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Strong Convergence of Nonexpansive Nonself-Mapping in Hilbert Space

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Abstract. Let C be a closed convex subset of a Hilbert space X, $T : C \to H$ a non-expansive nonself-mapping such that $F(T) \neq \emptyset$. $P : H \to C$ is a projective operator, $f : H \to C$ is a fixed contractive. In this paper, we study the convergence of the following type sequence generated by

$$x_{n+1} = P(\alpha_n f(x_n) + (1 - \alpha_n)Tx_n)$$

and prove the sequence $\{x_n\}$ converges strongly to a fixed point of $F(T)$ under the weaker condition $Tx_{n+1} - Tx_n \rightharpoonup 0, n \to \infty$. The result improves the result of Wittmann.

Keywords: Fixed point; non-expansive nonself-mapping; strong convergence; Banach limit

1. Introduction and Preliminaries

Throughout the paper, let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. Similarly, $x_n \to x$ will symbolize strong convergence. Let C be a nonempty closed convex subset of H,and $T : C \to H$ be a non-expansive nonself-mapping such that $F(T) \neq \emptyset$.In 1992, Marino and Trombetta(see reference[3]) consider the contraction $S_t : C \to C$ and $U_t : C \to C$ defined by

$$S_t(x) = tPTx + (1 - t)u \quad x \in C$$

and

$$U_t(x) = P(tTx + (1 - t)u) \quad x \in C$$

for each $t \in [0, 1)$ and each fixed $u \in C$, where $P : H \to C$ is a projective operator, then S_t and U_t has a unique fixed point. By Banach contractive theorem, there is a unique $x_t \in F(S_t)$, $y_t \in F(U_t)$ such that

$$x_t = tPTx_t + (1 - t)u \tag{1.1}$$

and

$$y_t = P(tTy_t + (1-t)u). \tag{1.2}$$

In 1995, Xu and Yin(see reference[2]) proved that if $T : C \to H$ is a non-expansive nonself-mapping, then the sequence $\{x_t\}$ defined by (1.1) strongly converges to $P_{F(T)}u$, the sequence $\{y_t\}$ defined by (1.2) strongly converges to $P_{F(T)}u$, where $P : H \to C$ is the nearest point projection of H onto C, i.e. for each $x \in H$, Px is the unique element of C that satisfies $\|x - Px\| = d(x, C) : \inf_{c\in C} \|x - c\|$. This result which has been proved by Takahashi and Kim(see reference[4]) is true in Banach space too.

In 1992, Wittmann proved the following theorem(see reference[1]):
Let H be a Hilbert space, C be a nonempty closed convex subset of H,and $T : C \to C$ be a non-expansive mapping, and $F(T) \neq \phi$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n \quad for \;\; n = 1, 2, 3 \cdots$$

where $\{\alpha_n\} \in (0, 1)$ and satisfies

$$lim_{n\to\infty}\alpha_n = 0, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, and \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then $\{x_n\}$ strongly converges to $Px \in F(T)$, where $P : C \to F(T)$ is a projective operator.

Definition 1.1. (see reference[5]) μ is called a Banach limit if μ is a continuous linear functional on l^∞ satisfying:

$(i)\, \|\mu(e)\| = 1 = \mu(1), e = (1, 1, 1 \cdot\cdot\cdot)$
$(ii) \mu_n(a_n) = \mu_n(a_{n+1}), \forall a_n \in (a_0, a_1, \cdots) \in l^\infty$
$(iii) \liminf_{n\to\infty} a_n \le \mu(a_n) \le \limsup_{n\to\infty} a_n, \forall a_n \in (a_0, a_1, \cdots) \in l^\infty$

According to time and circumstances, we use $\mu_n(a_n)$ instead of $\mu(a_0, a_1, \cdots)$.

Further, we know the following result:

Lemma 1.2. *(see reference [1]) For a given* $a \in R$*, for all* $\{a_n\} \in l^\infty$ *satisfies* $\mu_n(a_n) \le a$*, if* $limsup_{n\to\infty}(a_{n+1} - a_n) \le 0$*, then* $limsup_{n\to\infty} a_n \le a$*.*

Lemma 1.3. *Let* $\{\alpha_n\}$ *be a sequence of nonnegative real numbers satisfying the property*

$$\alpha_{n+1} \le (1-\beta_n)\alpha_n + o(\beta_n), \forall n \ge 0$$

where $\{\beta_n\} \in (0, 1)$ *and satisfies* $\sum_{n=0}^{\infty} \beta_n = \infty$*, then* $lim_{n\to\infty}\alpha_n = 0$*.*

We now define

$$U_t x = P(tf(x) + (1-t)Tx), \forall x \in C,$$

where $P : H \to C$ is a projective operator and $f : C \to H$, for some $\gamma \in (0, \frac{1}{2}]$ is a fixed contractive mapping i.e. $\forall x, y \in C$ such that $\|f(x) - f(y)\| \le \gamma \|x - y\|$. By Banach contractive theorem, there is a unique $x_t \in C$ such that

$$x_t = P(tf(x_t) + (1-t)Tx_t). \tag{1.3}$$

If $F(T) \neq \phi$, then $x_t \to u \in F(T)$, as $t \to 0^+$. Now we can have the following result:

$$(i)Tx_t \to u, t \to 0^+$$
$$(ii)tf(x_t) + (1-t)Tx_t - x_t \to 0, t \to 0^+$$

2. Main Results

Theorem 2.1. *Let C be a closed convex subset of a Hilbert space H,$T : C \to H$ be a nonexpansive mapping with $F(T) \neq \phi$,Let $f : H \to C$ is a fixed contractive mapping. The sequence $\{x_n\}$ is defined by*

$$x_{n+1} = P[\alpha_n f(x_n) + (1-\alpha_n)Tx_n], n = 1, 2, \dots$$

where $P : H \to C$ is projective operator and $\alpha_n \in (0,1)$ satisfies

$$lim_{n\to\infty}\alpha_n = 0, and, \sum_{n=1}^{\infty} \alpha_n = \infty$$

, if $Tx_{n+1} - Tx_n \rightharpoonup 0, n \to \infty$. Then $\{x_n\}$ strongly converges to $u = lim_{t\to 0}x_t \in F(T)$, where $\{x_t\}$ is defined by (1.3).

Proof. First we show $\{x_n\}$ is bounded. Let $q \in F(T)$, then

$$\begin{aligned} \|x_{n+1} - q\| &= \|P[\alpha_n f(x_n) + (1-\alpha_n)Tx_n] - Pq\| \\ &\leq \|\alpha_n(f(x_n) - q) + (1-\alpha_n)(x_n - q)\| \\ &\leq \alpha_n(\|f(x_n) - f(q)\| + \|f(q) - q\|) + (1-\alpha_n)\|x_n - q\| \\ &\leq (1 - (1-\beta)\alpha_n)\|x_n - q\| + \alpha_n \|f(q) - q\| \\ &\leq M, \end{aligned}$$

where $M = \max\{\|x_n - q\|, \frac{1}{1-\beta}\|f(q) - q\|\}$,so we have $\{x_n\}$ is bounded, and so are $\{f(x_n)\}$ and $\{Tx_n\}$.

1. $\mu_n\langle f(u) - u, Tx_n - u\rangle \leq 0$

Indeed

$$\begin{aligned} \|x_{n+1} - PTx_n\| &= \|P[\alpha_n f(x_n) + (1-\alpha_n)Tx_n] - PTx_n\| \\ &\leq \|\alpha_n f(x_n) + (1-\alpha_n)Tx_n - Tx_n\| \\ &\leq \alpha_n \|f(x_n) - Tx_n\| \end{aligned}$$

Since $\lim_{n\to\infty} \alpha_n = 0$, so we have $\|x_{n+1} - PTx_n\| \to 0$ as $n \to \infty$. Noting that in the Hilbert space H, there holds that the inequality $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$,for all $x, y \in H$, then we have

$$\begin{aligned} \|x_t - x_{n+1}\|^2 &= \|x_t - PTx_n + PTx_n - x_{n+1}\|^2 \\ &\leq \|x_t - PTx_n\|^2 + 2\langle PTx_n - x_{n+1}, x_t - x_{n+1}\rangle \\ &\leq \|P[tf(x_t) + (1-t)Tx_t] - PTx_n\|^2 \\ &+ 2\|PTx_n - x_{n+1}\|\|x_t - x_{n+1}\| \\ &\leq \|tf(x_t) + (1-t)Tx_t - Tx_n\|^2 + 2\|PTx_n - x_{n+1}\|\|x_t - x_{n+1}\| \end{aligned} \quad (2.1)$$

In the above inequality we set $y = tf(x_t) + (1-t)Tx_t$, then

$$\begin{aligned}
\|tf(x_t) + (1-t)Tx_t - Tx_n\|^2 &= \|t(f(x_t) - Tx_n) + (1-t)(Tx_t - Tx_n)\|^2 \\
&\leq (1-t)^2 \|Tx_t - Tx_n\|^2 \\
&+ 2t\langle f(x_t) - Tx_n, y - Tx_n\rangle \\
&= (1-t)^2 \|Tx_t - Tx_n\|^2 \\
&+ 2t\langle f(x_t) - Tx_n, y - Tx_t + Tx_t - Tx_n\rangle \\
&= (1-t)^2 \|Tx_t - Tx_n\|^2 \\
&+ 2t\langle f(x_t) - Tx_n, y - Tx_t\rangle \\
&+ 2t\langle f(x_t) - Tx_t, Tx_t - Tx_n\rangle \\
&+ 2t\langle Tx_t - Tx_n, Tx_t - Tx_n\rangle \\
&\leq (1+t^2) \|x_t - x_n\|^2 \\
&+ 2t\langle f(x_t) - Tx_n, y - Tx_t\rangle \\
&+ 2t\langle f(x_t) - Tx_t, Tx_t - Tx_n\rangle.
\end{aligned} \tag{2.2}$$

From(2.1)(2.2) we can get

$$\begin{aligned}
2t\langle f(x_t) - Tx_t, Tx_n - Tx_t\rangle &\leq (1+t^2) \|x_t - x_n\|^2 + 2t\langle f(x_t) - Tx_n, y - Tx_t\rangle \\
&+ 2\|PTx_n - x_{n+1}\| \|x_t - x_{n+1}\| - \|x_t - x_{n+1}\|^2
\end{aligned}$$

Now we apply Banach limit to the above inequality,

$$\begin{aligned}
2t\mu_n\langle f(x_t) - Tx_t, Tx_n - Tx_t\rangle &\leq (1+t^2)\mu_n(\|x_t - x_n\|^2) + 2t\mu_n\langle f(x_t) - Tx_n, y - Tx_t\rangle \\
&- \mu_n(\|x_t - x_{n+1}\|^2) \\
&= t^2\mu_n(\|x_t - x_n\|^2) + 2t\mu_n\langle f(x_t) - Tx_n, y - Tx_t\rangle
\end{aligned}$$

i.e.

$$\langle f(x_t) - Tx_t, Tx_n - Tx_t\rangle \leq \frac{t}{2}\mu_n(\|x_t - x_n\|^2) + \mu_n\langle f(x_t) - Tx_n, y - Tx_t\rangle$$

So we have $\limsup_{t\to 0} \mu_n\langle f(x_t) - Tx_t, Tx_n - Tx_t\rangle \leq 0$, i.e.

$$\mu_n\langle f(u) - u, Tx_n - u\rangle \leq 0.$$

2. we shall prove $limsup_{n\to\infty}\langle f(u) - u, Tx_n - u\rangle \leq 0$.

We set $a_n = \langle f(u) - u, Tx_n - u\rangle$, since $\mu_n(a_n) \leq 0$ for all Banach limit and $\{Tx_n\}$ is bounded, now we take the subsequence $\{Tx_{n_j}\}$ of $\{Tx_n\}$, such that $limsup_{n\to 0}(a_{n+1} - a_n) = lim_{j\to\infty}(a_{n_j+1} - a_{n_j})$, from $Tx_{n_j+1} - Tx_{n_j} \rightharpoonup 0$, as $j \to \infty$, then

$$\begin{aligned}
limsup_{n\to\infty}(a_{n+1} - a_n) &= lim_{j\to\infty}(a_{n_j+1} - a_{n_j}) \\
&= \lim_{j\to\infty}\langle f(u) - u, Tx_{n_j+1} - Tx_{n_j}\rangle = 0
\end{aligned}$$

From lemma1.1 we have

$$limsup_{n\to\infty} a_n \leq 0.$$

i.e.

$$limsup_{n\to\infty}\langle f(u) - u, Tx_n - u\rangle \leq 0.$$

Let $\theta_n = \max\{\langle f(u) - u, Tx_n - u\rangle, 0\}$, then $\theta_n \to 0$ as $n \to \infty$.

3. Finally we prove $x_n \to u$ as $n \to \infty$.

$$\begin{aligned}
\|x_{n+1} - u\|^2 &= \|P[\alpha_n f(x_n) + (1-\alpha_n)Tx_n] - u\|^2 \\
&\leq \|\alpha_n(f(x_n) - u) + (1-\alpha_n)(Tx_n - u)\|^2 \\
&= \alpha_n^2 \|f(x_n - u)\|^2 + (1-\alpha_n)^2 \|Tx_n - u\|^2 \\
&+ 2\alpha_n(1-\alpha_n)\langle f(x_n) - u, Tx_n - u\rangle \\
&\leq (1-\alpha_n)^2 \|x_n - u\|^2 + \alpha_n^2 \|f(x_n) - u\|^2 \\
&+ 2\alpha_n(1-\alpha_n)(\langle f(x_n) - f(u), Tx_n - u\rangle + \langle f(u) - u, Tx_n - u\rangle) \\
&\leq (1-\alpha_n)^2 \|x_n - u\|^2 + \alpha_n^2 \|f(x_n) - u\|^2 + 2\alpha_n(1-\alpha_n)\beta \|x_n - u\|^2 \\
&+ 2\alpha_n(1-\alpha_n)\langle f(u) - u, Tx_n - u\rangle \\
&\leq (1-\alpha_n) \|x_n - u\|^2 + \alpha_n^2 \|f(x_n) - u\|^2 + 2\alpha_n(1-\alpha_n)\theta_n \\
&\leq (1-\alpha_n) \|x_n - u\|^2 + o(\alpha_n)
\end{aligned}$$

From lemma1.2 we can have

$$x_n \to u, as, n \to \infty$$

The proof is complete. □

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