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Covering and distortion theorems for planar harmonic univalent mappings

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Abstract. In this paper, we investigate Clunie and Sheil-Small's covering theorems for sense-preserving planar harmonic univalent mappings defined in the unit disk. Our results significantly improve the earlier known result. Also, we obtain a distortion theorem for fully starlike harmonic mappings in the unit disk.

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1. Introduction and main results. The present article is mainly concerned with the class $\mathcal{S}_H^0$ of sense-preserving planar harmonic univalent mappings $f = h+\overline{g}$ defined on the unit disk $\mathbb{D} = \{z : |z| < 1\}$, where h and g are holomorphic functions in $\mathbb{D}$ normalized in a standard form: $h(0) = g(0) = g'(0) = 0$ and $h'(0) = 1$, see [8]. If $g(z)$ is identically zero on the decomposition of $f(z)$, then the class $\mathcal{S}_H^0$ in this case reduces to the classical family $\mathcal{S}$ of normalized holomorphic univalent functions $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $\mathbb{D}$.

Recall that the Jacobian J_f of a harmonic function $f = h + \overline{g}$ is given by

$$J_f = |f_z|^2 - |f_{\overline{z}}|^2 = |h'|^2 - |g'|^2,$$

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and so f is locally univalent and sense-preserving in $\mathbb{D}$ if and only if $|g'(z)| < |h'(z)|$ in $\mathbb{D}$ (see [11]). We refer to [1,3,8,9,11–13] for the theory of planar harmonic mappings.

For $a \in \mathbb{C}$, let $\mathbb{D}(a,r) = \{z : |z-a| < r\}$. In particular, we use $\mathbb{D}_r$ to denote the disk $\mathbb{D}(0,r)$ and $\mathbb{D}$ for the unit disk $\mathbb{D}_1$. For a harmonic function f defined in $\mathbb{D}$, we use the following standard notations:

$$\Lambda_f(z) = \max_{0\leq\theta\leq 2\pi} |f_z(z) + e^{-2i\theta} f_{\overline{z}}(z)| = |f_z(z)| + |f_{\overline{z}}(z)|$$

and

$$\lambda_f(z) = \min_{0\leq\theta\leq 2\pi} |f_z(z) + e^{-2i\theta} f_{\overline{z}}(z)| = \big|\,|f_z(z)| - |f_{\overline{z}}(z)|\,\big|.$$

Thus, for a sense-preserving harmonic function $f = h + \overline{g}$, one has $J_f(z) = \Lambda_f(z)\lambda_f(z)$ and the dilatation ω of f defined by $\omega = g'/h'$ is analytic in $\mathbb{D}$ such that $|\omega(z)| < 1$ in $\mathbb{D}$.

The following conjecture is in [13] (see also [9, p. 97]).

Conjecture 1.1. *If $f \in \mathcal{S}_H^0$, then the range $f(\mathbb{D})$ contains a disk $\mathbb{D}_{\frac{1}{6}}$.*

Also, the following result is known from [8, Theorem 4.4].

Theorem A. *Each function $f \in \mathcal{S}_H^0$ satisfies the inequality*

$$|f(z)| \geq \frac{1}{4}\frac{|z|}{(1+|z|)^2},\ z \in \mathbb{D}.$$

In particular, the range $f(\mathbb{D})$ contains the disk $\mathbb{D}_{\frac{1}{16}}$.

Let ψ be a non-constant holomorphic function in $\mathbb{D}$. For $z_0 \in \mathbb{D}$, a disk $\mathbb{D}(\psi(z_0), r)$ is called a *univalent disk* contained in $\psi(\mathbb{D})$ if there is a subdomain D of $\mathbb{D}$ such that ψ is univalent in D and $\psi(D) = \mathbb{D}(\psi(z_0), r)$. The radius of the largest univalent disk with center $\psi(z_0)$ is denoted by $r(z_0, \psi)$. Let $\mathcal{B}_\psi = \sup\{r(z,\psi) : z \in \mathbb{D}\}$. The locally univalent Bloch constant is defined by

$$\mathcal{B}_{loc} = \inf\{\mathcal{B}_\psi : \ \psi'(0) = 1 \quad \text{and} \quad \psi'(z) \neq 0 \quad \text{for } z \in \mathbb{D}\}.$$

Let BH_{loc} denote the class of all locally biholomorphic functions in $\mathbb{D}$ satisfying the conditions

$$\psi(0) = \psi'(0) - 1 = 0 \quad \text{and} \quad (1-|z|^2)|\psi'(z)| \leq 1 \quad \text{for } z \in \mathbb{D}.$$

In [10], Landau proved that

$$\mathcal{B}_{loc} = \inf\{\mathcal{B}_\psi : \ \psi \in BH_{loc}\}.$$

In [2, Theorem 1], the authors proved $\mathcal{B}_{loc} > \frac{1}{2} + 2 \times 10^{-8}$, but the best bound of $\mathcal{B}_{loc}$ is unknown.

The first aim of this paper is to consider Conjecture 1.1, and we improve Theorem A in the following form.

Theorem 1.2. *If $f \in \mathcal{S}_H^0$, then $f(\mathbb{D})$ contains a disk with the radius $R \geq \frac{\mathcal{B}_{loc}}{4} > \frac{\rho_0}{4}$, where $\rho_0 = \frac{1}{2} + 2 \times 10^{-8}$.*

For convenience, we denote by $\mathcal{H}(\mathbb{D})$ the set of all univalent and sense-preserving harmonic mappings $f = h + \overline{g}$ in $\mathbb{D}$ satisfying the normalization $h(0) = g(0) = 0$, where h and g are holomorphic in $\mathbb{D}$. The functions h and g are referred to by us as the holomorphic and anti-holomorphic parts of f, respectively. Let $\mathcal{S}_H$ denote the subclass of $\mathcal{H}(\mathbb{D})$ with $h'(0) = 1$. Thus, $\mathcal{S}_H^0 = \{f = h + \overline{g} \in \mathcal{S}_H : g'(0) = 0\}$. It is well-known that $\mathcal{S}_H^0$ is compact (see [8]).

A function $f \in \mathcal{H}(\mathbb{D})$ is said to be *a fully starlike* harmonic mapping if it maps every circle $|z| = r < 1$ onto a curve bounds a domain starlike with respect to the origin (see [6, p. 138]).

In [7], Chuaqui and Hernández discussed the relationship between the images of the linear connectivity of $\mathbb{D}$ under the harmonic mappings $f = h + \overline{g}$ and under their corresponding holomorphic counterparts h, where h and g are holomorphic in $\mathbb{D}$. For the extensive discussions on this topic, see [4,5]. The second aim of the article is to discuss the relationship between f and its holomorphic counterpart h, where $f = h + \overline{g} \in \mathcal{H}(\mathbb{D})$.

Theorem 1.3. *Let $f = h + \overline{g} \in \mathcal{H}(\mathbb{D})$ be a fully starlike harmonic mapping with $f_{\overline{z}}(0) = 0$, where h and g are holomorphic in $\mathbb{D}$. Then for all $r \in (0, 1)$ and $z \in \overline{\mathbb{D}}$,*

$$\frac{1}{1+r}|f(rz)| \leq |h(rz)| \leq \frac{1}{1-r}|f(rz)|. \tag{1.1}$$

In particular, one has

$$|f(z)| \leq 2|h(z)|, \quad z \in \mathbb{D}, \tag{1.2}$$

and if $h'(0) = 1$, then

$$|h(z)| \geq \frac{1}{8}\frac{|z|}{(1+|z|)^2}, \quad z \in \mathbb{D}. \tag{1.3}$$

Note that (1.2) follows from (1.1), whereas (1.3) is a simple consequence of (1.2) and Theorem A. Thus, it suffices to prove (1.1). The proofs of Theorems 1.2 and 1.3 will be presented in Section 2.

2. The proofs of the main results. First, we recall that (cf. [9, p. 5]) a sense-preserving univalent harmonic mapping is *K-quasiconformal*, $K \in [1, \infty)$, if $\Lambda_f(z) \leq K\lambda_f(z)$ for $z \in \mathbb{D}$.

Proof of Theorem 1.2. Let $f \in \mathcal{S}_H^0$. For $z \in \mathbb{D}$, let $\omega(z) = \overline{f_{\overline{z}}(z)}/f_z(z)$. Then ω is holomorphic in $\mathbb{D}$, $\omega(0) = 0$, and $|\omega(z)| < 1$ for $z \in \mathbb{D}$. Thus, by the Schwarz lemma, $|\omega(z)| \leq |z|$ for $z \in \mathbb{D}$. Therefore, for any fixed $r \in (0, 1)$ and $z \in \mathbb{D}_r$, we have

$$\frac{\Lambda_f(z)}{\lambda_f(z)} = \frac{1+|\omega(z)|}{1-|\omega(z)|} \leq \frac{1+r}{1-r} = K_r.$$

For $\zeta \in \mathbb{D}$, let

$$F(\zeta) = r^{-1}f(r\zeta) = H(\zeta) + \overline{G(\zeta)},$$

where

$$H(\zeta) = r^{-1}h(r\zeta) \text{ and } G(\zeta) = r^{-1}g(r\zeta).$$

It is easy to see that F is a K_r-quasiconformal harmonic mapping in $\mathbb{D}$ satisfying $F_\zeta(0) = H'(0) = 1$ and $F_{\overline{\zeta}}(0) = H(0) = G(0) = 0$. Since F is sense-preserving, we see that H is a locally biholomorphic function. For all $\rho < \mathcal{B}_{loc}$, there is a disk $\mathbb{D}(\xi_0, \rho)$ which is the biholomorphic image of a subdomain Π of $\mathbb{D}$ under H. For $\xi \in \mathbb{D}(\xi_0, \rho)$, let $\Phi(\xi) = F\big(H^{-1}(\xi)\big)$. Then $\Phi(\mathbb{D}(\xi_0, \rho)) = F(\Pi)$. By calculations, we have

$$\Phi_\xi = \frac{F_\zeta}{H'} \text{ and } \Phi_{\overline{\xi}} = \frac{F_{\overline{\zeta}}}{\overline{H'}},$$

which implies that Φ is also a K_r-quasiconformal harmonic mapping in $\mathbb{D}(\xi_0, \rho)$. Since $\Phi_\xi = F_\zeta/H' \equiv 1$ and $|\Phi_{\overline{\xi}}| \leq r$, we find that for each $\xi' \in \mathbb{D}(\xi_0, \rho)$ with $\xi' \neq \xi_0$,

$$\begin{aligned}
|\Phi(\xi') - \Phi(\xi_0)| &= \left| \int_{[\xi_0,\ \xi']} \Phi_\xi(\xi) d\xi + \Phi_{\overline{\xi}}(\xi) d\overline{\xi} \right| \\
&\geq \left| \int_{[\xi_0,\ \xi']} \Phi_\xi(\xi) d\xi \right| - \left| \int_{[\xi_0,\ \xi']} \Phi_{\overline{\xi}}(\xi) d\overline{\xi} \right| \\
&\geq (1-r)|\xi' - \xi_0|,
\end{aligned}$$

where $[\xi_0, \xi']$ denotes the line segment with endpoints ξ_0 and ξ'. If ξ' tends to the boundary $\partial\mathbb{D}(\xi_0, \rho)$, then $f(\mathbb{D})$ contains a disk with the radius $R \geq \rho r(1-r)$. Since

$$\max_{0<r<1} \big[r(1-r)\big] = \frac{1}{4},$$

by letting ρ tend to $\mathcal{B}_{loc}$, we conclude that $f(\mathbb{D})$ contains a disk with the radius $R \geq \frac{\mathcal{B}_{loc}}{4} > \frac{\rho_0}{4}$, where $\rho_0 = \frac{1}{2} + 2 \times 10^{-8}$. The proof of this theorem is complete. □

Proof of Theorem 1.3. For $z \in \mathbb{D}$ and a fixed $r \in (0, 1)$, let

$$F(z) = r^{-1}f(rz) = H(z) + \overline{G(z)},$$

where $H(z) = r^{-1}h(rz)$ and $G(z) = r^{-1}g(rz)$. Then F is also fully starlike and $F_{\overline{z}}(0) = G(0) = H(0) = 0$. For $z \in \mathbb{D}$, let $w(z) = \overline{F_{\overline{z}}(z)}/F_z(z)$. Then $w(0) = 0$ and for $z \in \mathbb{D}$,

$$|w(z)| = \frac{|F_{\overline{z}}(z)|}{|F_z(z)|} = \frac{|f_{\overline{z}}(rz)|}{|f_z(rz)|} < r. \tag{2.1}$$

This implies that f is a K_r-quasiconformal harmonic mapping in $\mathbb{D}_r$, where $K_r = (1+r)/(1-r)$. Differentiating both sides of the equation $F^{-1}(F(z)) = z$ yields the relations

$$(F^{-1})_\zeta H' + (F^{-1})_{\overline{\zeta}} G' = 1 \text{ and } (F^{-1})_\zeta \overline{G'} + (F^{-1})_{\overline{\zeta}} \overline{H'} = 0,$$

where $\zeta = F(z)$. Solving these two equations gives

$$(F^{-1})_\zeta = \frac{\overline{H'}}{J_F} \quad \text{and} \quad (F^{-1})_{\overline{\zeta}} = -\frac{\overline{G'}}{J_F}, \tag{2.2}$$

where J_F denotes the Jacobian of F. Since $F(\mathbb{D})$ is fully starlike with respect to the origin, for each point $z \in \mathbb{D}$ and $t \in [0,1]$, $\varphi(t) = F(z)t \in F(\mathbb{D})$. Let $\gamma = F^{-1} \circ \varphi$ and $F(z) = |F(z)|e^{i\theta_0}$. Define

$$H(F^{-1}(\varsigma)) = \varsigma - \overline{G(F^{-1}(\varsigma))},$$

where $\varsigma \in F(\mathbb{D})$. By the chain rule, we have

$$\tau_\varsigma = G' \cdot (F^{-1})_\varsigma \quad \text{and} \quad \tau_{\overline{\varsigma}} = G' \cdot (F^{-1})_{\overline{\varsigma}}, \tag{2.3}$$

where $\tau = G \circ F^{-1}$. Applying (2.1) and (2.2) to (2.3), we obtain

$$|\tau_\varsigma| + |\tau_{\overline{\varsigma}}| = \frac{|w|}{1-|w|} \le \frac{r}{1-r}.$$

Hence we have

$$\begin{aligned} |H(z)| &\le \int_\varphi \left(1 + |G_\varsigma(F^{-1}(\varsigma))| + |G_{\overline{\varsigma}}(F^{-1}(\varsigma))|\right) |d\varsigma| \\ &= \int_\varphi (1 + |\tau_\varsigma(\varsigma)| + |\tau_{\overline{\varsigma}}(\varsigma)|)\, |d\varsigma| \\ &\le |F(z)| \int_0^1 \left(1 + \frac{r}{1-r}\right) dt \\ &= \frac{1}{1-r}|F(z)|, \end{aligned}$$

which implies

$$|H(z)| \le \frac{1}{1-r}|F(z)|, \quad \text{i.e.} \quad |h(rz)| \le \frac{1}{1-r}|f(rz)|.$$

Next, we prove the first inequality in (1.1). Applying (2.2), we see that $\mathrm{Re}\left[e^{-i\theta_0}\overline{G(z)}\right]$

$$\begin{aligned} &= \mathrm{Re}\left[e^{-i\theta_0}\left(\overline{\int_\gamma G'(z)\, dz}\right)\right] \\ &= \mathrm{Re}\left[e^{-i\theta_0}\left(\overline{\int_0^1 G'(\gamma(t))\, \frac{d}{dt}\gamma(t)\, dt}\right)\right] \\ &= \mathrm{Re}\left\{e^{-i\theta_0}\left[\overline{\int_0^1 G'(\gamma(t))\Big(\varphi'(t)\frac{\partial}{\partial\zeta}F^{-1}(\varphi(t)) + \overline{\varphi'(t)}\frac{\partial}{\partial\overline{\zeta}}F^{-1}(\varphi(t))\Big)\, dt}\right]\right\} \end{aligned}$$

$$= |F(z)|\text{Re}\left\{e^{-i\theta_0}\left[\overline{\int_0^1 \frac{G'(\gamma(t))\overline{H'(\gamma(t))}e^{i\theta_0} - |G'(\gamma(t))|^2 e^{-i\theta_0}}{J_F(\gamma(t))}\,dt}\right]\right\}$$

$$\leq |F(z)|\int_0^1 \frac{|H'(\gamma(t))\overline{G'(\gamma(t))}e^{-2i\theta_0}| - |G'(\gamma(t))|^2}{J_F(\gamma(t))}\,dt$$

$$\leq |F(z)|\int_0^1 \frac{|w(\gamma(t))|}{1+|w(\gamma(t))|}\,dt$$

$$\leq \frac{r|F(z)|}{1+r},$$

which gives

$$\text{Re}\left\{\frac{\overline{G(z)}}{F(z)}\right\} \leq \frac{r}{1+r}.$$

Hence, we conclude that

$$\frac{|H(z)|}{|F(z)|} \geq \text{Re}\left\{\frac{H(z)}{F(z)}\right\} = 1 - \text{Re}\left\{\frac{\overline{G(z)}}{F(z)}\right\} \geq \frac{1}{1+r},$$

and therefore

$$|h(rz)| \geq \frac{1}{1+r}|f(rz)|.$$

The proof of this theorem is complete. □

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