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AN UPPER BOUND OF THE ESSENTIAL NORM OF COMPOSITION OPERATORS BETWEEN WEIGHTED BERGMAN SPACES*

Zhihua CHEN (陈志华)[1] *Liangying JIANG* (江良英)[2] *Qiming YAN* (颜启明)[1†]
1. Department of Mathematics, Tongji University, Shanghai 200092, China
2. Department of Applied Mathematics, Shanghai Finance University, Shanghai 201209, China
E-mail: *zzzhhc@tongji.edu.cn; liangying1231@163.com; yan_qiming@hotmail.com*

Abstract In this paper, we define the generalized counting functions in the higher dimensional case and give an upper bound of the essential norms of composition operators between the weighted Bergman spaces on the unit ball in terms of these counting functions. The sufficient condition for such operators to be bounded or compact is also given.

Key words essential norm; composition operator; weighted Bergman space

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1 Introduction

Let $D(=B_1)$ denote the unit disc of $\mathbb{C}$ and let φ be a holomorphic function on D with $\varphi(D)\subset D$. Then $C_\varphi f=f\circ\varphi$ defines a composition operator C_φ on the space of holomorphic functions in D.

For $p>0$, the Hardy space $H^p(D)$ is the space of functions f that are analytic on D and satisfy

$$\|f\|^p_{H^p(D)}=\sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{\mathrm{i}\theta})|^p\mathrm{d}\theta=\lim_{r\to1^-}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{\mathrm{i}\theta})|^p\mathrm{d}\theta<\infty.$$

The boundedness of C_φ on $H^p(D)$ for any $\varphi: D\to D$ is a consequence of Littlewood subordination principle. In 1973, Shapiro and Taylor [1] gave the following necessary condition for the compactness of C_φ on $H^2(D)$.

Theorem A If C_φ is compact on $H^2(D)$, then φ cannot have a finite angular derivative at any point of ∂D.

In 1987, Shapiro [2] considered the essential norm of composition operators C_φ on $H^2(D)$ and gave the following sufficient and necessary condition which involves the Nevanlinna counting function of those φ.

Theorem B Let $\|C_\varphi\|_e$ denote the essential norm of C_φ, regarded as an operator on $H^2(D)$. Then $\|C_\varphi\|^2_e=\limsup\limits_{|w|\to1^-}\frac{N_\varphi(w)}{(-\log|w|)}$, where $N_\varphi(w)=\sum\limits_{z\in\varphi^{-1}(w)}-\log|z|$ is the Nevanlinna

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†Corresponding author: Qiming YAN.

counting function and $\varphi^{-1}(w)$ denotes the sequence of $\varphi-$preimages of w, each point being repeated in the sequence according to its multiplicity. In particular, C_φ is compact on $H^2(D)$ if and only if $\lim\limits_{|w|\to 1^-}\frac{N_\varphi(w)}{(-\log|w|)}=0$.

In [3], Luo and Li extended Shapiro's result to the case of composition operator $C_\varphi: H^p(D)\to H^q(D)$ with $1<p\le q<\infty$.

Theorem C Let φ be a holomorphic self-map of D. For $1<p\le q<\infty$, $C_\varphi: H^p(D)\to H^q(D)$ is the composition operator induced by φ. If C_φ is bounded, then there exist constants C_1 and C_2 such that

$$C_1\limsup_{|w|\to 1^-}\frac{N_\varphi(w)}{(-\log|w|)^{\frac{q}{p}}}\le \|C_\varphi\|_e^q\le C_2\limsup_{|w|\to 1^-}\frac{N_\varphi(w)}{(-\log|w|)^{\frac{q}{p}}}.$$

For the case of several complex variables, denote by $B_N(r)=\{z\in\mathbb{C}^N:|z|<r\}$ the ball of $\mathbb{C}^N$ with radius r. Let $B_N=B_N(1)$ be the unit ball and $B_N(r)=rB_N$. Set $\partial B_N(r)=\{z\in\mathbb{C}^N:|z|=r\}$. Let $\mathrm{d}\tau$ be the Euclidean volume element of $\mathbb{C}^N=\mathbb{R}^{2N}$ with $\int_{B_N(r)}\mathrm{d}\tau=\frac{\pi^N}{N!}r^{2N}$. We have $\mathrm{d}\tau=r^{2N-1}dr\wedge\mathrm{d}\sigma$, where $\mathrm{d}\sigma$ is the induced volume element on ∂B_N. Let $\mathrm{d}\sigma_r=r^{2N-1}\mathrm{d}\sigma$ be the volume element of $\partial B_N(r)$ with $\int_{\partial B_N(r)}\mathrm{d}\sigma_r=\frac{2\pi^N}{(N-1)!}r^{2N-1}$.

For each p $(0<p<\infty)$, the Hardy space $H^p(B_N)$ is defined by

$$H^p(B_N)=\left\{f\in H(B_N):\sup_{0<r<1}\frac{(N-1)!}{2\pi^N}\int_{\partial B_N}|f(r\xi)|^p\mathrm{d}\sigma<\infty\right\}$$

and

$$\|f\|^p_{H^p(B_N)}=\sup_{0<r<1}\frac{(N-1)!}{2\pi^N}\int_{\partial B_N}|f(r\xi)|^p\mathrm{d}\sigma.$$

In B_N, many self-maps induce unbounded composition operators on the Hardy spaces of B_N. It is hard to give some sufficient condition for boundedness of C_φ, even φ is univalent and analytic in $\overline{B_N}$ cannot guarantee that C_φ is bounded.

In [4], MacCluer and Shapiro showed that

Theorem D Let $\varphi: B_N\to B_N$ be univalent with $\Omega_\varphi(z)=\frac{\|(\frac{\partial\varphi}{\partial z})\|^2}{|\det(\frac{\partial\varphi}{\partial z})|^2}$ is bounded in B_N, where $(\frac{\partial\varphi}{\partial z})$ is the Jacobi matrix of the map φ and $\|(\frac{\partial\varphi}{\partial z})\|^2=\sum\limits_{i,j=1}^{N}|\frac{\partial\varphi_i}{\partial z_j}|^2$. Then C_φ is bounded on $H^2(B_N)$. Furthermore, C_φ is compact on $H^2(B_N)$ if and only if φ has no finite angular derivative at any point of ∂B_N.

We note that if $\Omega_\varphi(z)$ is a well-defined function on B_N, then $\det(\frac{\partial\varphi}{\partial z})$ must be a nowhere zero function on B_N (cf. [5]).

In [5, 6], the authors gave an upper bound of the essential norms of composition operators between Hardy spaces of the unit ball in terms of the counting function in the higher dimensional value distribution theory defined by Professor Chern [7]. The sufficient condition for such operators to be bounded or compact is also given as follows.

Theorem E Let $\varphi(z)=(\varphi_1(z),\cdots,\varphi_N(z)):B_N\to B_N$ be a holomorphic map. For $1\le p\le q<\infty$, $C_\varphi: H^p(B_N)\to H^q(B_N)$ is the composition operator induced by φ. Assume that $a\le\Omega_\varphi(z)\le b$ on B_N with $a,b\in\mathbb{R}^+$. If

$$\limsup_{|w|\to 1^-}\frac{N_\varphi(w)}{(1-|w|)^{\frac{Nq}{p}-(N-1)}}=A<+\infty,$$

then C_φ is a bounded operator and there exists a positive constant C such that

$$\|C_\varphi\|_e^q \le C \limsup_{|w|\to 1^-} \frac{N_\varphi(w)}{(1-|w|)^{\frac{Nq}{p}-(N-1)}}.$$

Furthermore, C_φ is compact if $\lim\limits_{|w|\to 1^-} \frac{N_\varphi(w)}{(1-|w|)^{\frac{Nq}{p}-(N-1)}} = 0$. Here, the counting function

$$N_\varphi(w) = \frac{1}{2N-2} \sum_{z\in\varphi^{-1}(w)} \left(\frac{1}{|z|^{2N-2}} - 1\right)$$

for $w \in B_N \setminus \{\varphi(0)\}$.

In [2], Shapiro also studied composition operators between weighted Bergman spaces on D. Denote Lebesgue measure on D by $\mathrm{d}A$, normalized so that $\int_D \mathrm{d}A = 1$. For $\gamma > -1$, define the measure $\mathrm{d}A_\gamma$ on D by $\mathrm{d}A_\gamma = (1-|w|^2)^\gamma \mathrm{d}A$. For $0 < p < \infty$ and $\gamma > -1$, we define the weighted Bergman space $A_\gamma^p(D)$ to be those functions f analytic on D and satisfying

$$\|f\|_{A_\gamma^p(D)}^p = \int_D |f(w)|^p \mathrm{d}A_\gamma < \infty,$$

and the Bergman space $A^p(D)$ consists of analytic functions such that

$$\|f\|_{A^p(D)}^p = \int_D |f|^p \mathrm{d}A < \infty.$$

It is clear that if $p \le q$, then $H^q(D) \subset H^p(D)$ and $A^q(D) \subset A^p(D)$, and $H^p(D) \subset A^p(D)$ for all p. Moreover, when $p < q$, each of these inclusions is proper.

Shapiro also introduced the generalized counting functions $N_{\varphi,\gamma}(w)$, defined for $\gamma > 0$ by

$$N_{\varphi,\gamma}(w) = \sum_{z\in\varphi^{-1}(w)} (\log(1/|z|))^\gamma,$$

and used them to give the estimate of essential norm of the composition operators on the weighted Bergman spaces as follows.

Theorem F Let $C_\varphi : A_\alpha^2(D) \to A_\alpha^2(D)$ be the composition operator. For each $\alpha > -1$, there exist positive constants C_1 and C_2, which depend only on α, such that

$$C_1 \limsup_{|w|\to 1^-} \frac{N_{\varphi,\alpha+2}(w)}{(-\log|w|)^{\alpha+2}} \le \|C_\varphi\|_e^2 \le C_2 \limsup_{|w|\to 1^-} \frac{N_{\varphi,\alpha+2}(w)}{(-\log|w|)^{\alpha+2}}.$$

In 1996, Smith [8] studied composition operators $C_\varphi : A_\alpha^p(D) \to A_\beta^q(D)$ $(0 < p \le q < \infty$, $\alpha, \beta > -1)$, and gave the necessary and sufficient condition for such operators to be bounded or compact as follows.

Theorem G Let $0 < p \le q < \infty$ and let φ be an analytic self-map of D. Then $C_\varphi : A_\alpha^p(D) \to A_\beta^q(D)$ is bounded if and only if

$$N_{\varphi,\beta+2}(w) = O\left((\log(1/|w|)^{(\alpha+2)q/p}\right) \quad (|w| \to 1^-),$$

and C_φ is compact if and only if

$$N_{\varphi,\beta+2}(w) = o\left((\log(1/|w|)^{(\alpha+2)q/p}\right) \quad (|w| \to 1^-).$$

In this paper, we will define the generalized counting functions in the higher dimensional case and give an estimate of the essential norms of the composition operators between the weighted Bergman spaces on B_N $(N > 1)$.

The weighted Bergman space $A^p_\gamma(B_N)$, $\gamma > -1$, is the set of those holomorphic functions f on B_N such that

$$\|f\|^p_{A^p_\gamma(B_N)} = c_\gamma \int_{B_N} |f(w)|^p (1-|w|^2)^\gamma \mathrm{d}\tau_w < \infty,$$

where $c_\gamma = \frac{\Gamma(N+\gamma+1)}{\pi^N \Gamma(\gamma+1)}$.

Our result is stated as follows.

Theorem 1.1 Let $\varphi(z) = (\varphi_1(z), \cdots, \varphi_N(z)) : B_N \to B_N$ be a holomorphic map. For $1 \le p \le q < \infty$ and $\alpha, \beta > -1$, $C_\varphi : A^p_\alpha(B_N) \to A^q_\beta(B_N)$ is the composition operator induced by φ. Assume that $a \le \Omega_\varphi(z) \le b$ on B_N with $a, b \in \mathbb{R}^+$. If

$$\limsup_{|w|\to 1^-} \frac{N_{\varphi,\beta+2}(w)}{(1-|w|)^{\frac{(N+\alpha+1)q}{p}-(N-1)}} = A < +\infty,$$

then C_φ is a bounded operator and there exists a positive constant C such that

$$\|C_\varphi\|^q_e \le C \limsup_{|w|\to 1^-} \frac{N_{\varphi,\beta+2}(w)}{(1-|w|)^{\frac{(N+\alpha+1)q}{p}-(N-1)}}.$$

Furthermore, C_φ is compact if

$$\lim_{|w|\to 1^-} \frac{N_{\varphi,\beta+2}(w)}{(1-|w|)^{\frac{(N+\alpha+1)q}{p}-(N-1)}} = 0.$$

(The counting function $N_{\varphi,\beta+2}(w)$ in Theorem 1.1 will be introduced in the next section.)

2 The Counting Functions in Value Distribution Theory

In the classical Nevanlinna theory for one variable, for a meromorphic function f and $w \in \mathbb{C} \cup \{\infty\} = \mathbb{P}^1(\mathbb{C})$, the Nevanlinna counting function is defined by

$$N_f(r,w) = n_f(0,w)\log r + \int_0^r (n_f(t,w) - n_f(0,w))\frac{\mathrm{d}t}{t},$$

where $n_f(t,w)$ is the number of f taking value w on the closed disc $\overline{B_1(t)}$ with counting multiplicity and $n_f(0,w)$ is the multiplicity at $z=0$. It is easy to check that, for $w \in \mathbb{C}$,

$$N_f(r,w) = \mathrm{ord}_0(f-w)\log r + \sum_{z\in B_1(r), z\neq 0} \mathrm{ord}_z(f-w)\log\frac{r}{|z|}.$$

Now, we consider an entire function $\varphi : B_1 \to B_1$. For any $w \in \mathbb{C}\setminus\{\varphi(0)\}$ and $0 < r < 1$,

$$N_\varphi(r,w) = \sum_{z\in B_1(r), z\neq 0} \mathrm{ord}_z(\varphi - w)\log\frac{r}{|z|} = \sum_{j=1}^{n_\varphi(r,w)} \log\frac{r}{|z_j|}.$$

In [2], Shapiro defined

$$N_\varphi(w) = \lim_{r\to 1^-} N_\varphi(r,w) = \sum_{z\in\varphi^{-1}(w)} \log\frac{1}{|z|},$$

and defined the generalized counting function

$$N_{\varphi,\gamma}(w) = \sum_{z\in\varphi^{-1}(w)} (\log(1/|z|))^\gamma \quad \text{for } \gamma > 0,$$

where each point in $\varphi^{-1}(w)$ is repeated in the sequence according to its multiplicity.

We now recite the counting function in the higher dimensional value distribution theory introduced by Chern in 1960 (see [7]).

Let f be a holomorphic map from $\mathbb{C}^N$ into $\mathbb{P}^N(\mathbb{C})$ with reduced representation $f = [f_0 : f_1 : \cdots : f_N]$, where $f_0, f_1, \cdots, f_N$ are holomorphic functions on $\mathbb{C}^N$ without common zeros.

For a point A in $\mathbb{P}^N(\mathbb{C})$ with $f^{-1}(A) \cap B_N(r)$ consisting only of a finite number of points, let $n_f(r, A)$ be the number of times that A is covered by $f(B_N(r))$, counting multiplicities.

For $A \in \mathbb{P}^N(\mathbb{C}) \setminus \{f(0)\}$, the counting function is defined as

$$N_f(r, A) = \int_0^r \frac{n_f(t, A)}{t^{2N-1}} \mathrm{d}t.$$

A simple calculation shows that

$$N_f(r, A) = \frac{1}{2N-2} \sum_{z \in B_N(r) \cap f^{-1}(A)} \left(\frac{1}{|z|^{2N-2}} - \frac{1}{r^{2N-2}} \right),$$

where each point is repeated according to its multiplicity.

Now, we consider that $\varphi = (\varphi_1, \cdots, \varphi_N)$ be the holomorphic map from B_N into B_N with $\det(\frac{\partial \varphi}{\partial z}) \neq 0$ on B_N. Then, φ can be regarded as a holomorphic map from $\mathbb{C}^N$ into $\mathbb{P}^N(\mathbb{C})$ with reduced representation $[1 : \varphi_1 : \cdots : \varphi_N]$. For any $w = (w_1, \cdots, w_N) \in B_N \setminus \{\varphi(0)\}$ (or $w = [1 : w_1 : \cdots : w_N] \in \mathbb{P}^N(\mathbb{C}) \setminus \{[1 : \varphi_1(0) : \cdots : \varphi_N(0)]\}$), $\varphi^{-1}(w) \cap B_N(r)$ consists only finite number points, where $0 < r < 1$. We can consider

$$N_\varphi(r, w) = \frac{1}{2N-2} \sum_{z \in \varphi^{-1}(w) \cap B_N(r)} \left(\frac{1}{|z|^{2N-2}} - \frac{1}{r^{2N-2}} \right).$$

Since $N_\varphi(r, w)$ increases with r, let

$$N_\varphi(w) := \lim_{r \to 1^-} N_\varphi(r, w) = \frac{1}{2N-2} \sum_{z \in \varphi^{-1}(w)} \left(\frac{1}{|z|^{2N-2}} - 1 \right).$$

Now, we define the generalized counting function in the higher dimensional case as follows:

$$N_{\varphi,\gamma}(w) := \frac{1}{2N-2} \sum_{z \in \varphi^{-1}(w)} \left(\frac{1}{|z|^{2N-2}} - 1 \right) (1 - |z|)^{\gamma - 1} \quad \text{for } \gamma > 0.$$

3 Some Lemmas

First, we give the estimates of $|f|$ and $|\mathrm{grad} f|$ for $f \in A^p_\gamma(B_N)$.

Lemma 3.1 If $f \in A^p_\gamma(B_N)$ $(0 < p < \infty, \gamma > -1)$, then

(i) $|f(z)| \leq \frac{C_1 \|f\|_{A^p_\gamma(B_N)}}{(1-|z|)^{\frac{N+\gamma+1}{p}}}$,

(ii) $|\mathrm{grad} f(z)| \leq \frac{C_2 \|f\|_{A^p_\gamma(B_N)}}{(1-|z|)^{\frac{N+\gamma+1}{p}+1}}$,

where C_1 and C_2 are independent of f.

Proof The proof of (i) can be found in [9].

Since $f \in A^p_\gamma(B_N)$, by Theorem 2.17 in [10], we have

$$\int_{B_N} \left| \frac{\partial f}{\partial z_j} \right|^p (1 - |z|^2)^{\gamma + p} \mathrm{d}\tau_z < \infty \quad \text{for any } j = 1, \cdots, N.$$

i.e., $\frac{\partial f}{\partial z_j} \in A^p_{\gamma+p}(B_N)$ for $j = 1, \cdots, N$. Moreover, we know that $\int_{B_N} |f(z)|^p(1-|z|^2)^\gamma \mathrm{d}\tau_z$ is comparable to the quantity

$$\sum_{j=1}^N \left|\frac{\partial f}{\partial z_j}(0)\right| + \sum_{j=1}^N \int_{B_N} \left|\frac{\partial f}{\partial z_j}\right|^p (1-|z|^2)^{\gamma+p}\mathrm{d}\tau_z.$$

Therefore $\left\|\frac{\partial f}{\partial z_j}\right\|_{A^p_{\gamma+p}(B_N)} \le C\|f\|_{A^p_\gamma(B_N)}$. Now, using (i), we have

$$\left|\frac{\partial f}{\partial z_j}\right| \le \frac{C_1\left\|\frac{\partial f}{\partial z_j}\right\|_{A^p_{\gamma+p}(B_N)}}{(1-|z|)^{\frac{N+\gamma+p+1}{p}}} \le \frac{C_1 C\|f\|_{A^p_\gamma(B_N)}}{(1-|z|)^{\frac{N+\gamma+1}{p}+1}} \quad \text{for } j = 1, \cdots, N.$$

That is

$$|\mathrm{grad} f(z)| \le \frac{C_2\|f\|_{A^p_\gamma(B_N)}}{(1-|z|)^{\frac{N+\gamma+1}{p}+1}}.$$

Lemma 3.1 is proved. □

In order to estimate

$$\begin{aligned}\|f\circ\varphi\|^p_{A^p_\gamma(B_N)} &= c_\gamma \int_{B_N} |f\circ\varphi(z)|^p(1-|z|^2)^\gamma \mathrm{d}\tau_z \\ &= c_\gamma \int_0^1 r_0^{2N-1}\int_{\partial B_N(r_0)} |f\circ\varphi(z)|^p(1-|z|^2)^\gamma \mathrm{d}\sigma_z \mathrm{d}r_0 \\ &= c_\gamma \int_0^1 r_0^{2N-1}(1-r_0^2)^\gamma\left(\int_{\partial B_N(r_0)} |f\circ\varphi(z)|^p \mathrm{d}\sigma_z\right)\mathrm{d}r_0, \end{aligned} \tag{3.1}$$

we need the following well-known Green formula.

Green Formula Let U and V be C^2 real functions on $\overline{D} \subset \mathbb{R}^m$, where D is a domain with smooth boundary ∂D. Then

$$\int_D (U\triangle V - V\triangle U)\mathrm{d}\tau = \int_{\partial D}\left(U\frac{\partial V}{\partial n} - V\frac{\partial U}{\partial n}\right)\mathrm{d}\sigma,$$

where $\mathrm{d}\tau$ is the volume form on $\mathbb{R}^m$, $\mathrm{d}\sigma$ is the induced volume form on ∂D and $\frac{\partial V}{\partial n}(\frac{\partial U}{\partial n})$ is the outward normal derivative of $V(U)$ on ∂D.

Lemma 3.2 For any $0 < r_0 < 1$, we have

$$\begin{aligned}\int_{\partial B_N(r_0)} |f\circ\varphi(z)|^p \mathrm{d}\sigma_z = {}& \frac{2}{N-1}\int_{B_N(r_0)}\left(\frac{1}{|z|^{2N-2}} - \frac{1}{|r_0|^{2N-2}}\right)\sum_{k=1}^N \frac{\partial^2}{\partial z_k \partial \bar{z}_k}|f\circ\varphi|^p \mathrm{d}\tau_z \\ &+ \frac{2\pi^N}{(N-1)!}|f(\varphi(0))|^p. \end{aligned} \tag{3.2}$$

Proof We apply the Green formula for $U = \frac{1}{|z|^{2N-2}} - \frac{1}{r_0^{2N-2}}$ and $V = |f\circ\varphi|^p$ on $B_N(r_0)\setminus B_N(\varepsilon)$ with $\varepsilon \to 0^+$. We have

$$\begin{aligned}&\int_{B_N(r_0)\setminus B_N(\varepsilon)} (U\triangle V - V\triangle U)\mathrm{d}\tau_z \\ &= 4\int_{B_N(r_0)\setminus B_N(\varepsilon)}\left(\frac{1}{|z|^{2N-2}} - \frac{1}{|r_0|^{2N-2}}\right)\sum_{k=1}^N \frac{\partial^2}{\partial z_k \partial \bar{z}_k}|f\circ\varphi|^p \mathrm{d}\tau_z \\ &\to 4\int_{B_N(r_0)}\left(\frac{1}{|z|^{2N-2}} - \frac{1}{|r_0|^{2N-2}}\right)\sum_{k=1}^N \frac{\partial^2}{\partial z_k \partial \bar{z}_k}|f\circ\varphi|^p \mathrm{d}\tau_z \quad (\varepsilon \to 0^+),\end{aligned}$$

$$\int_{\partial B_N(r_0)}\left(U\frac{\partial V}{\partial n}-V\frac{\partial U}{\partial n}\right)\mathrm{d}\sigma_{r_0z}=-\int_{\partial B_n(r_0)}|f\circ\varphi|^p\left(-(2N-2)\frac{1}{r_0^{2N-1}}\right)\mathrm{d}\sigma_{r_0z}$$
$$=(2N-2)\int_{\partial B_N(r_0)}|f\circ\varphi|^p\mathrm{d}\sigma_z,$$

and

$$-\int_{\partial B_N(\varepsilon)}\left(U\frac{\partial V}{\partial n}-V\frac{\partial U}{\partial n}\right)\mathrm{d}\sigma_{\varepsilon z}$$
$$=-\int_{\partial B_N(\varepsilon)}\left(\frac{1}{\varepsilon^{2N-2}}-\frac{1}{r_0^{2N-2}}\right)\frac{\partial V}{\partial n}\mathrm{d}\sigma_{\varepsilon z}+\int_{\partial B_N(\varepsilon)}|f\circ\varphi|^p\left(-(2N-2)\frac{1}{\varepsilon^{2N-1}}\right)\mathrm{d}\sigma_{\varepsilon z}$$
$$\to-(2N-2)\frac{2\pi^N}{(N-1)!}|f(\varphi(0))|^p\qquad(\varepsilon\to0^+).$$

Hence,

$$\int_{\partial B_N(r_0)}|f\circ\varphi|^p\mathrm{d}\sigma_z=\frac{2}{N-1}\int_{B_N(r_0)}\left(\frac{1}{|z|^{2N-2}}-\frac{1}{|r_0|^{2N-2}}\right)\sum_{k=1}^N\frac{\partial^2}{\partial z_k\partial\bar z_k}|f\circ\varphi|^p\mathrm{d}\tau_z$$
$$+\frac{2\pi^N}{(N-1)!}|f(\varphi(0))|^p.$$

□

Lemma 3.3 Let $(w_1,\cdots,w_N)=\varphi(z_1,\cdots,z_N)$ be a holomorphic self-map of B_N with $N>1$. Assume that $\Omega_\varphi(z)\le b$ on B_N with $b\in\mathbb{R}^+$. If $f(w)\in H(B_N)$, then

$$\|f\circ\varphi\|^p_{A^p_\gamma(B_N)}\le\frac{2^{\gamma+1}bNp^2c_\gamma}{\gamma+1}\int_{B_N}|f(w)|^{p-2}|\mathrm{grad}f(w)|^2N_{\varphi,\gamma+2}(w)\mathrm{d}\tau_w$$
$$+\frac{2^{\gamma+1}\pi^Nc_\gamma}{(N-1)!(\gamma+1)}|f(\varphi(0))|^p.\tag{3.3}$$

Proof By (3.1) and (3.2), we have

$$\|f\circ\varphi\|^p_{A^p_\gamma(B_N)}=c_\gamma\int_0^1r_0^{2N-1}(1-r_0^2)^\gamma\left(\int_{\partial B_N(r_0)}|f\circ\varphi|^p\mathrm{d}\sigma_z\right)\mathrm{d}r_0$$
$$\le2^\gamma c_\gamma\int_0^1r_0^{2N-1}(1-r_0)^\gamma\left(\int_{\partial B_N(r_0)}|f\circ\varphi|^p\mathrm{d}\sigma_z\right)\mathrm{d}r_0$$
$$=2^\gamma c_\gamma\int_0^1r_0^{2N-1}(1-r_0)^\gamma\left[\frac{2}{N-1}\int_{B_N(r_0)}\left(\frac{1}{|z|^{2N-2}}-\frac{1}{r_0^{2N-2}}\right)\sum_{k=1}^N\frac{\partial^2}{\partial z_k\partial\bar z_k}|f\circ\varphi|^p\mathrm{d}\tau_z\right.$$
$$\left.+\frac{2\pi^N}{(N-1)!}|f(\varphi(0))|^p\right]\mathrm{d}r_0$$
$$=\frac{2^{\gamma+1}c_\gamma}{N-1}\int_{B_N}\sum_{k=1}^N\frac{\partial^2}{\partial z_k\partial\bar z_k}|f\circ\varphi|^p\left(\int_{|z|}^1r_0^{2N-1}(1-r_0)^\gamma\left(\frac{1}{|z|^{2N-2}}-\frac{1}{r_0^{2N-2}}\right)\mathrm{d}r_0\right)\mathrm{d}\tau_z$$
$$+\frac{2^{\gamma+1}\pi^Nc_\gamma}{(N-1)!}|f(\varphi(0))|^p\int_0^1r_0^{2N-1}(1-r_0)^\gamma\mathrm{d}r_0$$
$$\le\frac{2^{\gamma+1}c_\gamma}{N-1}\int_{B_N}\sum_{k=1}^N\frac{\partial^2}{\partial z_k\partial\bar z_k}|f\circ\varphi|^p\left(\frac{1}{|z|^{2N-2}}-1\right)\left(\int_{|z|}^1r_0^{2N-1}(1-r_0)^\gamma\mathrm{d}r_0\right)\mathrm{d}\tau_z$$
$$+\frac{2^{\gamma+1}\pi^Nc_\gamma}{(N-1)!(\gamma+1)}|f(\varphi(0))|^p.$$

On the other hand,

$$\int_{|z|}^{1} r_0^{2N-1}(1-r_0)^\gamma \mathrm{d}r_0 = \frac{(1-|z|)^{\gamma+1}|z|^{2N-1}}{\gamma+1} + \frac{2N-1}{\gamma+1}\int_{|z|}^{1}(1-r_0)^{\gamma+1}r_0^{2N-2}\mathrm{d}r_0$$
$$\leq \frac{1}{\gamma+1}(1-|z|)^{\gamma+1} + \frac{2N-1}{\gamma+1}(1-|z|)^{\gamma+1} = \frac{2N}{\gamma+1}(1-|z|)^{\gamma+1}.$$

Hence,

$$\begin{aligned}
&\|f\circ\varphi\|^p_{A^p_\gamma(B_N)} \leq \frac{2^{\gamma+2}Nc_\gamma}{(N-1)(\gamma+1)}\int_{B_N}\sum_{k=1}^{N}\frac{\partial^2}{\partial z_k\partial\bar z_k}|f\circ\varphi|^p\left(\frac{1}{|z|^{2N-2}}-1\right)(1-|z|)^{\gamma+1}\mathrm{d}\tau_z\\
&+\frac{2^{\gamma+1}\pi^N c_\gamma}{(N-1)!(\gamma+1)}|f(\varphi(0))|^p\\
&=\frac{2^\gamma Np^2c_\gamma}{(N-1)(\gamma+1)}\int_{B_N}\left(\frac{1}{|z|^{2N-2}}-1\right)(1-|z|)^{\gamma+1}|f\circ\varphi(z)|^{p-2}\\
&\times \mathrm{grad}f\cdot\left(\frac{\partial\varphi}{\partial z}\right)\cdot\overline{\left(\frac{\partial\varphi}{\partial z}\right)}^T\cdot\overline{\mathrm{grad}f}^T\mathrm{d}\tau_z+\frac{2^{\gamma+1}\pi^N c_\gamma}{(N-1)!(\gamma+1)}|f(\varphi(0))|^p\\
&\leq\frac{2^\gamma bNp^2c_\gamma}{(N-1)(\gamma+1)}\int_{B_N}\left(\frac{1}{|z|^{2N-2}}-1\right)(1-|z|)^{\gamma+1}|f\circ\varphi(z)|^{p-2}\\
&\times|\mathrm{grad}f|^2\left|\det\left(\frac{\partial\varphi}{\partial z}\right)\right|^2\mathrm{d}\tau_z+\frac{2^{\gamma+1}\pi^N c_\gamma}{(N-1)!(\gamma+1)}|f(\varphi(0))|^p\\
&=\frac{2^{\gamma+1}bNp^2c_\gamma}{\gamma+1}\int_{B_N}|f(w)|^{p-2}|\mathrm{grad}f(w)|^2N_{\varphi,\gamma+2}(w)\mathrm{d}\tau_w+\frac{2^{\gamma+1}\pi^N c_\gamma}{(N-1)!(\gamma+1)}|f(\varphi(0))|^p.
\end{aligned}$$

□

Lemma 3.4 For $f\in A^p_\gamma(B_N)$ $(0<p<\infty,\ \gamma>-1)$ and $r_1\geq\frac12$, we have

$$\int_{B_N\setminus r_1B_N}|f|^{p-2}|\mathrm{grad}f|^2(1-|w|)^{\gamma+2}\mathrm{d}\tau_w\leq\frac{8(\gamma+1)(\gamma+2)(N-1)}{p^2c_\gamma}\|f\|^p_{A^p_\gamma(B_N)}.\tag{3.4}$$

Proof Using (3.1) and (3.2) for $\varphi(z)\equiv z$, we get

$$\begin{aligned}
&\|f\|^p_{A^p_\gamma(B_N)}\\
&=\frac{2c_\gamma}{N-1}\int_{B_N}\sum_{k=1}^{N}\frac{\partial^2}{\partial z_k\partial\bar z_k}|f|^p\left(\int_{|z|}^1 r_0^{2N-1}(1-r_0^2)^\gamma\left(\frac{1}{|z|^{2N-2}}-\frac{1}{r_0^{2N-2}}\right)\mathrm{d}r_0\right)\mathrm{d}\tau_z\\
&+\frac{2\pi^Nc_\gamma}{(N-1)!}|f(0)|^p\int_0^1 r_0^{2N-1}(1-r_0^2)^\gamma\mathrm{d}r_0\\
&\geq\frac{p^2c_\gamma}{2(N-1)}\int_{B_N}|f|^{p-2}|\mathrm{grad}f|^2\left(\int_{|z|}^1 r_0^{2N-1}(1-r_0)^\gamma\left(\frac{1}{|z|^{2N-2}}-\frac{1}{r_0^{2N-2}}\right)\mathrm{d}r_0\right)\mathrm{d}\tau_z\\
&=\frac{p^2c_\gamma}{2(N-1)}\int_{B_N}|f|^{p-2}|\mathrm{grad}f|^2\left(\int_{|w|}^1 r_0^{2N-1}(1-r_0)^\gamma\left(\frac{1}{|w|^{2N-2}}-\frac{1}{r_0^{2N-2}}\right)\mathrm{d}r_0\right)\mathrm{d}\tau_w\\
&\geq\frac{p^2c_\gamma}{2(N-1)}\int_{B_N\setminus r_1B_N}|f|^{p-2}|\mathrm{grad}f|^2\left(\int_{|w|}^1 r_0(1-r_0)^\gamma\left(\frac{r_0^{2N-2}-|w|^{2N-2}}{|w|^{2N-2}}\right)\mathrm{d}r_0\right)\mathrm{d}\tau_w
\end{aligned}$$

for $r_1\geq 1/2$. By

$$\frac{r_0^{2N-2}-|w|^{2N-2}}{|w|^{2N-2}}=\frac{(r_0-|w|)(r_0+|w|)\sum\limits_{k=0}^{N-2}r_0^{2(N-1-k)}|w|^{2k}}{|w|^{2N-2}}$$

$$\geq (r_0-|w|)|w| \geq \frac{1}{2}(r_0-|w|),$$

we have

$$\int_{|w|}^1 r_0(1-r_0)^\gamma\left(\frac{r_0^{2N-2}-|w|^{2N-2}}{|w|^{2N-2}}\right)\mathrm{d}r_0 \geq \frac{1}{2}\int_{|w|}^1 r_0(1-r_0)^\gamma(r_0-|w|)\mathrm{d}r_0$$
$$=\frac{1}{2}(1-|w|)\int_{|w|}^1 r_0(1-r_0)^\gamma\mathrm{d}r_0-\frac{1}{2}\int_{|w|}^1 r_0(1-r_0)^{\gamma+1}\mathrm{d}r_0,$$

$$\begin{aligned}(1-|w|)\int_{|w|}^1 r_0(1-r_0)^\gamma\mathrm{d}r_0 &= \frac{1}{\gamma+1}(1-|w|)^{\gamma+2}|w|+\frac{1-|w|}{\gamma+1}\int_{|w|}^1(1-r_0)^{\gamma+1}\mathrm{d}r_0\\ &= \frac{1}{\gamma+1}(1-|w|)^{\gamma+2}|w|+\frac{1}{(\gamma+1)(\gamma+2)}(1-|w|)^{\gamma+3},\end{aligned}$$

and

$$\begin{aligned}-\int_{|w|}^1 r_0(1-r_0)^{\gamma+1}\mathrm{d}r_0 &= -\frac{1}{\gamma+2}|w|(1-|w|)^{\gamma+2}-\frac{1}{\gamma+2}\int_{|w|}^1(1-r_0)^{\gamma+2}\mathrm{d}r_0\\ &= -\frac{1}{\gamma+2}|w|(1-|w|)^{\gamma+2}-\frac{1}{(\gamma+2)(\gamma+3)}(1-|w|)^{\gamma+3}.\end{aligned}$$

Hence, for $|w| \geq \frac{1}{2}$,

$$\begin{aligned}&\int_{|w|}^1 r_0(1-r_0)^\gamma\left(\frac{r_0^{2N-2}-|w|^{2N-2}}{|w|^{2N-2}}\right)\mathrm{d}r_0\\ \geq\ &\frac{1}{2}(1-|w|)\int_{|w|}^1 r_0(1-r_0)^\gamma\mathrm{d}r_0-\frac{1}{2}\int_{|w|}^1 r_0(1-r_0)^{\gamma+1}\mathrm{d}r_0\\ =\ &\frac{1}{2}\left(\frac{1}{(\gamma+1)(\gamma+2)}-\frac{1}{(\gamma+2)(\gamma+3)}\right)(1-|w|)^{\gamma+3}+\frac{1}{2}\left(\frac{1}{\gamma+1}-\frac{1}{\gamma+2}\right)|w|(1-|w|)^{\gamma+2}\\ \geq\ &\frac{1}{4(\gamma+1)(\gamma+2)}(1-|w|)^{\gamma+2}.\end{aligned}$$

We have

$$\|f\|^p_{A^p_\gamma(B_N)} \geq \frac{p^2c_\gamma}{8(\gamma+1)(\gamma+2)(N-1)}\int_{B_N\setminus r_1B_N}|f|^{p-2}|\mathrm{grad}f|^2(1-|w|)^{\gamma+2}\mathrm{d}\tau_w.$$

i.e.,

$$\int_{B_N\setminus r_1B_N}|f|^{p-2}|\mathrm{grad}f|^2(1-|w|)^{\gamma+2}\mathrm{d}\tau_w \leq \frac{8(\gamma+1)(\gamma+2)(N-1)}{p^2c_\gamma}\|f\|^p_{A^p_\gamma(B_N)}.$$

□

4 The Proof of Theorem 1.1

Throughout this section, C denotes a positive constant independent of f and φ, whose value may change from one occurrence to the next one.

First, we show that $C_\varphi : A^p_\alpha(B_N) \to A^q_\beta(B_N)$ $(1 \leq p \leq q < \infty,\ \alpha, \beta > -1)$ is bounded.

By the assumption

$$\limsup_{|w|\to 1^-}\frac{N_{\varphi,\beta+2}(w)}{(1-|w|)^{\frac{(N+\alpha+1)q}{p}-(N-1)}}=A,$$

for any fixed $\varepsilon > 0$, we can find an r_1 with $\frac{1}{2} \le r_1 < 1$ such that

$$\frac{N_{\varphi,\beta+2}(w)}{(1-|w|)^{\frac{(N+\alpha+1)q}{p}-(N-1)}} < A + \varepsilon,$$

where $|w| \ge r_1$.

By (3.3), we obtain

$$\begin{aligned}
&\|f\circ\varphi\|^q_{A^q_\beta(B_N)}\\
\le\ & \frac{2^{\beta+1}\pi^N c_\beta}{(N-1)!(\beta+1)}|f\circ\varphi(0)|^q + \frac{2^{\beta+1}bNq^2c_\beta}{\beta+1}\int_{B_N}|f(w)|^{q-2}|\mathrm{grad} f(w)|^2 N_{\varphi,\beta+2}(w)\mathrm{d}\tau_w\\
=\ & \frac{2^{\beta+1}\pi^N c_\beta}{(N-1)!(\beta+1)}|f\circ\varphi(0)|^q + \frac{2^{\beta+1}bNq^2c_\beta}{\beta+1}\int_{B_N\setminus r_1B_N}|f(w)|^{q-2}|\mathrm{grad} f(w)|^2 N_{\varphi,\beta+2}(w)\mathrm{d}\tau_w\\
&+\frac{2^{\beta+1}bNq^2c_\beta}{\beta+1}\int_{r_1B_N}|f(w)|^{q-2}|\mathrm{grad} f(w)|^2 N_{\varphi,\beta+2}(w)\mathrm{d}\tau_w\\
=\ & \frac{2^{\beta+1}\pi^N c_\beta}{(N-1)!(\beta+1)}|f\circ\varphi(0)|^q\\
&+\frac{2^{\beta+1}bNq^2c_\beta}{\beta+1}\int_{B_N\setminus r_1B_N}|f(w)|^{q-p}|f(w)|^{p-2}|\mathrm{grad} f(w)|^2 N_{\varphi,\beta+2}(w)\mathrm{d}\tau_w\\
&+\frac{2^{\beta+1}bNq^2c_\beta}{\beta+1}\int_{r_1B_N}|f(w)|^{q-2}|\mathrm{grad} f(w)|^2 N_{\varphi,\beta+2}(w)\mathrm{d}\tau_w
\end{aligned}$$

for any $f \in A^p_\alpha(B_N)$. By (i) of Lemma 3.1, we have

$$|f\circ\varphi(0)|^q \le \frac{C\|f\|^q_{A^p_\alpha(B_N)}}{(1-|\varphi(0)|)^{\frac{(N+\alpha+1)q}{p}}}$$

and

$$\begin{aligned}
&\frac{2^{\beta+1}bNq^2c_\beta}{\beta+1}\int_{B_N\setminus r_1B_N}|f(w)|^{q-p}|f(w)|^{p-2}|\mathrm{grad} f|^2 N_{\varphi,\beta+2}(w)\mathrm{d}\tau_w\\
\le\ & C\|f\|^{q-p}_{A^p_\alpha(B_N)}\int_{B_N\setminus r_1B_N}|f(w)|^{p-2}|\mathrm{grad} f|^2(1-|w|)^{\alpha+2}\frac{N_{\varphi,\beta+2}(w)}{(1-|w|)^{\frac{(N+\alpha+1)(q-p)}{p}+(\alpha+2)}}\mathrm{d}\tau_w\\
\le\ & C\cdot(A+\varepsilon)\|f\|^{q-p}_{A^p_\alpha(B_N)}\int_{B_N\setminus r_1B_N}|f(w)|^{p-2}|\mathrm{grad} f|^2(1-|w|)^{\alpha+2}\mathrm{d}\tau_w\\
\le\ & C\|f\|^q_{A^p_\alpha(B_N)},
\end{aligned}$$

where the last inequality is provided by (3.4). On the other hand, by the condition $a \le \Omega_\varphi$, we have $\int_{B_N} N_\varphi(w)\mathrm{d}\tau_w \le \frac{\pi^N}{2a(N-1)!}$ (cf. [5]). Since

$$N_{\varphi,\beta+2}(w) = \frac{1}{2N-2}\sum_{z\in\varphi^{-1}(w)}\left(\frac{1}{|z|^{2N-2}}-1\right)(1-|z|)^{\beta+1} \le N_\varphi(w),$$

we have

$$\int_{B_N} N_{\varphi,\beta+2}(w)\mathrm{d}\tau_w \le \int_{B_N} N_\varphi(w)\mathrm{d}\tau_w \le \frac{\pi^N}{2a(N-1)!}. \tag{4.1}$$

By (ii) of Lemma 3.1, we have

$$\frac{2^{\beta+1}bNq^2c_\beta}{\beta+1}\int_{r_1B_N}|f(w)|^{q-2}|\mathrm{grad} f|^2 N_{\varphi,\beta+2}(w)\mathrm{d}\tau_w$$

$$\leq \frac{C\|f\|^q_{A^p_\alpha(B_N)}}{(1-r_1)^{\frac{(N+\alpha+1)q}{p}+2}} \int_{r_1 B_N} N_{\varphi,\beta+2}(w)\mathrm{d}\tau_w \leq C\|f\|^q_{A^p_\alpha(B_N)}, \tag{4.2}$$

where the last inequality is provided by (4.1).

Hence, we have

$$\|f\circ\varphi\|^q_{A^q_\beta(B_N)} \leq \frac{C}{(1-|\varphi(0)|)^{\frac{(N+\alpha+1)q}{p}}}\|f\|^q_{A^p_\alpha(B_N)},$$

which implies that C_φ is a bounded operator.

Next, we estimate the essential norm of C_φ.

For a bounded operator $C_\varphi : A^p_\alpha(B_N) \to A^q_\beta(B_N)$ $(1 \leq p \leq q < \infty)$, the essential norm $\|C_\varphi\|_e$ of C_φ is defined to be the distance from C_φ to the set of the compact operators $K : A^p_\alpha(B_N) \to A^q_\beta(B_N)$, namely,

$$\|C_\varphi\|_e := \inf\{\|C_\varphi - K\| : K \text{ is a compact operator}\}.$$

For $1 \leq p < \infty$, we define $K_n := C_{\varphi_n}$ for any $n \geq 2$, where $\varphi_n = \frac{n-1}{n}z$. It is easy to deduce that $\|K_n\| \leq 1$ and K_n is compact on every $A^p_\alpha(B_N)$ (see [11]). Put $R_n := I - K_n$ for $n \geq 2$, similar to the proof of Proposition 2.3 in [11], we have

$$\frac{1}{2}\limsup_{n\to\infty}\|R_n C_\varphi\| \leq \|C_\varphi\|_e \leq \liminf_{n\to\infty}\|R_n C_\varphi\|. \tag{4.3}$$

Consider $f \in A^p_\alpha(B_N)$ with $\|f\|_{A^p_\alpha(B_N)} = 1$. Using (3.3) for any $\frac{1}{2} \leq r_1 < 1$, we have

$$\begin{aligned}
&\|R_n C_\varphi f\|^q_{A^q_\beta(B_N)}\\
\leq\ & \frac{2^{\beta+1}\pi^N c_\beta |R_n f(\varphi(0))|^q}{(N-1)!(\beta+1)} + \frac{2^{\beta+1}bNq^2c_\beta}{\beta+1}\int_{B_N}|R_n f(w)|^{q-2}|\mathrm{grad}R_n f(w)|^2 N_{\varphi,\beta+2}(w)\mathrm{d}\tau_w\\
=\ & \frac{2^{\beta+1}\pi^N c_\beta |R_n f(\varphi(0))|^q}{(N-1)!(\beta+1)} + \frac{2^{\beta+1}bNq^2c_\beta}{\beta+1}\int_{B_N\setminus r_1B_N}|R_n f(w)|^{q-2}|\mathrm{grad}R_n f(w)|^2 N_{\varphi,\beta+2}(w)\mathrm{d}\tau_w\\
&+\frac{2^{\beta+1}bNq^2c_\beta}{\beta+1}\int_{r_1B_N}|R_n f(w)|^{q-2}|\mathrm{grad}R_n f(w)|^2 N_{\varphi,\beta+2}(w)\mathrm{d}\tau_w.
\end{aligned} \tag{4.4}$$

By (i) of Lemma 3.1 and (3.4), we hold that

$$\begin{aligned}
&\frac{2^{\beta+1}bNq^2c_\beta}{\beta+1}\int_{B_N\setminus r_1B_N}|R_n f(w)|^{q-2}|\mathrm{grad}R_n f|^2 N_{\varphi,\beta+2}(w)\mathrm{d}\tau_w\\
\leq\ & C\|R_n f\|^{q-p}_{A^p_\alpha(B_N)}\int_{B_N\setminus r_1B_N}|R_n f(w)|^{p-2}|\mathrm{grad}R_n f|^2(1-|w|)^{\alpha+2}\frac{N_{\varphi,\beta+2}(w)}{(1-|w|)^{\frac{(N+\alpha+1)q}{p}-(N-1)}}\mathrm{d}\tau_w\\
\leq\ & C\|f\|^q_{A^p_\alpha(B_N)}\cdot\sup_{r_1\leq|w|<1}\frac{N_{\varphi,\beta+2}(w)}{(1-|w|)^{\frac{(N+\alpha+1)q}{p}-(N-1)}}\\
\leq\ & C\cdot\sup_{r_1\leq|w|<1}\frac{N_{\varphi,\beta+2}(w)}{(1-|w|)^{\frac{(N+\alpha+1)q}{p}-(N-1)}},
\end{aligned} \tag{4.5}$$

where $\|R_n\| \leq 2$ since $\|K_n\| \leq 1$ and $R_n = I - K_n$. On the other hand, we have, for $w \in r_1B_N$,

$$\begin{aligned}
|R_n f(w)| &= \left|f(w) - f\left(\frac{n-1}{n}w\right)\right| = \left|\int_{\frac{n-1}{n}}^1 \frac{\mathrm{d}f(tw)}{\mathrm{d}t}\mathrm{d}t\right| = \left|\int_{\frac{n-1}{n}}^1 \sum_{j=1}^N w_j\frac{\partial f}{\partial w_j}(tw)\mathrm{d}t\right|\\
&\leq \frac{1}{n}\sum_{j=1}^N \sup_{z\in r_1B_N}\left|\frac{\partial f}{\partial w_j}(z)\right|.
\end{aligned}$$

For all $z \in r_1 B_N$ and any fixed $\delta \in (r_1, 1)$, $\left|\frac{\partial^m f}{\partial w^m}(z)\right| \le C \cdot \sup\{|f(z)| : |z| \le \delta\}$, where $m = (m_1, \cdots, m_N)$ is a multi-index (see the proof of Lemma 2.4 in [10]). Hence, by (i) of Lemma 3.1,

$$|R_n f(w)| \le \frac{1}{n} \sum_{j=1}^{N} \sup_{z \in r_1 B_N} \left|\frac{\partial f}{\partial w_j}(z)\right| \le \frac{C}{n} \sup\left\{|f(z)| : |z| \le \frac{1+r_1}{2}\right\}$$
$$\le \frac{C}{n} \cdot \frac{\|f\|_{A^p_\alpha(B_N)}}{(1-r_1)^{\frac{N+\alpha+1}{p}}} \le \frac{C}{n(1-r_1)^{\frac{N+\alpha+1}{p}}}$$

for $w \in r_1 B_N$. Similarly, we have, for $1 \le k \le N$ and $w \in r_1 B_N$,

$$\left|\frac{\partial}{\partial w_k} R_n f(w)\right| = \left|\int_{\frac{n-1}{n}}^{1} \sum_{j=1}^{N} \left(\delta_{kj} \frac{\partial f}{\partial w_j}(tw) + t w_j \frac{\partial^2 f}{\partial w_j \partial w_k}(tw)\right) \mathrm{d}t\right| \le \frac{C}{n(1-r_1)^{\frac{N+\alpha+1}{p}}}.$$

Hence,

$$\frac{2^{\beta+1} b N q^2 c_\beta}{\beta+1} \int_{r_1 B_N} |R_n f(w)|^{q-2} |\mathrm{grad} R_n f|^2 N_{\varphi,\beta+2}(w) \mathrm{d}\tau_w$$
$$\le \left(\frac{C}{n(1-r_1)^{\frac{N+\alpha+1}{p}}}\right)^q \int_{r_1 B_N} N_{\varphi,\beta+2}(w) \mathrm{d}\tau_w \to 0 \tag{4.6}$$

and $|R_n f(\varphi(0))| \to 0$ as $n \to \infty$.

Combining (4.3), (4.4), (4.5), (4.6) and letting $n \to \infty$, we get

$$\|C_\varphi\|_e^q \le C \cdot \sup_{r_1 \le |w| < 1} \frac{N_{\varphi,\beta+2}(w)}{(1-|w|)^{\frac{(N+\alpha+1)q}{p} - (N-1)}}.$$

Let $r_1 \to 1^-$. Then

$$\|C_\varphi\|_e^q \le C \cdot \limsup_{|w| \to 1^-} \frac{N_{\varphi,\beta+2}(w)}{(1-|w|)^{\frac{(N+\alpha+1)q}{p} - (N-1)}}.$$

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