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ON CERTAIN INEQUALITIES FOR SOME REGULAR FUNCTIONS DEFINED ON THE UNIT DISC

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In this paper we obtain some inequalities for some regular functions f defined on the unit disc. Our results include or improve several previous results.

1. Introduction.

Let $A(p)$ denote the class of function $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ which are regular in the unit disc $E = \{z : |z| < 1\}$. If $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$ and $h(z) = z^p + \sum_{n=p+1}^{\infty} c_n z^n$ belong to $A(p)$, we define the Hadamard product or convolution of g and h by $(g*h)(z) = z^p + \sum_{n=p+1}^{\infty} b_n c_n z^n$. $z \in E$. For $f \in A(p)$, define

$$D^{n+p-1} f(z) = f(z) * \left(\frac{z^p}{(1-z)^{n+p}} \right) , \tag{1}$$

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where n is any integer greater than $-p$. Then

$$D^{n+p-1}f(z) = \left\{ \frac{z^p(z^{n-1}f(z))^{(n+p-1)}}{(n+p-1)!} \right\} . \tag{2}$$

It can be shown that (2) yields the following identity

$$z(D^{n+p-1}f(z))' = (n+p)D^{n+p}f(z) - nD^{n+p-1}f(z) . \tag{3}$$

In this note we give certain inequalities for $f \in A(p)$ which satisfies the condition

$$\mathrm{Re}\left\{\frac{D^{n+p-1}f(z)}{z^p}\right\} > \alpha \tag{4}$$

and for the integral (5) of functions satisfying (4),

$$F(z) = \frac{p+c}{z^c}\int_0^z u^{c-1}f(u)du . \tag{5}$$

For the existence of the integral in (5), the power represents the principal branch. These inequalities include or improve several results given by Bernardi [1], Goel and Sohi [2], Jack [3], Libera [4], Obradovic [5,6], Owa [7], Shukla and Kumar [8], Singh and Singh [9], Soni [10] and Strohacker [11].

To prove the inequalities, we need the following lemma of Jack [3].

LEMMA. *Let $w(z)$ be regular in the unit disc, with $w(0) = 0$. Then if $|w|$ attains its maximum value on the circle $|z| = r$ at a point z_1, we have*

$$z_1w'(z_1) = kw(z_1)$$

where k is real and $k \geqq 1$.

2. Main results.

THEOREM 1. *Let $f \in A(p)$ for some $p \in N$ and satisfy the condition*

$$Re\left\{ \frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} \right\} > \alpha , \quad z \in E , \tag{6}$$

for some integer n greater than $-p$ and $\alpha < 1$, then

$$Re\left\{ \frac{D^{n+p}F(z)}{D^{n+p-1}F(z)} \right\} > \gamma(\alpha,n,p,c) ,$$

where F *is defined as in (5),* $c+p>0$, $c \geqq 2(1-\alpha)(n+p)-(p+1)$ *and*

$$\gamma(\alpha,n,p,c) = \{2(n-c)+2\alpha(p+n)-1+\sqrt{(2(c-n)+2\alpha(p+n)-1)^2+8(p+c)}\}/(4(p+n)) .$$

For $a=[2(c+p-1)(n+p-1)-1]/[2(n+p)(c+p-1)]$ and $c \geqq -p+2$ in Theorem 1, it is easy to obtain that $\gamma(\alpha,n,p,c) = (n+p-1)/(n+p)$, thus Theorem 1 implies a result by Soni [10].

COROLLARY 1. *Let* $f \in A(p)$ *satisfy the condition*

$$Re\{ \frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} \} > [2(c+p-1)(n+p-1)-1]/[2(n+p)(c+p-1)]$$

$z \in E$, p *a positive integer,* n *any integer greater than* $-p$ *and* $c \geqq -p+2$.

Then

$$Re\{ \frac{D^{n+p}F(z)}{D^{n+p-1}F(z)} \} > (n+p-1)/(n+p) .$$

The second corollary which can be derived from Theorem 1 was proved by Obradovic [6]. Take $p = 1$, $n = 0$ and $\alpha < 1$ and note that

$$\frac{zf'(z)}{f(z)} = \frac{Df(z)}{f(z)} = -c + \frac{z^c f(z)}{\int_0^z u^{c-1}f(u)du}$$

and $\gamma(\alpha,0,1,c)+c = [2c+2a-1+\sqrt{(2c+2\alpha-1)^2+8(c+1)}]/4$, so we get:

COROLLARY 2. *Let* $f \in S^*(a)$, $0 \leqq \alpha < 1$, *and let* $c > max\{-1,-2\alpha\}$, *then we have*

$$Re \frac{z^c f(z)}{\int_0^z u^{c-1}f(u)du} > [2c+2\alpha-1+\sqrt{(2c+2\alpha-1)^2+8(c+1)}]/4, \quad z \in E ,$$

where $S^*(\alpha) = \{f \in A(1) : Re[(zf'(z))/f(z)] > \alpha\}$.

Proof of Theorem 1. Suppose $f \in A(p)$ satisfies the conditions in the theorem and write

$$\frac{D^{n+p}F(z)}{D^{n+p-1}F(z)} = \frac{1-(2\gamma-1)w(z)}{1-w(z)} \tag{7}$$

where $\gamma = \gamma(\alpha,n,p,c)$. By (3),

$$z(D^{n+p-1}F(z))' = (n+p)D^{n+p}F(z) - nD^{n+p-1}F(z)$$
$$= \left[(n+p)\frac{(1-(2\gamma-1)w(z))}{1-w(z)} - n\right]D^{n+p-1}F(z) .$$

It is easy to use (5) and (7) to derive that

$$(8)\quad (c+p)D^{n+p}f(z) = z(D^{n+p}F(z))' + c(D^{n+p}F(z))$$
$$= z\left[\frac{(1-(2\gamma-1)w(z))}{1-w(z)}D^{n+p-1}F(z)\right]' + cD^{n+p}F(z)$$
$$= D^{n+p-1}F(z)\left\{\frac{2(1-\gamma)zw'(z)}{(1-w(z))^2} + \frac{1-(2\gamma-1)w(z)}{1-w(z)}\left[(n+p)\frac{(1-(2\gamma-1)w(z))}{1-w(z)} - n + c\right]\right\},$$

$$(c+p)D^{n+p-1}f(z) = z(D^{n+p-1}F(z))' + c(D^{n+p-1}F(z))$$
$$= D^{n+p-1}F(z)\left[(n+p)\frac{(1-(2\gamma-1)w(z))}{1-w(z)} - n + c\right] .$$

From the above two identities, we conclude that

$$\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} = \frac{2(1-\gamma)zw'(z)}{(1-w(z))[p+c-((2\gamma-1)(n+p)+c-n]w(z)} + \frac{1-(2\gamma-1)w(z)}{1-w(z)} .$$

If $|w(z)| \not< 1$, there exists $z_1 \in E$, so that $|w(z_1)| = 1$, then by Jack's lemma, there exists $k \geqq 1$, such that $z_1w'(z_1) = kw(z_1)$.

Write $w(z_1) = u + iv$ and take the real part of the above identity. After some computations, one has

$$(9)\quad \mathrm{Re}\left\{\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} - \alpha\right\}\bigg|\, z = z_1$$
$$= \gamma - \alpha + 2(1-\gamma)k\ \mathrm{Re}\left\{\frac{u+iv}{(1-(u+iv))[p+c-((2\gamma-1)(n+p)+c-n)(u+iv)]}\right\}$$
$$= \gamma - \alpha + 2(1-\gamma)k\left\{\frac{(u-1+iv)(a-bu+ibv)}{2(1-u)((a-bu)^2+b^2v^2}\right\}$$

$$= \gamma - \alpha + \left\{\frac{-(1-\gamma)k(a+b)}{a^2 - 2abu + b^2}\right\}$$

where we write $a = p + c$, $b = (2\gamma - 1)(n+p) + c - n$. Put

$$g(u) = (a+b)/(a^2 - 2abu + b^2) \ .$$

The condition $c \geq 2(1-\alpha)(n+p) - (p+1)$ and the definition of $\gamma(\alpha,n,p,c)$ imply $b \geq 0$ and $\gamma < 1$, also $a = p + c > 0$. Then $g(u)$ is increasing and thus $1/(a+b) = g(-1) \leq g(u)$. We have from (9) and $k \geq 1$ that

$$\operatorname{Re}\left.\left\{\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} - \alpha\right\}\right|_{z=z_1} \leq \gamma - \alpha + \frac{-(1-\gamma)}{a+b}$$

$$= [2(n+p)\gamma^2 + (2(c-n) - 2\alpha(n+p) + 1)\gamma + 2\alpha(n-c) - 1]/(a+b) = 0 \ .$$

since γ is a root of the polynomial

$$2(n+p)x^2 + (2(c-n) - 2\alpha(n+p) + 1)x + 2\alpha(n-c) - 1 = 0 \ .$$

This contradicts assumption (6), so the proof is completed.

We prove the remaining theorems for $p = 1$ only, the general case can be obtained by the same method.

THEOREM 2. *Suppose* $f \in A(1)$, $\alpha < 1 \leq \beta$, $n \in N_0$ *and*

$$Re\{\frac{D^{n+1}f(z)}{D^n f(z)}\} > \alpha$$

then

$$Re\left(\frac{D^n f(z)}{z}\right)^{\frac{1}{2(n+1)(1-\alpha)\beta}} > \frac{\beta}{1+\beta}$$

for $z \in E$.

Proof. Suppose f satisfies the conditions in the theorem, let $\gamma = \beta/(1+\beta)$ and let $w(z)$ be a regular function such that

$$\left(\frac{D^n f(z)}{z}\right)^{\frac{1}{2(n+1)(1-\alpha)\beta}} = \frac{1-(2\gamma-1)w(z)}{1-w(z)} \tag{10}$$

then $w(0) = 0$. The theorem will follow if we can show that $|w(z)| < 1$ in E .

Now by differentiating (10) logarithmically, we get

$$\frac{-(2\gamma-1)zw'(z)}{1-(2\gamma-1)w(z)} - \frac{-zw'(z)}{1-w(z)} = \frac{1}{2(1-\alpha)\beta}\left\{\frac{D^{n+1}f(z)}{D^n f(z)} - 1\right\} ,$$

where we use identity (3) for $p = 1$, so

(11) $$\frac{D^{n+1}f(z)}{D^n f(z)} = \frac{4(1-\alpha)\beta(1-\gamma)zw'(z)}{[1-w(z)][1-(2\gamma-1)w(z)]} + 1 .$$

If $|w(z)| \not< 1$ in E , by Jack's lemma, there exist $z_1 \in E$ and a real $k \geqq 1$, such that $|w(z_1)| = 1 \geqq |w(z)|$, $\forall |z| \leqq |z_1| = r < 1$, and $z_1 w'(z_1) = kw(z_1)$. Let $w(z_1) = u + iv$, then

(12) $$\mathrm{Re}\ \{z_1 w'(z_1)/[(1-w(z_1))(1-(2\gamma-1)w(z_1))]\}$$

$$= -\gamma k/[2(2\gamma^2 - 2\gamma + 1 - (2\gamma-1)u)] .$$

Put $$g(u) = 1/[2\gamma^2 - 2\gamma + 1 - (2\gamma-1)u]$$

$$g(-1) = 1/(2\gamma^2),\ g(1) = 1/(2(1-\gamma)^2) .$$

$\gamma \geqq 1/2$ implies $g(u)$ is an increasing function of u ,

$$1/(2\gamma^2) \leqq g(u) \leqq 1/(2(1-\gamma)^2) .$$

Applying (11) and (12), one has

$$\mathrm{Re}\ \left\{\frac{D^{n+1}f(z)}{D^n f(z)} - \alpha\right\}\Bigg|_{z=z_1}$$

$$=[4(1-\alpha)\beta(1-\gamma)(-\gamma k)]/[2(2\gamma^2-2\gamma+1-(2\gamma-1)u)] + 1 - \alpha$$

$$= 1 - \alpha - 2(1-\alpha)\beta(1-\gamma)\gamma k g(u)$$

$$\leqq 1 - \alpha - (1-\alpha)\beta(1-\gamma)/\gamma = 0$$

which contradicts the assumption. Thus the theorem is proved.

Replacing n by $n+1$ and $\alpha = 1/2$, $\beta = 1$ in Theorem 2, we obtain:

COROLLARY 3. *If* $f \in A(1)$, $z \in E$ *and*

$$Re\ \left\{\frac{D^{n+2}f(z)}{D^{n+1}f(z)}\right\} > 1/2\ ,\ \text{then}\ Re\left(\frac{D^{n+1}f(z)}{z}\right)^{\frac{1}{n+2}} \quad 1/2 .$$

This is Theorem 3 of [9] by Singh and Singh. Under the condition of

Corollary 3, taking $n = 0$, we have the known result of Strohacker [11], that is, $\text{Re}\{1 + zf''(z)/f'(z)\} > 0$ implies $\text{Re}\{\sqrt{f'(z)}\} > 1/2$. By considering $n = 0$, $\beta = 1$ in Theorem 2, one obtains:

COROLLARY 4. *If* $f \in A(1)$ *and*

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha\text{ , then } Re\left(\frac{f(z)}{z}\right)^{\frac{1}{2(1-\alpha)}} > 1/2\text{ , } z \in E\text{ .}$$

This is Theorem 2 of Jack [3]. Recently Obradovic [5] proved the following result which can be derived from Theorem 2 by taking $n = 0$ and $\beta = 1/(2(1-\alpha))$.

COROLLARY 5. *If* $f \in A(1)$ *and*

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha\text{ , then } Re\left(\frac{f(z)}{z}\right) > 1/(3-2\alpha)\text{, } z \in E\text{ .}$$

Note that $f \in K(\alpha)$, denoting the class

$$\{f \in A(1) : \text{Re}(1 + zf''(z)/f'(z)) > \alpha\}\text{ ,}$$

is equivalent to $\text{Re}(D^2f/Df(z)) > (1+\alpha)/2$. So replacing α by $(1+\alpha)/2$, n by 1 and β by $1/(1-\alpha)$ in Theorem 2, one obtains Corollary 6 which was proved by Obradovic and Owa [7].

COROLLARY 6. *If* $f \in K(\alpha)$ *, then* $Re\sqrt{f'(z)} > 1/(2-\alpha)$.

THEOREM 3. *Suppose* $Re\, c > -1$, $\alpha < 1$, $n \in N_0 = N \cup \{0\}$, $f \in A(1)$ *and satisfies*

$$Re\ \left\{\frac{D^{n+1}f(z)}{z}\right\} > \alpha,\ \text{ for } z \in E,\ \text{ then } Re\ \left\{\frac{D^{n+1}F(z)}{z}\right\} > \gamma(\alpha,c)\text{ ,}$$

for $z \in E$ *, where* $\gamma(\alpha,c) = \{\alpha+Re[1/(2(c+1))]\}/\{1+Re[1/(2(c+1))]\}$ *and* $F(z)$ *is defined as in (5) for* $p = 1$.

Proof. As in Theorem 1, we assume the function f satisfies the conditions in the theorem and write

$$\frac{D^{n+1}F(z)}{z} = \frac{1-(2\gamma-1)w(z)}{1-w(z)} \tag{13}$$

where $\gamma = \gamma(\alpha, c)$ and $w(z)$ a regular function in E, then $w(0) = 0$. It is sufficient to show that $|w(z)| < 1,\ \forall\, z \in E$.

Taking the logarithmic derivative of (13), one obtains

(14)
$$\frac{z(D^{n+1}F(z))'}{D^{n+1}F(z)} - 1 = \left\{\frac{2(1-\gamma)zw'(z)}{[1-w(z)][1-(2\gamma-1)w(z)]}\right\}.$$

From (13), (14) and (8) with $p = 1$, we have

(15)
$$(c+1)\left(\frac{D^{n+1}f(z)}{z}\right) = (D^{n+1}F(z))' + c\,\frac{D^{n+1}F(z)}{z}$$
$$= \frac{2(1-\gamma)zw'(z)}{(1-w(z))^2} + (1+c)\left(\frac{1-(2\gamma-1)w(z)}{1-w(z)}\right).$$

If $|w(z)| \not< 1$, there exists $z_1 \in E$, so that $|w(z)| \leqq |w(z_1)| = 1$, for all $z \in E$, then by Jack's lemma, there exists $k \geqq 1$, such that

$$z_1w'(z_1) = kw(z_1).$$

Write $w(z_1) = u + iv$ so that $z_1w'(z_1)/(1-w(z_1))^2 = -k/(2(1-u)$ and take the real part of (15). After some computations we have

$$\operatorname{Re}\left(\frac{D^{n+1}f(z)}{z} - \alpha\right)\Bigg|_{z=z_1} = \frac{-(1-\gamma)k}{1-u}\cdot\operatorname{Re}\left(\frac{1}{c+1}\right) + \gamma - \alpha$$
$$\leqq \frac{-(1-\gamma)}{2}\cdot\operatorname{Re}\left(\frac{1}{c+1}\right) + \gamma - \alpha$$
$$= \gamma\left\{1 + \frac{1}{2}\operatorname{Re}\left(\frac{1}{c+1}\right)\right\} - \alpha - \frac{1}{2}\operatorname{Re}\left(\frac{1}{c+1}\right)$$
$$= 0,$$

which contradicts the assumption. So $|w(z)| < 1$, for $z \in E$. Thus we have proved the theorem.

From (3) and (8),

$$\frac{D^{n+2}F(z)}{z} = (c+1)\,\frac{D^{n+1}f(z)}{z} + (n+1-c)\,\frac{D^{n+1}F(z)}{z}$$

A few computations lead to:

COROLLARY 7. *If $-1 < c \leqq n+1$ and under the conditions of Theorem 3, then we have*

(16)
$$\operatorname{Re}\frac{D^{n+2}F(z)}{z} > \alpha + [(n+1-c)(1-\alpha)]/[(n+2)(2c+3)].$$

This result improves the estimations of theorem 3.3 of Shukla and Kumar [8].

If $-1 < c \leq n+1$ and $\alpha = -(n+1-c)/[(c+1)(2n+5)] > -1/2(c+1))$, (16) implies $\mathrm{Re}\{D^{n+2}F(z)/z\} > 0$. From this result, considering real c and α replaced by 0 and $-1/(2(c+1))$ respectively in Theorem 3, we derive the following two Corollaries which are theorems 2 and 4 obtained by Singh and Singh in [9].

COROLLARY 8. *If* $f \in M_n(0)$, *then* (i) $\forall c > -1$, $F \in M_n(0)$; (ii) *for* $-1 < c \leq n+1$, $F \in M_{n+1}(0)$, *where* $n \in N_0$ *and*

$$M_n(\alpha) = \{f \in A(1) : Re[D^{n+1}f(z)/z] > \alpha\ ,\ z \in E\}\ .$$

COROLLARY 9. *If* $f \in A(1)$ *and* $Re\ [D^{n+1}f(z)/z] > -1/(2(c+1))$, $c > -1$, *then* $F \in M_n(0)$.

Since $\gamma(\alpha,c) = \alpha + \{(1-\alpha)\mathrm{Re}[1/(2(c+1))]\}/\{1+\mathrm{Re}[1/(2(c+1))]\} > \alpha$, Theorem 3 may be rewritten as: If $f \in M_n(\alpha)$, then $F \in M_n(\gamma(\alpha,c)) \subset M_n(\alpha)$ which implies the theorem 2 of Goel and Sohi [2]. When $n = \alpha = 0$, we have Bernardi's result [1]: If $\mathrm{Re}f'(z) > 0$, then $\mathrm{Re}F'(z) > 0$. If $c = 1$, we have a result of Libera [4].

COROLLARY 10. *If* $f \in A(1)$, $c > -1$, $0 \leq \alpha < 1$ *and*

$$Re[D^{n+1}f(z)/z] > \alpha - (1-\alpha)/(2(1+c)) \quad \text{then} \quad F \in M_n(\alpha)\ .$$

This is Theorem 3 in [2] which can be derived from our Theorem 3 by taking $\alpha = \alpha - (1-\alpha)/(2(1+c))$.

Obradovic [5,6] recently gave the following two results which can also be obtained from Theorem 3 by taking $n = c = \alpha = 0$ and $n = -1$ respectively. Note that $D^0f(z) = f(z)$.

COROLLARY 11. *Let* $f \in A(1)$, *then* $Re\{f'(z)\} > 0$ *implies* $Re\{f(z)/z\} > 1/3$.

COROLLARY 12. *Let* $f \in A(1)$, $\alpha < 1$ *and* $c > -1$, *then for* $z \in E$ $Re\{f(z)/z\} > \alpha$ *implies*

$$Re\left(\frac{c+1}{z^{c+1}}\int^{z} t^{c-1}f(t)dt\right) > \alpha + (1-\alpha)/(3+2c)\ .$$

We state the following theorem which is proved by a similar method. It improves a result of Goel and Sohi [2].

THEOREM 4. *Suppose* $\alpha < 1$, $f \in A(1)$ *and* $Re[D^{n+2}f(z)/z] > \alpha$, *then*

$$[D^{n+1}f(z)/z] > [2(n+2)\alpha + 1]/(2n+5), n \in N_0 .$$

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