-1)}(G_{p-k+1})_{\bar z}(z)+\sum_{k=1}^{p}(k-1)z|z|^{2(k-2)}G_{p-k+1}(z)\right|\Bigg\}$$

from which we obtain that

$$B_f \geqslant \sup_{z\in\mathbb{D}}\big(1-|z|^2\big)\left|\left|\sum_{k=1}^{p}|z|^{2(k-1)}(G_{p-k+1})_z(z)\right|-\left|\sum_{k=1}^{p}|z|^{2(k-1)}(G_{p-k+1})_{\bar z}(z)\right|\right|. \tag{3.4}$$

For the proof of (3.2), we simply adopt the method of the proof of [9, Theorem 3]. Using the above expression for B_f, it follows easily that

$$\begin{aligned} B_f &\leqslant \sup_{z\in\mathbb{D}}\big(1-|z|^2\big)\left[\sum_{k=1}^{p}|z|^{2(k-1)}\big(\big|(G_{p-k+1})_z(z)\big|+\big|(G_{p-k+1})_{\bar z}(z)\big|\big)+2\sum_{k=1}^{p}(k-1)|z|^{2k-3}\big|G_{p-k+1}(z)\big|\right] \\ &\leqslant \sup_{z\in\mathbb{D}}\left[\frac{4M}{\pi}\sum_{k=1}^{p}|z|^{2(k-1)}+2M\sum_{k=1}^{p}(k-1)\big(1-|z|^2\big)|z|^{2k-3}\right] \\ &\leqslant 2M\max_{z\in\mathbb{D}}\phi_p\big(|z|\big), \end{aligned}$$

where $\phi_p(y)$ is defined by (3.3). A computation gives

$$\phi_p(y)=\frac{2}{\pi}\left(\frac{1-y^{2p}}{1-y^2}\right)+y\left(\frac{1-y^{2p}}{1-y^2}-py^{2(p-1)}\right)$$

and therefore, we see that there is a unique root y_0 of $\phi_p'(y)=0$ lying in the unit interval $(0,1)$ such that

$$B_f \leqslant 2M\phi_p(y_0).$$

The proof of this theorem easily follows. □

We remark that, when $p=1$, (3.1) (resp. (3.2)) is a generalization of [9, Theorem 1] (resp. [9, Theorem 3]).

Theorem 2. *Let $f(z)=\sum_{k=1}^{p}|z|^{2(k-1)}G_{p-k+1}(z)$, where $f(0)=G_p(0)=J_f(0)-1=0$ and for each $k\in\{1,\dots,p\}$,*

(i) *G_{p-k+1} is harmonic in $\mathbb{D}$, and*
(ii) *$|G_{p-k+1}(z)|\leqslant M$ in $\mathbb{D}$ for some $M\geqslant 1$.*

Then there is a positive number ρ_0 such that f is univalent in $\mathbb{D}_{\rho_0}$, where ρ_0 $(0<\rho_0<1)$ is a unique root of the equation $A(\rho)=0$ and

$$A(\rho)=\lambda-\frac{T\rho(2-\rho)}{(1-\rho)^2}-\frac{4M}{\pi(1-\rho)^2}\sum_{k=1}^{p-1}\rho^{2k}-2M\sum_{k=1}^{p-1}k\rho^{2k-1},$$

or equivalently,

$$A(\rho)=\lambda-\frac{T\rho(2-\rho)}{(1-\rho)^2}-\frac{4M(1-\rho^{2p})}{\pi(1-\rho)^2(1-\rho^2)}-2M\rho\left(\frac{1-p\rho^{2p-2}+(p-1)\rho^{2p}}{(1-\rho^2)^2}\right).$$

Here $\lambda:=\lambda(M)$ is defined by (2.3) and

$$T:=T(M)=\begin{cases}\sqrt{2M^2-2} & \text{if } 1\leqslant M\leqslant \pi/\sqrt{\pi^2-8},\\ \frac{4M}{\pi} & \text{if } M>\pi/\sqrt{\pi^2-8}.\end{cases} \tag{3.5}$$

Moreover, the range $f(\mathbb{D}_{\rho_0})$ contains a univalent disk $\mathbb{D}_{R_0}$, where

$$R_0=\rho_0\left[\lambda-\frac{T\rho_0}{1-\rho_0}-\frac{4M}{\pi(1-\rho_0)}\sum_{k=1}^{p-1}\rho_0^{2k}\right].$$

Proof. For each $k\in\{1,\dots,p\}$, we may represent the harmonic functions $G_{p-k+1}(z)$ in series form as

$$G_{p-k+1}(z)=a_{0,p-k+1}+\sum_{j=1}^{\infty}a_{j,p-k+1}z^j+\sum_{j=1}^{\infty}\bar b_{j,p-k+1}\bar z^j.$$

We consider the cases $k=1$ and $2 \leqslant k \leqslant p$ separately. Thus, $k=1$ gives

$$G_p(z) = a_{0,p} + \sum_{j=1}^{\infty} a_{j,p} z^j + \sum_{j=1}^{\infty} \bar{b}_{j,p} \bar{z}^j.$$

Then, because $G_p(0)=0$, it follows that $a_{0,p}=0$. Using Lemmas A and 1, we have

$$|a_{n,p}| + |b_{n,p}| \leqslant T(M),$$

where $T(M)$ is defined by (3.5). For $2 \leqslant k \leqslant p$, we use Lemma 1 and obtain that

$$|a_{j,p-k+1}| + |b_{j,p-k+1}| \leqslant \frac{4M}{\pi} \quad \text{for each } j \geqslant 1.$$

Also, we observe that

$$J_f(0) = \left|(G_p)_z(0)\right|^2 - \left|(G_p)_{\bar{z}}(0)\right|^2 = J_{G_p}(0) = 1$$

and hence, by Theorem B(2), Lemma A and (1.2), we have

$$\lambda_f(0) \geqslant \lambda(M),$$

where $\lambda(M)$ is defined by (2.3).

Now, we fix ρ with $0<\rho<1$. To prove the univalency of f, we choose two distinct points z_1, z_2 in $\mathbb{D}_\rho$. Let $\gamma = \{(z_2-z_1)t+z_1\colon\ 0 \leqslant t \leqslant 1\}$. Then

$$\begin{aligned}
\left|f(z_1)-f(z_2)\right| &= \left|\int_\gamma f_z(z)\,dz + f_{\bar{z}}(z)\,d\bar{z}\right| \\
&\quad - \left|\int_\gamma (G_p)_z(z)\,dz + (G_p)_{\bar{z}}(z)\,d\bar{z} + \sum_{k=1}^{p-1}\int_\gamma |z|^{2k}\big[(G_{p-k})_z(z)\,dz \right. \\
&\quad \left. + (G_{p-k})_{\bar{z}}(z)\,d\bar{z}\big] + \sum_{k=1}^{p-1}\int_\gamma kG_{p-k}(z)\big(\bar{z}^k z^{k-1}\,dz + \bar{z}^{k-1}z^k\,d\bar{z}\big)\right| \\
&\geqslant \left|\int_\gamma (G_p)_z(0)\,dz + (G_p)_{\bar{z}}(0)\,d\bar{z}\right| \\
&\quad - \left|\sum_{k=1}^{p-1}\int_\gamma |z|^{2k}\big[(G_{p-k})_z(z)\,dz + (G_{p-k})_{\bar{z}}(z)\,d\bar{z}\big]\right| \\
&\quad - \left|\sum_{k=1}^{p-1}\int_\gamma kG_{p-k}(z)\big(\bar{z}^k z^{k-1}\,dz + \bar{z}^{k-1}z^k\,d\bar{z}\big)\right| \\
&\quad - \left|\int_\gamma \big[(G_p)_z(z) - (G_p)_z(0)\big]\,dz + \big[(G_p)_{\bar{z}}(z) - (G_p)_{\bar{z}}(0)\big]\,d\bar{z}\right| \\
&\geqslant |z_1-z_2|\Bigg[\lambda(M) - \sum_{n=2}^{\infty} n\big(|a_{n,p}|+|b_{n,p}|\big)\rho^{n-1} - \sum_{k=1}^{p-1} 2kM\rho^{2k-1} \\
&\quad - \sum_{k=1}^{p-1}\rho^{2k}\sum_{n=1}^{\infty} n\big(|a_{n,p-k}|+|b_{n,p-k}|\big)\rho^{n-1}\Bigg] \\
&\geqslant |z_1-z_2|A(\rho),
\end{aligned}$$

where

$$A(\rho) = \lambda(M) - \frac{T(M)\rho(2-\rho)}{(1-\rho)^2} - \frac{4M}{\pi(1-\rho)^2}\sum_{k=1}^{p-1}\rho^{2k} - 2M\sum_{k=1}^{p-1}k\rho^{2k-1}.$$

Table 1
Values of ρ_0 and R_0 for Theorem 2, and the values of ρ_0' and R_0' of [18, Theorem 2.4].

M	p	$\rho_0 = \rho_0(M, p)$	$R_0 = R_0(M, \rho_0(M, p))$	ρ_0'	R_0'
1.1296	1	0.281269	0.277343	0.2812697	0.1137838
2	1	0.0716508	0.0416831	0.0716508	0.0145914
2.2976	1	0.0537573	0.0273103	0.0537575	0.0094419
3	1	0.0326028	0.0127324	0.0312014	0.0041490

Table 2
Values of ρ_0 and R_0 for Theorem 2 and the corresponding values of ρ_2 and R_2 of [1, Theorem 1].

M	p	$\rho_0 = \rho_0(M, p)$	$R_0 = R_0(M, \rho_0(M, p))$	ρ_2	R_2
1	2	0.336401	0.263358	0.1111923	0.0564158
1.1296	2	0.154845	0.0802659	0.0901011	0.0406733
2	2	0.0420325	0.0117913	0.0313502	0.0081254
2.2976	2	0.0315535	0.00768366	0.0239999	0.0054270
3	2	0.0187875	0.00351868	0.0142671	0.0024785
1	3	0.306459	0.248657	–	–
1.1296	3	0.151744	0.0792771	–	–
2	3	0.0419713	0.0117812	–	–
2.2976	3	0.0315276	0.00767996	–	–
3	3	0.0187819	0.00351805	–	–

It is not difficult to verify that $A(\rho)$ is decreasing,

$$\lim_{\rho\to 0+} A(\rho) = \lambda(M) \quad \text{and} \quad \lim_{\rho\to 1-} A(\rho) = -\infty.$$

Hence there exists a unique root $\rho_0 \in (0,1)$ of the equation $A(\rho) = 0$. This shows that $|f(z_1) - f(z_2)| > 0$ for any two distinct points z_1, z_2 in $|z| < \rho_0$, which proves the univalency of f in the disk $\mathbb{D}_{\rho_0}$.

Finally, proceeding exactly in the same way, we consider any z with $|z| = \rho_0$. Then, we have

$$\begin{aligned} \big|f(z)\big| &\geqslant |a_{1,p}z + b_{1,p}\bar{z}| - \left|\sum_{n=2}^{\infty} a_{n,p}z^n + \bar{b}_{n,p}\bar{z}^n\right| - \left|\sum_{k=1}^{p-1} |z|^{2k} G_{p-k}(z)\right| \\ &\geqslant \rho_0\left[\lambda(M) - \frac{T(M)\rho_0}{1-\rho_0} - \frac{4M}{\pi(1-\rho_0)}\sum_{k=1}^{p-1}\rho_0^{2k}\right]. \end{aligned}$$

The proof of this theorem is complete. □

We remark that when $p = 1$ (resp. $p = 2$), Theorem 2 improved [18, Theorem 2.4] (resp. [1, Theorem 1]). It is important to note that in the case $p = 1$ of Theorem 2, we just need to remove the term(s) that involves the summation sign. By a simplification, this case leads to the following corollary and we omit its proof (see Table 2 for computational values obtained with the help of `Mathematica` package). In Table 1, the left columns refer to values obtained from Theorem 2 for the case $p = 1$ (i.e. Corollary 1) while the right two columns correspond to value of Liu obtained in [18, Theorem 2.4].

Corollary 1. *Let f be harmonic in $\mathbb{D}$ such that $|f(z)| \leqslant M$ for some $M \geqslant 1$ and $f(0) = J_f(0) - 1 = 0$. Then f is univalent in $\mathbb{D}_{\rho_0}$ with*

$$\rho_0 = 1 - \sqrt{T/(\lambda + T)}.$$

Moreover, the range $f(\mathbb{D}_{\rho_0})$ contains a univalent disk $\mathbb{D}_{R_0}$, where

$$R_0 = \big(1 - \sqrt{T/(\lambda+T)}\big)\big[\lambda - T + \sqrt{T(\lambda+T)}\big].$$

Here $\lambda = \lambda(M)$ and $T = T(M)$ are defined by (2.3) *and* (3.5)*.*

Theorem 3. *Let $f(z) = |z|^{2(p-1)}G(z)$, where $p > 1$, G be harmonic with $G(0) = J_G(0) - 1 = 0$, and $|G(z)| \leqslant M$ in $\mathbb{D}$ for some positive constant M. Then f is univalent in $\mathbb{D}_{\rho_1}$, where*

$$\rho_1 = \frac{\lambda + 2T - \sqrt{4T^2 + \lambda T}}{\lambda + 3T}. \tag{3.6}$$

Moreover, the range $f(\mathbb{D}_{\rho_1})$ *contains a univalent disk* $\mathbb{D}_{R_1}$*, where*

$$R_1 = \frac{(\lambda + 2T - \sqrt{4T^2 + \lambda T})^{2p}}{(\lambda + 3T)^{2p-1}}. \tag{3.7}$$

Here $\lambda = \lambda(M)$ *and* $T = T(M)$ *are defined by* (2.3) *and* (3.5).

Proof. Let

$$G(z) = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \bar{b}_n \bar{z}_n, \quad z \in \mathbb{D}.$$

Then, using Lemmas A and 1 we have

$$|a_n| + |b_n| \leqslant T(M) \quad \text{for } n \geqslant 2.$$

Note that

$$J_G(0) = |a_1|^2 - |b_1|^2 = 1$$

and hence, by Theorem B(2), Lemma A and (1.2), we have

$$\lambda_G(0) \geqslant \lambda(M).$$

Fix ρ with $0 < \rho < 1$. To prove the univalency of f, we choose two distinct points z_1, z_2 in $\mathbb{D}_\rho$. Let $\gamma = \{(z_2 - z_1)t + z_1\colon\ 0 \leqslant t \leqslant 1\}$. Then

$$\begin{aligned}
\big|f(z_1) - f(z_2)\big| &= \left| \int_\gamma f_z(z)\,dz + f_{\bar{z}}(z)\,d\bar{z} \right| \\
&= \Bigg| \int_\gamma p|z|^{2(p-1)}\big(G_z(0)\,dz + G_{\bar{z}}(0)\,d\bar{z}\big) \\
&\quad + \int_\gamma (p-1)|z|^{2(p-1)} \left[\frac{G - zG_z(0)}{z}\,dz + \frac{G - \bar{z}G_{\bar{z}}(0)}{\bar{z}}\,d\bar{z} \right] \\
&\quad + \int_\gamma |z|^{2(p-1)} \big[\big(G_z - G_z(0)\big)\,dz + \big(G_z - G_z(0)\big)\,d\bar{z}\big] \Bigg| \\
&\geqslant |z_1 - z_2| \left(\int_0^1 |z|^{2(p-1)}\,dt \right) \left[\lambda(M) - 2\sum_{n=2}^{\infty} \big(|a_n| + |b_n|\big)\rho^{n-1} - \sum_{n=2}^{\infty} n\big(|a_n| + |b_n|\big)\rho^{n-1} \right] \\
&\geqslant |z_1 - z_2| \left(\int_0^1 |z|^{2(p-1)}\,dt \right) B(\rho),
\end{aligned}$$

where

$$B(\rho) = \lambda - T\frac{2\rho}{1-\rho} - T(M)\left(\frac{1}{(1-\rho)^2} - 1 \right).$$

Here $\lambda = \lambda(M)$ and $T = T(M)$ are defined by (2.3) and (3.5). If we can show that there exists a unique ρ_1 in the interval $(0, 1)$ such that $B(\rho_1) = 0$, then we would get that $f(z_1) \neq f(z_2)$ and so, $f(z)$ would be univalent for $\mathbb{D}_{\rho_1}$. Thus, for the proof of the first part, it suffices to show that ρ_1 is given by (3.6). In order to obtain this, we see that $B(\rho) = 0$ is equivalent to $C(\rho) = 0$, where

$$C(\rho) = \rho^2(\lambda + 3T) - 2\rho(\lambda + 2T) + \lambda.$$

We observe that $C(0) > 0 > C(1)$ and $C(4/3) > 0$, and so it is easy to see that the unique root ρ_1 that lies in the interval $(0, 1)$ is given by (3.6).

Table 3
Values of ρ_1 and R_1 for Theorem 3 and the corresponding values of ρ_3 and R_3 of [1, Theorem 2] for $p = 2$.

M	p	$\rho_1 = \rho_1(M, p)$	$R_1 = R_1(M, \rho_1(M, p))$	ρ_3	R_3
1.1296	2	0.18145	0.00316981	0.070212	0.000181608
2	2	0.0381711	0.0000164341	0.023814	3.9856×10^{-6}
2.2976	2	0.0281849	5.75395×10^{-6}	0.0181752	1.54162×10^{-6}
3	2	0.0167757	9.28296×10^{-7}	0.0107616	2.44939×10^{-7}
1.1296	3	0.18145	0.000104363	–	–
2	3	0.0381711	2.3945×10^{-8}	–	–
2.2976	3	0.0281849	4.57087×10^{-9}	–	–
3	3	0.0167757	2.61245×10^{-10}	–	–

Furthermore, for the proof of the second part, we see that for any z with $|z| = \rho_1$,

$$\left|f(z)\right| = \rho_1^{2(p-1)}\left|\sum_{n=1}^{\infty} a_n z^n + \bar{b}_n \bar{z}^n\right| \geqslant \rho_1^{2(p-1)}\left[|a_1 z + \bar{b}_1 \bar{z}| - \left|\sum_{n=2}^{\infty} a_n z^n + \bar{b}_n \bar{z}^n\right|\right]$$
$$\geqslant \rho_1^{2p-1}\left[\lambda(M) - T(M)\frac{\rho_1}{1-\rho_1}\right].$$

Now, to complete the proof, we need to show that R_1 given by (3.7) is obtained from

$$R_1 = \rho_1^{2p-1}\left[\lambda - T\frac{\rho_1}{1-\rho_1}\right]. \tag{3.8}$$

In order to verify this, we compute that

$$\frac{\rho_1}{1-\rho_1} = \frac{\lambda + 2T - \sqrt{4T^2 + \lambda T}}{\sqrt{4T^2 + \lambda T} + T}.$$

Multiplying the numerator and the denominator of the right-hand side expression by the quantity $\sqrt{4T^2 + \lambda T} - T$, and then simplifying the resulting expression yields that

$$\frac{\rho_1}{1-\rho_1} = -2 + \frac{\sqrt{4T^2 + \lambda T}}{T}$$

and therefore,

$$\lambda - T\frac{\rho_1}{1-\rho_1} = \lambda + 2T - \sqrt{4T^2 + \lambda T}.$$

Using this and (3.6), we easily see that R_1 given by (3.7) satisfies the desired condition (3.8). The proof of this theorem is finished. □

We remark that, when $p = 2$, Theorem 3 is an improvement of [1, Theorem 2] and for $p \geqslant 3$ the result is new. In Table 3, the left columns refer to values obtained from Theorem 3 for the case $p = 2$ and $p = 3$ while the right two columns correspond to the values obtained from [1, Theorem 2] for the case $p = 2$. In particular, when $M = 1$, Theorem 3 is sharp in which case the harmonic mapping f is affine.

References

[1] Z. Abdulhadi, Y. Abu Muhanna, Landau's theorem for biharmonic mappings, J. Math. Anal. Appl. 338 (2008) 705–709.
[2] Z. Abdulhadi, Y. Abu Muhanna, S. Khoury, On univalent solutions of the biharmonic equations, J. Inequal. Appl. 5 (2005) 469–478.
[3] Z. Abdulhadi, Y. Abu Muhanna, S. Khoury, On some properties of solutions of the biharmonic equation, Appl. Math. Comput. 177 (2006) 346–351.
[4] H. Chen, P.M. Gauthier, W. Hengartner, Bloch constants for planar harmonic mappings, Proc. Amer. Math. Soc. 128 (2000) 3231–3240.
[5] Sh. Chen, S. Ponnusamy, X. Wang, Landau's theorem for certain biharmonic mappings, Appl. Math. Comput. 208 (2009) 427–433.
[6] Sh. Chen, S. Ponnusamy, X. Wang, Properties of some classes of planar harmonic and planar biharmonic mappings, Complex Anal. Oper. Theory (2010), doi:10.1007/s11785-010-0061-x.
[7] Sh. Chen, S. Ponnusamy, X. Wang, Coefficient estimates, Landau's theorem and Bohr's inequality for harmonic mappings, submitted for publication.
[8] J.G. Clunie, T. Sheil-Small, Harmonic univalent functions, Ann. Acad. Sci. Fenn. Ser. A I 9 (1984) 3–25.
[9] F. Colonna, The Bloch constant of bounded harmonic mappings, Indiana Univ. Math. J. 38 (1989) 829–840.
[10] M. Dorff, M. Nowark, Landau's theorem for planar harmonic mappings, Comput. Methods Funct. Theory 4 (2004) 151–158.
[11] P. Duren, Harmonic Mappings in the Plane, Cambridge Univ. Press, 2004.
[12] A. Grigoryan, Landau and Bloch theorems for planar harmonic mappings, Complex Var. Elliptic Equ. 51 (2006) 81–87.
[13] E. Heinz, On one-to-one harmonic mappings, Pacific J. Math. 9 (1959) 101–105.
[14] S.A. Khuri, Biorthogonal series solution of Stokes flow problems in sectorial regions, SIAM J. Appl. Math. 56 (1996) 19–39.
[15] E. Landau, Über die Bloch'sche Konstante und zwei verwandte Weltkonstanten, Math. Z. 30 (1929) 608–634.

[16] W.E. Langlois, Slow Viscous Flow, Macmillan Company, 1964.
[17] M. Liu, Landau theorems for biharmonic mappings, Complex Var. Elliptic Equ. 53 (2008) 843–855.
[18] M. Liu, Landau's theorem for planar harmonic mappings, Comput. Math. Appl. 57 (7) (2009) 1142–1146.
[19] M. Liu, Estimates on Bloch constants for planar harmonic mappings, Sci. China Ser. A 52 (1) (2009) 87–93.
[20] X.Y. Liu, D. Minda, Distortion theorems for Bloch functions, Trans. Amer. Math. Soc. 333 (1) (1992) 325–338.
[21] D. Minda, The Bloch and Marden constants, in: Computational Methods and Function Theory, Valparaíso, 1989, in: Lecture Notes in Math., vol. 1435, Springer, Berlin, 1990, pp. 131–142.
[22] H. Xinzhong, Estimates on Bloch constants for planar harmonic mappings, J. Math. Anal. Appl. 337 (2008) 880–887.