Expert Systems with Applications 38 (2011) 13432–13440

Contents lists available at ScienceDirect

Expert Systems with Applications

journal homepage: www.elsevier.com/locate/eswa

An approach to group decision making with interval fuzzy preference relations based on induced generalized continuous ordered weighted averaging operator

Huayou Chen [a,b,*], Ligang Zhou [a]

[a] School of Mathematical Sciences, Anhui University, Hefei, Anhui 230039, China
[b] Fisher School of Business, Ohio State University, Columbus, OH 43210, USA

ARTICLE INFO

Keywords:
Group decision making
Aggregation operator
IGCOWA operator
C-IGCOWA operator
Interval fuzzy preference relation

ABSTRACT

The aim of this paper is to develop a new class of operator called the induced generalized continuous ordered weighted averaging (IGCOWA) operator, which extends the GOWA operator, the IGOWA operator and the ICOWG operator. We study the desirable properties of the IGCOWA operator and describe families of IGCOWA operator according to the special weighting vector and generalized mean parameter. Particularly, we present the consistency induced generalized continuous ordered weighted averaging (C-IGCOWA) operator, in which the consistency indicator is the induced ordering variable to reflect the importance of interval fuzzy preference relation. Finally, the C-IGCOWA operator is applied to group decision making with interval fuzzy preference relations in a numerical example.

1. Introduction

Yager (1988) proposed the ordered weighted averaging (OWA) operator, which provided a parameterized family of averaging operator. The OWA operator is lying between the max operator and the min operator, whose fundamental aspect is a reordering step in which the input arguments are rearranged in descending order and the weight vector is merely associated with its ordered position.

Yager and Kacprzyk (1997), Kacprzyk and Zadrozny (2001), Calvo, Mayor, and Mesiar (2002), Yager (2004a, 2004b), Xu (2005), and Liu (2006) used the OWA operator in a wide range of applications, such as decision making, expert systems, data mining, approximate reasoning, fuzzy system and control etc. Later, Chiclana, Herrera, and Herrera-Viedma (2001) and Xu and Da (2002, 2003) investigated the ordered weighted geometric (OWG) averaging operator, which was based on the OWA operator and on the geometric mean. We can see that the OWA operator and OWG operator can be used to aggregate the finite discrete argument collections.

However, the complexity and uncertainty of problems are becoming more progressive gradually. As a result, the input information arguments provided by decision makers are given in the form of interval numerical values rather than the exact values in the process of decision making. Therefore, Yager (2004a) developed a continuous ordered weighted averaging (COWA) operator by considering the situation where all the arguments to be aggregated were the interval values. A continuous ordered weighted averaging (COWA) operator is the extension of the OWA operator.

Yager and Xu (2006) proposed the continuous ordered weighted geometric (COWG) averaging operator based on the COWA operator and the geometric mean, and studied its application to decision making with interval multiplicative preference relation. In the last few years, the COWA operator had been paid much attention, e.g. Xu (2006b, 2006e) and Gong and Liang (2008), etc.

Chen, Liu, and Wang (2008) gave the definition of continuous ordered weighted harmonic (COWH) averaging operator on the basis of ordered weighted harmonic averaging operator, which is suitable for aggregating continuous interval argument with the cost type indicators. The difference between the COWA operator and the COWH operator is that the interval argument is aggregated by the ordered weighted harmonic (OWH) averaging operator not by the OWA operator.

Another interesting extension of OWA operator is the generalized OWA (GOWA) operator based on generalized mean in the OWA operator. The GOWA operator includes a wide range of mean operators, such as the OWA operator, the OWG operator, the OWH operator etc. The generalized OWA (GOWA) operator was introduced by Yager (2004b) and an additional parameter was added to the OWA operator controlling the power of the argument values to be aggregated. Xu (2006d) define various generalized induced linguistic aggregation operators, including generalized induced linguistic ordered weighted averaging (GILOWA) operator, generalized induced linguistic ordered weighted geometric (GILOWG) operator, generalized induced uncertain linguistic ordered weighted averaging (GIULOWA) operator (Xu, 2006c), etc.

* Corresponding author at: School of Mathematical Sciences, Anhui University, Hefei, Anhui 230039, China. Tel.: +86 551 5107459; fax: +86 1 614 377 2237.
E-mail addresses: huayouc@126.com, chen.1887@osu.edu (H. Chen).

0957-4174/$ - see front matter
doi:10.1016/j.eswa.2011.04.175

Mitchell and Estrakh (1997) gave a modified OWA operator in which the input arguments were not rearranged by their values but by a function of the arguments. Therefore, Yager and Filev (1999) introduced the induced ordered weighted averaging (IOWA) operator, which was a more general type of OWA operator. The reordering step of the IOWA is carried out with order-inducing variable values, rather than depending on the values of the arguments. Similarly, Xu and Da (2003) proposed the induced ordered weighted geometric (IOWG) averaging operator and Chen, Liu, and Sheng (2004) introduced the induced ordered weighted harmonic (IOWH) averaging operator, respectively.

Recently, Wu, Li, Li, and Duan (2009) has developed an aggregation operator called induced continuous ordered weighted geometric (ICOWG) operator, which is based on the IOWG operator and the COWG operator. Merigó and Gil-Lafuente (2009) presented the induced generalized ordered weighted averaging (IGOWA) operator, which used generalized means and order-inducing variables in the reordering process and extended the OWA operator. IGOWA operator includes the particular cases of both the generalized OWA and the induced OWA operator.

The aim of this paper is to provide a new class of operator called the induced generalized continuous ordered weighted averaging (IGCOWA) operator. We can obtain a generalization including the GOWA operator, the IGOWA operator and the ICOWG operator. We also study the properties and families of IGCOWA operator. Especially, we propose the consistency induced generalized continuous ordered weighted averaging (C-IGCOWA) operator, which can be applied to the group decision making with interval fuzzy preference relations.

This paper is organized as follows. In Section 2, we briefly review some basic concepts such as the OWA operator, the COWA operator, the COWG operator, the COWH operator, the GOWA operator, the IOWA operator, the ICOWG operator, the IGOWA operator. We define the ICOWA operator and the ICOWG operator. In Section 3, we present the IGCOWA operator and study some properties of the IGCOWA operator. In Section 4, we give different families of IGCOWA operator according the weighting vector and the parameter of generalized mean. In Section 5, the consistency induced generalized continuous ordered weighted averaging (C-IGCOWA) operator is introduced to aggregate interval fuzzy preference relations in the group decision making, and a selection process for group decision making problems based on C-IGCOWA operator is also given. An illustrative example is discussed in Section 6. Finally, Section 7 summarizes the main conclusions of the paper.

2. Preliminaries

In this section, we briefly describe some averaging operators based on the OWA operator.

2.1. OWA operator

The OWA operator proposed by Yager (1988) can be defined as follows:

Definition 2.1. An OWA operator of dimension n is a mapping $f_w : R^n \to R$, such that

$$f_w(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j \tag{1}$$

where b_j is the jth largest of the $(a_1, a_2, \ldots, a_n)$ and the w_j is the weight, satisfying $w_j \in [0,1]$, $j = 1, 2, \ldots, n$ and $\sum_{j=1}^{n} w_j = 1$.

We can obtain the OWA weights by

$$w_j = Q(j/n) - Q((j-1)/n), \quad j = 1, 2, \ldots, n \tag{2}$$

where Q is a basic unit-interval monotonic (BUM) function $Q : [0,1] \to [0,1]$, and it is monotonic, $Q(0) = 0$ and $Q(1) = 1$. These weights determined by Eq. (2) satisfy the conditions that $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

The ordered weighted geometric (OWG) averaging operator proposed by Xu and Da (2002, 2003) is defined in the following:

Definition 2.2. An OWA operator of dimension n is a mapping $g_w : R^n \to R$ associated with an exponential weighting vector $w = (w_1, w_2, \ldots, w_n)$, such that

$$g_w(a_1, a_2, \ldots, a_n) = \prod_{j=1}^{n} b_j^{w_j} \tag{3}$$

where $w_j \in [0,1]$, $\sum_{j=1}^{n} w_j = 1$, b_j is the jth largest of the $(a_1, a_2, \ldots, a_n)$.

2.2. COWA operator, COWG operator, and COWH operator

COWA operator developed by Yager (2004a) can be defined as follows:

Definition 2.3. A continuous ordered weighted averaging (COWA) operator is a mapping $F_Q : \Omega^+ \to R^+$ associated with BUM function$Q$, such that

$$F_Q([a,b]) = \int_0^1 \frac{dQ(y)}{dy}(b - y(b-a))dy \tag{4}$$

where $\Omega^+ = \{[a,b] | a \in R^+, b \in R^+, a < b\}$, R^+ is the set of positive real number.

Based on the BUM function Q, Yager (2004a) proved that the general form representation of $F_Q([a,b])$ could be obtained by following Theorem 2.1:

Theorem 2.1. *If $\lambda = \int_0^1 Q(y)dy$, then*

$$F_Q([a,b]) = (1-\lambda)a + \lambda b \tag{5}$$

where λ is called the attitudinal character of Q.

From Theorem 2.1, we know that $F_Q([a,b])$ is always the weighted arithmetical mean of end points based on the attitudinal character.

For example, if the BUM function $Q(y) = y^n$, then $\lambda = \int_0^1 Q(y)dy = \frac{1}{n+1}$. By Eq. (5), we get

$$F_Q([a,b]) = (1-\lambda)a + \lambda b = \frac{na + b}{n+1}$$

Yager and Xu (2006) developed continuous ordered weighted geometric averaging (COWG) operator, which is defined as follows:

Definition 2.4. A continuous ordered weighted geometric (COWG) averaging operator is a mapping $G_Q : \Omega^+ \to R^+$ associated with BUM function Q, such that

$$G_Q([a,b]) = b \cdot \left(\frac{a}{b}\right)^{\int_0^1 \frac{dQ(y)}{dy} y dy} \tag{6}$$

where $\Omega^+ = \{[a,b] | a \in R^+, b \in R^+, a < b\}$, R^+ is the set of positive real number.

Similarly, if $\lambda = \int_0^1 Q(y)dy$ is the attitudinal character of Q, then

$$G_Q([a,b]) = a^{1-\lambda} \cdot b^{\lambda} \tag{7}$$

That is, $G_Q([a,b])$ is the weighted geometric mean of end points of the interval $[a,b]$ based on the attitudinal character.

Chen et al. (2008) proposed the continuous ordered weighted harmonic averaging (COWH) operator, which is described in the following:

Definition 2.5. A continuous ordered weighted harmonic (COWH) averaging operator is a mapping $H_Q:\Omega^+\to R^+$ associated with BUM function Q, such that

$$H_Q([a,b])=\frac{1}{\int_0^1\frac{dQ(y)}{dy}\left(\frac{1}{b}+y\left(\frac{1}{a}-\frac{1}{b}\right)\right)dy} \tag{8}$$

where $\Omega^+=\{[a,b]|a\in R^+,b\in R^+,a<b\}$, R^+ is the set of positive real number.

Similarly, if $\lambda=\int_0^1 Q(y)dy$ is the attitudinal character of Q, then

$$H_Q([a,b])=\frac{1}{\frac{1-\lambda}{a}+\frac{\lambda}{b}}$$

The COWA operator, the COWG operator and the COWH operator can be applied to not only defuzzification, but also decision making with many types of interval preference relations.

2.3. GOWA operator

Yager (2004b) introduced the generalized OWA (GOWA) operator, which can be defined by an additional controlling parameter as follows:

Definition 2.6. A mapping $M:R^n\to R$ is called a generalized ordered weighted aggregation (GOWA) operator of dimension n if

$$M(a_1,a_2,\ldots,a_n)=\left(\sum_{j=1}^n w_j(b_j)^\gamma\right)^{1/\gamma} \tag{9}$$

where b_j is the jth largest of $a_1,a_2,\ldots,a_n$ and $\{w_j, j=1,2,\ldots,n\}$ is the collection of weights satisfying $w_j\in[0,1]$ and $\sum_{j=1}^n w_j=1$. γ is a parameter such that $\gamma\in(-\infty,+\infty)$ and $\gamma\neq 0$.

In addition to some properties and particular cases associated with the GOWA operator, Yager (2004b) proved that the GOWA operator is monotonic with respect to a_i, which is described in the following theorem:

Theorem 2.2. *if $a_i\geqslant\tilde{a}_i$ for all i, then*

$$M(a_1,a_2,\ldots,a_n)\geqslant M(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_n) \tag{10}$$

2.4. IOWA operator, IOWG operator and IOWH operator

Yager and Filev (1999) and Yager (2003) introduced the induced ordered weighted averaging (IOWA) operator:

Definition 2.7. An IOWA operator of dimension n is a mapping $IOWA_w:R^n\to R$ characterized by the n dimensional weighting vector $w=(w_1,w_2,\ldots,w_n)$ and a set of order inducing variables u_i by a formula of the following form:

$$IOWA_w(\langle u_1,a_1\rangle,\langle u_2,a_2\rangle,\ldots,\langle u_n,a_n\rangle)=\sum_{j=1}^n w_j a_{\sigma(j)} \tag{11}$$

where $w_j\in[0,1]$, $\sum_{j=1}^n w_j=1, \sigma:\{1,2,\ldots,n\}\to\{1,2,\ldots,n\}$ is a permutation, such that $u_{\sigma(j)}\geqslant u_{\sigma(j+1)}$, $j=1,2,\ldots,n-1$, $(a_{\sigma(1)},a_{\sigma(2)},\ldots,a_{\sigma(n)})$ is reordered for argument values $a_1,a_2,\ldots,a_n$ in decreasing order of the values of the u_j, $j=1,2,\ldots,n$.

Xu and Da (2003) proposed the induced ordered weighted geometric (IOWG) averaging operator.

Definition 2.8. An IOWG operator of dimension n is a mapping $IOWG_w:R^n\to R$ which is defined as

$$IOWG_w(\langle u_1,a_1\rangle,\langle u_2,a_2\rangle,\ldots,\langle u_n,a_n\rangle)=\prod_{j=1}^n (a_{\sigma(j)})^{w_j} \tag{12}$$

where $w=(w_1,w_2,\ldots,w_n)$ is weighting vector, and u_i is the order inducing variable, $\sigma:\{1,2,\ldots,n\}\to\{1,2,\ldots,n\}$ is a permutation, such that $u_{\sigma(j)}\geqslant u_{\sigma(j+1)}$, $j=1,2,\ldots,n-1$.

Chen et al. (2004) introduced the induced ordered weighted harmonic (IOWH) averaging operator, and gave its application to combination forecasting method.

Definition 2.9. A mapping $IOWH_w:R^n\to R$ is called an IOWH operator of dimension n, if

$$IOWH_w(\langle u_1,a_1\rangle,\langle u_2,a_2\rangle,\ldots,\langle u_n,a_n\rangle)=\frac{1}{\sum_{j=1}^n\frac{w_j}{a_{\sigma(j)}}} \tag{13}$$

where $w=(w_1,w_2,\ldots,w_n)$ is n dimensional weighting vector, and $(u_1,u_2,\ldots,u_n)$ is a set of order inducing variables, such that $u_{\sigma(j)}\geqslant u_{\sigma(j+1)}$, $j=1,2,\ldots,n-1$, $\sigma:\{1,2,\ldots,n\}\to\{1,2,\ldots,n\}$ is a permutation.

2.5. ICOWG operator and ICOWA operator

ICOWG operator developed by Wu et al. (2009) is defined as follows:

Definition 2.10. Suppose $[a_1,b_1],[a_2,b_2],\ldots,[a_n,b_n]$ is a set of interval numbers. An induced continuous ordered weighted geometric (ICOWG) operator is a mapping: $ICOWG_w:(\Omega^+)^n\to R^+$ associated with a weighting vector $w=(w_1,w_2,\ldots,w_n)$, such that $w_j\in[0,1]$, $\sum_{j=1}^n w_j=1$, which is defined to aggregate the set of second arguments of two tuples $\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle$ according to the following expression:

$$\begin{aligned}&ICOWG_w(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle)\\&=IOWG_w(\langle u_1,G_Q([a_1,b_1])\rangle,\langle u_2,G_Q([a_2,b_2])\rangle,\ldots,\langle u_n,G_Q([a_n,b_n])\rangle)\\&=\prod_{j=1}^n (G_Q([a_{\sigma(j)},b_{\sigma(j)}]))^{w_j}\end{aligned} \tag{14}$$

where $\sigma:\{1,2,\ldots,n\}\to\{1,2,\ldots,n\}$ is a permutation such that $u_{\sigma(j)}\geqslant u_{\sigma(j+1)}$, $\langle u_{\sigma(j)},G_Q([a_{\sigma(j)},b_{\sigma(j)}])\rangle$ is the two tuples with $u_{\sigma(j)}$, the jth highest value in the set $(u_1,u_2,\ldots,u_n)$, and $G_Q([a_{\sigma(j)},b_{\sigma(j)}])$ is calculated by Eq. (6) of COWG operator.

Similarly, we propose induced continuous ordered weighted averaging (ICOWA) operator, which is the extension of the IOWA operator and the COWA operator.

Definition 2.11. Let $[a_1,b_1],[a_2,b_2],\ldots,[a_n,b_n]$ be a set of interval numbers. A mapping $ICOWA_w$: $(\Omega^+)^n\to R^+$ is called the induced continuous ordered weighted averaging (ICOWA) operator, if

$$\begin{aligned}&ICOWA_w(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle)\\&=IOWA_w(\langle u_1,F_Q([a_1,b_1])\rangle,\langle u_2,F_Q([a_2,b_2])\rangle,\ldots,\langle u_n,F_Q([a_n,b_n])\rangle)\\&=\sum_{j=1}^n w_j F_Q([a_{\sigma(j)},b_{\sigma(j)}])\end{aligned} \tag{15}$$

where $\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle$ is the set of two tuples, $F_Q([a_{\sigma(j)},b_{\sigma(j)}])$ is calculated by Eq. (4) of COWA operator.

Weighting vector w satisfies $w_j \in [0,1]$, $\sum_{j=1}^{n} w_j = 1$; $\sigma : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ is a permutation with $u_{\sigma(j)} \geqslant u_{\sigma(j+1)}$.

2.6. IGOWA operator

Merigó and Gil-Lafuente (2009) presented the induced generalized ordered weighted averaging (IGOWA) operator:

Definition 2.12. An induced generalized ordered weighted averaging (IGOWA) of dimension n is a mapping $IGOWA : R^n \to R$ defined by an associated weighting vector $w = (w_1, w_2, \ldots, w_n)$ of dimension n, a set of order inducing variables $(u_1, u_2, \ldots, u_n)$ and a parameter γ, according to the following formula:

$$IGOWA_w(\langle u_1, a_1\rangle, \langle u_2, a_2\rangle, \ldots, \langle u_n, a_n\rangle) = \left(\sum_{j=1}^{n} w_j (a_{\sigma(j)})^{\gamma}\right)^{1/\gamma} \quad (16)$$

where $w_j \in [0,1]$, $\sum_{j=1}^{n} w_j = 1$, $\sigma : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ is a permutation such that $u_{\sigma(j)} \geqslant u_{\sigma(j+1)}, a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)}$ are reordered in decreasing order of the values of the $u_1, u_2, \ldots, u_n$.

3. The induced generalized continuous OWA operator (IGCOWA) and its properties

As we have already mentioned, the decision maker may provide the information of interval numbers due to uncertain decision-making environment, vague knowledge about the alternative and personal preference etc. It is necessary that we should study the induced generalized continuous OWA operator.

In this section we propose the induced generalized continuous OWA (IGCOWA) operator, which is an extension of the ICOWG operator, IGOWA operator and IGOWH operator. We will get a more general formulation of the aggregating all the interval values $[a_1,b_1],[a_2,b_2],\ldots,[a_n,b_n]$ in more complex situations based on an additional controlling parameter.

3.1. The IGCOWA operator

Definition 3.1. Suppose $[a_1,b_1],[a_2,b_2],\ldots,[a_n,b_n]$ is a set of interval numbers. An induced generalized continuous ordered weighted averaging (IGCOWA) operator is a mapping: $IGCOWA_w : (\Omega^+)^n \to R^+$ associated with a weighting vector $w = (w_1, w_2, \ldots, w_n)$, which is defined to aggregate the set of second arguments of two tuples $\langle u_1,[a_1,b_1]\rangle, \langle u_2,[a_2,b_2]\rangle, \ldots, \langle u_n,[a_n,b_n]\rangle$ according to the following expression:

$$\begin{aligned} & IGCOWA_w(\langle u_1,[a_1,b_1]\rangle, \langle u_2,[a_2,b_2]\rangle, \ldots, \langle u_n,[a_n,b_n]\rangle) \\ & = IGOWA_w(\langle u_1, F_Q([a_1,b_1])\rangle, \langle u_2, F_Q([a_2,b_2])\rangle, \ldots, \langle u_n, F_Q([a_n,b_n])\rangle) \\ & = \left(\sum_{j=1}^{n} w_j (F_Q([a_{\sigma(j)}, b_{\sigma(j)}]))^{\gamma}\right)^{1/\gamma} \end{aligned} \quad (17)$$

where $w_j \in [0,1]$, $\sum_{j=1}^{n} w_j = 1$, $\sigma : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ is a permutation such that $u_{\sigma(j)} \geqslant u_{\sigma(j+1)}$, γ is a parameter controlling the power of the argument values satisfying $\gamma \in (-\infty, +\infty)$ and $\gamma \neq 0$, and $F_Q([a_{\sigma(j)}, b_{\sigma(j)}])$ is calculated by Eq. (4).

Example 3.1. Assume we have the following collection of two tuples of arguments with order-inducing variables $\langle u_i,[a_i,b_i]\rangle$, $i = 1,2,3,4$, where $u_1 = 0.75$, $u_2 = 0.91$, $u_3 = 0.64$, $u_4 = 0.87$, $[a_1,b_1] = [0.42,0.64]$, $[a_2,b_2] = [0.45,0.73]$, $[a_3,b_3] = [0.72,0.95]$, $[a_4,b_4] = [0.56,0.84]$, the weighting vector $w = (0.25,0.35,0.25,0.15)$. If the BUM function $Q(y) = y^2$, then the attitudinal character $\lambda = \int_0^1 Q(y)dy = \int_0^1 y^2 dy = \frac{1}{3}$. Using Eq. (5), we have

$$F_Q([a_1,b_1]) = 0.49333, \quad F_Q([a_2,b_2]) = 0.54333$$
$$F_Q([a_3,b_3]) = 0.79667, \quad F_Q([a_4,b_4]) = 0.65333.$$

Thus by the order-inducing variables (u_1,u_2,u_3,u_4), we have

$$F_Q([a_{\sigma(1)}, b_{\sigma(1)}]) = 0.54333, \quad F_Q([a_{\sigma(2)}, b_{\sigma(2)}]) = 0.65333$$
$$F_Q([a_{\sigma(3)}, b_{\sigma(3)}]) = 0.49333, \quad F_Q([a_{\sigma(4)}, b_{\sigma(4)}]) = 0.79667$$

Suppose parameter $\gamma = \frac{1}{2}$, substituting weighting vector $w = (0.25,0.35,0.25,0.15)$ and $\gamma = \frac{1}{2}$ into Eq. (17), we get

$$\begin{aligned} & IGCOWA_w(\langle u_1,[a_1,b_1]\rangle, \langle u_2,[a_2,b_2]\rangle, \langle u_3,[a_3,b_3]\rangle, \langle u_4,[a_4,b_4]\rangle) \\ & = IGOWA_w(\langle u_1, F_Q([a_1,b_1])\rangle, \langle u_2, F_Q([a_2,b_2])\rangle, \langle u_3, F_Q([a_3,b_3])\rangle, \\ & \langle u_4, F_Q([a_4,b_4])\rangle) = (0.25 \times (0.54333)^{1/2} + 0.35 \times (0.65333)^{1/2} \\ & + 0.25 \times (0.49333)^{1/2} + 0.15 \times (0.79667)^{1/2})^2 = 0.6032 \end{aligned}$$

3.2. The properties of the IGCOWA operator

Theorem 3.1 (*Monotonicity*). *If $a_j \geqslant a'_j$, $b_j \geqslant b'_j$ for all $j = 1,2,\ldots,n$, then*

$$\begin{aligned} & IGCOWA_w(\langle u_1,[a_1,b_1]\rangle, \langle u_2,[a_2,b_2]\rangle, \ldots, \langle u_n,[a_n,b_n]\rangle) \\ & \geqslant IGCOWA_w(\langle u_1,[a'_1,b'_1]\rangle, \langle u_2,[a'_2,b'_2]\rangle, \ldots, \langle u_n,[a'_n,b'_n]\rangle) \end{aligned} \quad (18)$$

Proof. Let $\lambda = \int_0^1 Q(y)dy$ be the attitudinal character, where Q is a BUM function $Q : [0,1] \to [0,1]$, it follows that $\lambda \geqslant 0$. Since $a_j \geqslant a'_j, b_j \geqslant b'_j$ for all $j = 1,2,\ldots,n$, from Eq. (5), we have

$$F_Q([a_j,b_j]) \geqslant F_Q\left(\left[a'_j, b'_j\right]\right), \quad \text{for all } j = 1,2,\ldots,n$$

i.e.

$$F_Q([a_{\sigma(j)}, b_{\sigma(j)}]) \geqslant F_Q\left(\left[a'_{\sigma(j)}, b'_{\sigma(j)}\right]\right), \quad \text{for all } j = 1,2,\ldots,n$$

By Eq. (10) of Theorem 2.2, it follows that

$$\left(\sum_{j=1}^{n} w_j (F_Q([a_{\sigma(j)}, b_{\sigma(j)}]))^{\gamma}\right)^{1/\gamma} \geqslant \left(\sum_{j=1}^{n} w_j \left(F_Q\left(\left[a'_{\sigma(j)}, b'_{\sigma(j)}\right]\right)\right)^{\gamma}\right)^{1/\gamma}$$

From Eq. (17), we can see that the result follows, which completes the proof of Theorem. □

Theorem 3.2. *(Idempotency) If $[a_j, b_j] = [a, b]$, for all $j = 1,2,\ldots,n$, then*

$$IGCOWA_w(\langle u_1,[a_1,b_1]\rangle, \langle u_2,[a_2,b_2]\rangle, \ldots, \langle u_n,[a_n,b_n]\rangle) = (1-\lambda)a + \lambda b \quad (19)$$

where λ is the attitudinal character.

Proof. Since $[a_j,b_j] = [a,b]$, for all $j = 1,2,\ldots,n$, using Eq. (5), we have

$$F_Q([a_j,b_j]) = (1-\lambda)a_j + \lambda b_j = (1-\lambda)a + \lambda b, \quad j = 1,2,\ldots,n$$

i.e.,

$$F_Q([a_{\sigma(j)}, b_{\sigma(j)}]) = (1-\lambda)a + \lambda b, \quad \text{for all } j = 1,2,\ldots,n$$

From Eq. (17), we get

$$\begin{aligned} & IGCOWA_w(\langle u_1,[a_1,b_1]\rangle, \langle u_2,[a_2,b_2]\rangle, \ldots, \langle u_n,[a_n,b_n]\rangle) \\ & = \left(\sum_{i=1}^{n} w_i (F_Q([a_{\sigma(j)}, b_{\sigma(j)}]))^{\gamma}\right)^{1/\gamma} = \left(\sum_{i=1}^{n} w_i ((1-\lambda)a + \lambda b)^{\gamma}\right)^{1/\gamma} \\ & = (1-\lambda)a + \lambda b \end{aligned}$$

which completes the proof of Theorem. □

Theorem 3.3 (*Boundness*). *If* $[a_j, b_j]$, $j = 1,2,\ldots,n$ *is any set of interval numbers, then*

$$\min_j a_j \leqslant IGCOWA_w(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle) \leqslant \max_j b_j \quad (20)$$

Proof. Since $a_j \leqslant b_j$, $j = 1,2,\ldots,n$, using Eq. (5), we have

$$\min_j a_j \leqslant a_j \leqslant F_Q([a_j,b_j]) = (1-\lambda)a_j + \lambda b_j \leqslant b_j \leqslant \max_j b_j,$$
$$j = 1,2,\ldots,n$$

i.e.,

$$\min_j a_j \leqslant F_Q([a_{\sigma(j)},b_{\sigma(j)}]) \leqslant \max_j b_j, \quad j = 1,2,\ldots,n$$

By Eq. (10) of Theorem 2.2, it follows that

$$\min_j a_j = \left(\sum_{i=1}^n w_i(\min_j a_j)^\gamma\right)^{1/\gamma} \leqslant \left(\sum_{i=1}^n w_i(F_Q([a_{\sigma(j)},b_{\sigma(j)}]))^\gamma\right)^{1/\gamma}$$
$$\leqslant \left(\sum_{i=1}^n w_i(\max_j b_j)^\gamma\right)^{1/\gamma} = \max_j b_j$$

From Eq. (17), we can see that the result follows, which completes the proof of Theorem. □

Theorem 3.4 (*Commutativity*). *If* $\{\pi(1),\pi(2),\ldots,\pi(n)\}$ *is any permutation of* $\{1,2,\ldots,n\}$, *then*

$$IGCOWA_w(\langle u_{\pi(1)},[a_{\pi(1)},b_{\pi(1)}]\rangle,\langle u_{\pi(2)},[a_{\pi(2)},b_{\pi(2)}]\rangle,\ldots,\langle u_{\pi(n)},[a_{\pi(n)},b_{\pi(n)}]\rangle)$$
$$= IGCOWA_w(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle) \quad (21)$$

Proof. From Eq. (17) of Definition 3.1, we have

$$IGCOWA_w(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle)$$
$$= \left(\sum_{j=1}^n w_j(F_Q([a_{\sigma(j)},b_{\sigma(j)}]))^\gamma\right)^{1/\gamma}$$

$$IGCOWA_w(\langle u_{\pi(1)},[a_{\pi(1)},b_{\pi(1)}]\rangle,\langle u_{\pi(2)},[a_{\pi(2)},b_{\pi(2)}]\rangle,\ldots,\langle u_{\pi(n)},[a_{\pi(n)},b_{\pi(n)}]\rangle)$$
$$= \left(\sum_{j=1}^n w_j(F_Q([c_{\sigma(j)},d_{\sigma(j)}]))^\gamma\right)^{1/\gamma}$$

Since $\{\pi(1),\pi(2),\ldots,\pi(n)\}$ is any permutation of $\{1,2,\ldots,n\}$, we obtain $[a_{\sigma(j)},b_{\sigma(j)}] = [c_{\sigma(j)},d_{\sigma(j)}]$, $j = 1,2,\ldots,n$. Using Eq. (5), it follows that

$$F_Q([a_{\sigma(j)},b_{\sigma(j)}]) = F_Q([c_{\sigma(j)},d_{\sigma(j)}]), \quad j = 1,2,\ldots,n$$

So we can see that the result holds, which completes the proof of Theorem. □

4. Families of the induced generalized continuous OWA (IGCOWA) operator

We presented the IGCOWA operator, which is the extension of the many existing operators. The IGCOWA operator can be seen as a general model and frameworks that can be used in some possible situations, because the decision maker can choose the weighting vector w and a generalized parameter γ according to his/her preference and interests in the specific decision making problem. Let's discuss the different types of IGCOWA operator in this section.

4.1. Families of the IGCOWA operator from the different weighting vector

If we consider different cases of the weighting vector w, we could obtain different cases of aggregation operators, including the ICOWA operator, the composite function of GOWA operator and COWA operator.

Remark 1. If weighting vector $w^* = (1,0,\ldots,0,0)^T$, from Eq. (17) and (15), we have

$$IGCOWA_{w^*}(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle)$$
$$= \left(\sum_{j=1}^n w_j^*(F_Q([a_{\sigma(j)},b_{\sigma(j)}]))^\gamma\right)^{1/\gamma} = \left(1 \times (F_Q([a_{\sigma(1)},b_{\sigma(1)}]))^\gamma\right)^{1/\gamma}$$
$$= F_Q([a_{\sigma(1)},b_{\sigma(1)}])$$
$$= ICOWA_{w^*}(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle)$$

That is, ICOWA operator is a special of IGCOWA operator when weighting vector $w^* = (1,0,\ldots,0,0)^T$.

In fact, let $w_j = 1$, $w_k = 0$, for all $k = 1,2,\ldots,n$, and $k \neq j$, we can easily see that IGCOWA operator is reduced to the ICOWA operator.

Remark 2. If weighting vector $w = (\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n})^T$, using Eqs. (17), (4) and (10), we have

$$IGCOWA_w(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle)$$
$$= \left(\sum_{j=1}^n \frac{1}{n}(F_Q([a_{\sigma(j)},b_{\sigma(j)}]))^\gamma\right)^{1/\gamma} = \left(\sum_{j=1}^n \frac{1}{n}(F_Q([a_j,b_j]))^\gamma\right)^{1/\gamma}$$
$$= M(F_Q([a_1,b_1]),F_Q([a_2,b_2]),\ldots,F_Q([a_n,b_n]))$$

That is, IGCOWA operator is the composite function of GOWA operator and COWA operator when $w_j = 1/n$, for all $j = 1,2,\ldots,n$.

4.2. Families of the IGCOWA operator from the different parameter γ

Now, let's consider the form of the IGCOWA operator in some particular cases of γ. We will acquire a group of particular cases including the ICOWA operator, the ICOWG operator, and the ICOWH operator.

Remark 3. When $\gamma = 1$, the IGCOWA operator is reduced to the ICOWA operator. Using Eqs. (17) and (15), we have

$$IGCOWA_w(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle)$$
$$= \left(\sum_{j=1}^n w_j(F_Q([a_{\sigma(j)},b_{\sigma(j)}]))^\gamma\right)^{1/\gamma} = \sum_{j=1}^n w_jF_Q([a_{\sigma(j)},b_{\sigma(j)}])$$
$$= ICOWA_w(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle)$$

Remark 4. When $\gamma \to 0$, the IGCOWA operator is reduced to the composite function of IOWG operator and COWA operator.

Using the L'Hospital's rule, Eqs. (17), (12) and (4), we have

$$\lim_{\gamma\to 0} IGCOWA_w(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle)$$
$$=\lim_{\gamma\to 0}\left(\sum_{j=1}^n w_j(F_Q([a_{\sigma(j)},b_{\sigma(j)}]))^\gamma\right)^{1/\gamma}$$
$$=\lim_{\gamma\to 0}\exp\left(\log\left(\sum_{j=1}^n w_j(F_Q([a_{\sigma(j)},b_{\sigma(j)}]))^\gamma\right)^{1/\gamma}\right)$$
$$=\lim_{\gamma\to 0}\exp\left(\frac{\log\left(\sum_{j=1}^n w_j(F_Q([a_{\sigma(j)},b_{\sigma(j)}]))^\gamma\right)}{\gamma}\right)$$
$$=\exp\left(\lim_{\gamma\to 0}\frac{\log\left(\sum_{j=1}^n w_j(F_Q([a_{\sigma(j)},b_{\sigma(j)}]))^\gamma\right)}{\gamma}\right)$$
$$=\exp\left(\lim_{\gamma\to 0}\frac{\frac{1}{\sum_{j=1}^n w_j(F_Q([a_{\sigma(j)},b_{\sigma(j)}]))^\gamma}\sum_{j=1}^n w_j(F_Q([a_{\sigma(j)},b_{\sigma(j)}]))^\gamma\log(F_Q([a_{\sigma(j)},b_{\sigma(j)}]))}{1}\right)$$
$$=\exp\left(\sum_{j=1}^n w_j\log(F_Q([a_{\sigma(j)},b_{\sigma(j)}]))\right)=\exp\left(\log\left(\prod_{j=1}^n(F_Q([a_{\sigma(j)},b_{\sigma(j)}]))^{w_j}\right)\right)$$
$$=\prod_{j=1}^n(F_Q([a_{\sigma(j)},b_{\sigma(j)}]))^{w_j}$$
$$=IOWG_w(\langle u_1,F_Q([a_1,b_1])\rangle,\langle u_2,F_Q([a_2,b_2])\rangle,\ldots,\langle u_n,F_Q([a_n,b_n])\rangle).$$

It shows that the IGCOWA operator is reduced to the composite function of IOWG operator and COWA operator when $\gamma\to 0$.

Remark 5. When $\gamma = -1$, the IGCOWA operator becomes the composite function of IOWH operator and COWA operator.

Using Eqs. (17), (13) and (4), we have

$$IGCOWA_w(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle)$$
$$=\left(\sum_{j=1}^n w_j(F_Q([a_{\sigma(j)},b_{\sigma(j)}]))^\gamma\right)^{1/\gamma}=\frac{1}{\sum_{j=1}^n\frac{w_j}{F_Q([a_{\sigma(j)},b_{\sigma(j)}])}}$$
$$=IOWH_w(\langle u_1,F_Q([a_1,b_1])\rangle,\langle u_2,F_Q([a_2,b_2])\rangle,\ldots,\langle u_n,F_Q([a_n,b_n])\rangle)$$

which indicates that the IGCOWA operator is the composite function of IOWH operator and COWA operator when $\gamma = -1$.

4.3. Families of the IGCOWA operator from the degeneration of the interval arguments

Remark 6. If all of the interval numbers become the crisp numbers, i.e., $a_j = b_j$, for all $j = 1,2,\ldots,n$, then the IGCOWA operator is reduced to the IGOWA operator.

In fact, from Eq. (4), we have $F_Q([a_j,b_j]) = a_j$. Obviously, by Eqs. (17) and (16), it follows that

$$IGCOWA_w(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle)$$
$$=\left(\sum_{j=1}^n w_j(F_Q([a_{\sigma(j)},b_{\sigma(j)}]))^\gamma\right)^{1/\gamma}=\left(\sum_{j=1}^n w_j(a_{\sigma(j)})^\gamma\right)^{1/\gamma}$$
$$=IGOWA_w(\langle u_1,[a_1,b_1]\rangle,\langle u_2,[a_2,b_2]\rangle,\ldots,\langle u_n,[a_n,b_n]\rangle)$$

5. C-IGCOWA operator and its application to aggregate interval fuzzy preference relations in the group decision making

Wu et al. (2009) developed induced continuous ordered weighted geometric (ICOWG) operator to aggregate interval multiplicative preference relations. It is usually assumed that the multiplicative preference relations are reciprocal. Therefore, it is suitable for using ICOWG operator to aggregate interval multiplicative preference relations in order to maintain both the reciprocity and consistency properties in the collective interval multiplicative preference relation when the order-inducing values are unchanged. However, decision maker may provide the interval fuzzy preference relations, we cannot use ICOWG operator to aggregate this form of information. As a result, we introduce a new kind of operator called the consistency induced generalized continuous ordered weighted averaging (C-IGCOWA) operator to aggregate interval fuzzy preference relations for keeping the complementary properties in the collective interval fuzzy preference relation.

5.1. The consistency induced generalized continuous ordered weighted averaging (C-IGCOWA) operator and its related conceptions

Let $X = (x_1,x_2,\ldots,x_n)$ be a finite set of alternatives, $E = (e_1,e_2,\ldots,e_m)$ be a finite set of experts. An expert provides his/her preference which is described by a fuzzy preference relation (Herrera, Martínez, & Sánchez, 2005).

Definition 5.1. Let $R = (r_{ij})_{n\times n}$ be a matrix. If

$$r_{ij}\geqslant 0,\quad r_{ij}+r_{ji}=1,\quad r_{ii}=0.5,\quad \forall i,j=1,2,\ldots,n \tag{22}$$

then matrix R is called a fuzzy preference relation or complementary matrix, where r_{ij} denotes the preference degree of the alternative x_i over x_j.

Especially, $r_{ij} = 0.5$ indicates indifference between x_i and x_j; $r_{ij} > 0.5$ indicates that x_i is preferred to x_j, the larger r_{ij}, the greater the preference degree of the alternative x_i over x_j; $r_{ij} < 0.5$ indicates that x_j is preferred to x_i, the smaller r_{ij}, the greater the preference degree of the alternative x_j over x_i.

However, a decision maker cannot estimate their preference with an exact numerical value because they have vague knowledge about the preference degrees of one alternative over another, Xu (2002, 2004, 2006a) introduced the notion of interval fuzzy preference relation.

Definition 5.2. An interval fuzzy preference relation on the set X is defined as matrix $\widetilde{R}=(\tilde{r}_{ij})_{n\times n}\subset X\times X$, $\tilde{r}_{ij}=[r_{ij}^L,r_{ij}^U]$, satisfying

$$r_{ij}^U+r_{ji}^L=1,\quad r_{ji}^U+r_{ij}^L=1,\quad \forall i,j=1,2,\ldots,n,\ i\neq j,$$
$$r_{ii}^U=r_{ii}^L=0.5,\ i=j \tag{23}$$

where $r_{ij}^U\geqslant r_{ij}^L\geqslant 0$, $\tilde{r}_{ij}$ indicates the interval-valued preference degree of the alternative x_i over x_j, r_{ij}^L and r_{ij}^U are the lower and upper limits of $\tilde{r}_{ij}$, respectively.

Definition 5.3. Suppose that $\widetilde{R}^{(k)}=(\tilde{r}_{ij}^{(k)})_{n\times n}$ is the interval fuzzy preference relations provided by kth expert, $k = 1,2,\ldots,m$, Let $F_Q(\widetilde{R}^{(k)})=(F_Q(\tilde{r}_{ij}^{(k)}))_{n\times n}$, matrix $F_Q(\widetilde{R}^{(k)})$ is called the expected value fuzzy preference relations corresponding to the interval fuzzy preference relations $\widetilde{R}^{(k)}$, where

$$F_Q(\tilde{r}_{ij}^{(k)})=(1-\lambda)r_{ij}^L+\lambda r_{ij}^U,\quad \text{for } i\leqslant j,$$
$$F_Q(\tilde{r}_{ij}^{(k)})=1-F_Q(\tilde{r}_{ji}^{(k)}),\ \text{for } i>j, \tag{24}$$

Definition 5.3 indicates that the expected values of fuzzy preference relations are obtained by aggregating the interval fuzzy preference relations using the COWA operator.

It is obvious that the expected value fuzzy preference relations $F_Q(\widetilde{R}^{(k)})=(F_Q(\tilde{r}_{ij}^{(k)}))_{n\times n}$ are complementary matrices by Eqs. (24) and (22).

Herrera-Viedma, Herrera, Chiclana, and Luque (2004) presented the additive transitivity property of the fuzzy preference relations to define a new characterization of the consistency property, which is used to construct consistent fuzzy preference relations from a set of $n-1$ preference data. They proved the following the theorem:

Theorem 5.1. *A fuzzy preference relation $R = (r_{ij})_{n\times n}$ is consistent if and only if*

$$r_{ij} + r_{jk} + r_{ki} = \frac{3}{2}, \quad \forall i < j < k \tag{25}$$

and Eq. (25) is equivalent to the following Eq. (26)

$$r_{i(i+1)} + r_{(i+1)(i+2)} + \cdots + r_{(j-1)j} + r_{ji} = \frac{j-i+1}{2}, \quad \forall i < j, \tag{26}$$

Using this characterization method, Chiclana, Herrera-Viedma, Herrera, and Alonso (2007) gave a procedure to construct a consistent fuzzy preference relation from a non-consistent fuzzy preference relation. Suppose that a set of $n-1$ preference data of fuzzy preference relations $r_{12}, r_{23}, \ldots, r_{(n-1)n}$ is given, let

$$\hat{r}_{ij} = \begin{cases} r_{ij} & \text{if } i \leqslant j \leqslant i+1, \\ (r_{i(i+1)} + r_{(i+1)(i+2)} + \cdots + r_{(j-1)j}) - \frac{j-(i+1)}{2} & \text{if } j > i+1, \\ 1 - \hat{r}_{ij} & \text{if } j < i. \end{cases} \tag{27}$$

Then $\widehat{R} = (\hat{r}_{ij})_{n\times n}$ is a consistent fuzzy preference relation by the Theorem 5.1.

Let $F_Q(\widetilde{R}^{(k)}) = \left(F_Q\left(\tilde{r}_{ij}^{(k)}\right)\right)_{n\times n}$ be the expected value fuzzy preference relations corresponding to the interval fuzzy preference relations $\widetilde{R}^{(k)}$, we use Eq. (27) to construct the consistent fuzzy preference relation corresponding to $F_Q(\widetilde{R}^{(k)})$, which is denoted as $\widehat{F}_Q(\widetilde{R}^{(k)}) = (\widehat{F}_Q(\tilde{r}_{ij}^{(k)}))_{n\times n}$, where

$$\widehat{F}_Q(\tilde{r}_{ij}^{(k)}) = \begin{cases} F_Q(\tilde{r}_{ij}^{(k)}) & \text{if } i \leqslant j \leqslant i+1, \\ (F_Q(\tilde{r}_{i(i+1)}^{(k)}) + F_Q(\tilde{r}_{(i+1)(i+2)}^{(k)}) + \cdots + F_Q(\tilde{r}_{(j-1)j}^{(k)})) - \frac{j-(i+1)}{2} & \text{if } j > i+1, \\ 1 - \widehat{F}_Q(\tilde{r}_{ji}^{(k)}) & \text{if } j < i. \end{cases} \tag{28}$$

Chiclana et al. (2007) proposed the measure of the consistency as the order inducing variable because it can reflect the reliability of the information sources provided by the each expert in the group decision making. Motivated by this idea, we propose the consistency induced generalized continuous ordered weighted averaging (C-IGCOWA) operator to aggregate interval fuzzy preference relations.

First, we introduce the consistency indicator of the interval fuzzy preference relations which reflects the reliability of the information sources provided by the each expert in the group decision making.

Definition 5.4. Let

$$CI^{(k)} = 1 - \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}(F_Q(\tilde{r}_{ij}^{(k)}) - \widehat{F}_Q(\tilde{r}_{ij}^{(k)}))^2}, \quad k = 1,2,\ldots,m \tag{29}$$

We call $CI^{(k)}$ the consistency indicator of the interval fuzzy preference relations $\widetilde{R}^{(k)}$ given by the kth expert.

The closer $CI^{(k)}$ is to 1, the more consistent and reliable the information provided by the kth expert is. That is, the interval fuzzy preference relations with the more consistent indicator should be more important. Therefore, we can use the consistency indicator $CI^{(k)}$ as the order-inducing variables of the interval fuzzy preference relations to be aggregated in the process of group decision making.

Definition 5.5. An consistency induced generalized continuous ordered weighted averaging (C-IGCOWA) operator can be defined as follows

$$\begin{aligned} &C-IGCOWA_w(\langle CI^{(1)}, \tilde{r}_{ij}^{(1)}\rangle, \langle CI^{(2)}, \tilde{r}_{ij}^{(2)}\rangle, \ldots, \langle CI^{(m)}, \tilde{r}_{ij}^{(m)}\rangle) \\ &= IGOWA_w(\langle CI^{(1)}, F_Q(\tilde{r}_{ij}^{(1)})\rangle, \langle CI^{(2)}, F_Q(\tilde{r}_{ij}^{(2)})\rangle, \ldots, \langle CI^{(m)}, F_Q(\tilde{r}_{ij}^{(m)})\rangle) \\ &= \left(\sum_{k=1}^{m} w_k (F_Q(\tilde{r}_{ij}^{(\sigma(k))}))^{\gamma}\right)^{1/\gamma} \end{aligned} \tag{30}$$

where $\sigma : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ is a permutation such that $CI^{(\sigma(j))} \geqslant CI^{(\sigma(j+1))}$, γ is a parameter controlling the power of the argument values satisfying $\gamma \in (-\infty, +\infty)$ and $\gamma \neq 0$, and $\tilde{r}_{ij}^{(k)}$ indicates the interval-valued preference degree of the alternative x_i over x_j provided by kth expert, $k = 1,2,\ldots,m$, $F_Q(\tilde{r}_{ij}^{(\sigma(k))})$ is calculated by COWA operator.

According to the principle that the interval fuzzy preference relation is more important with the larger consistent indicator, the weighting vector $w = (w_1, w_2, \ldots, w_n)$ should be given by the following formula:

$$w_k = \frac{CI^{\sigma(k)}}{\sum_{j=1}^{m} CI^{\sigma(j)}}, \quad k = 1,2,\ldots,m \tag{31}$$

Definition 5.6. Let

$$\begin{cases} a_{ij} = C-IGCOWA_w(\langle CI^{(1)}, \tilde{r}_{ij}^{(1)}\rangle, \langle CI^{(2)}, \tilde{r}_{ij}^{(2)}\rangle, \ldots, \langle CI^{(m)}, \tilde{r}_{ij}^{(m)}\rangle) & \text{for } i \leqslant j, \\ a_{ij} = 1 - a_{ji} & \text{for } i > j. \end{cases} \tag{32}$$

Then matrix $A = (a_{ij})_{n\times n}$ is defined as the collective fuzzy preference relations of interval fuzzy preference relations $\widetilde{R}^{(k)}$, $k = 1,2,\ldots,m$ based on C-IGCOWA operator.

It is obvious that the collective fuzzy preference relations $A = (a_{ij})_{n\times n}$ is complementary by Eq. (32), i.e.,

$$a_{ij} + a_{ji} = 1, \text{ for } i,j = 1,2,\ldots,n.$$

Xu (2001) proposed a simple and practical formula for deriving the priority vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ from a fuzzy preference relation $A = (a_{ij})_{n\times n}$, where

$$\omega_i = \frac{\sum_{j=1}^{n} a_{ij} + \frac{n}{2} - 1}{n(n-1)}, \quad i = 1,2,\ldots,n \tag{33}$$

We can use the priority vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ to get the optimal solution set of alternatives of the group decision making problems, that is

$$X^* = \{x_{i^*} | x_{i^*} \in X, \omega_{i^*} = max_{1\leqslant i \leqslant n}\omega_i\} \tag{34}$$

5.2. A selection process for group decision making problems based on C-IGCOWA operator

Chiclana, Herrera, and Herrera-Viedma (1998) proposed two phases to choose the scheme for group decision making problems, which include aggregation and exploitation. In this section, we will propose a selection process for group decision making (GDM) problems based on C-IGCOWA operator, which is expressed as follows in steps.

Step 1. Suppose that $X = (x_1, x_2, \ldots, x_n)$ is a finite set of alternatives, $E = (e_1, e_2, \ldots, e_m)$ is a finite set of experts. Expert e_k provides his/her interval fuzzy preference relations $\widetilde{R}^{(k)} = (\tilde{r}_{ij}^{(k)})_{n\times n}$, $k = 1, 2, \ldots, m$, where $\tilde{r}_{ij}^{(k)}$ satisfies Eq. (23).

Step 2. Calculating the expected value fuzzy preference relations $F_Q(\widetilde{R}^{(k)})$ corresponding to the interval fuzzy preference relations $\widetilde{R}^{(k)}$ according to Eq. (24), where the BUM function $Q(y)$ is given.

Step 3. Constructing the consistent fuzzy preference relation $\widehat{F}_Q(\widetilde{R}^{(k)})$ corresponding to $F_Q(\widetilde{R}^{(k)})$ by Eq. (28).

Step 4. Calculating the consistency indicator $CI^{(k)}$of the interval fuzzy preference relations $\widetilde{R}^{(k)}$ by Eq. (29).

Step 5. Calculating the weighting vector $w = (w_1, w_2, \ldots, w_n)$ associated with the C-IGCOWA operator by Eq. (31).

Step 6. Utilizing the C-IGCOWA operator to aggregate the interval fuzzy preference relations $\widetilde{R}^{(k)}$ by Eq. (30), $k = 1, 2, \ldots, m$, and obtaining the collective fuzzy preference relations matrix $A = (a_{ij})_{n\times n}$ by Eq. (32).

Step 7. Calculating the priority vector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ of the collective fuzzy preference relations$A = (a_{ij})_{n\times n}$ by Eq. (33) and obtaining the best solution set of alternatives of the GDM problem according to Eq. (34).

6. Illustrative Example

Xu (2004) gave an example of a group decision making problem involves the evaluation of four schools of a university, which is used to show the result that the synthetic interval fuzzy preference relation is of acceptable compatibility under the condition that the each of the interval fuzzy preference relations is of acceptable compatibility. However, the evaluation result of four schools of a university was not given. In this section, we use these arguments and the C-IGCOWA operator to rank the four schools of a university.

There are four decision makers $E = (e_1, e_2, e_3, e_4)$. Each decision maker compares these four schools $S = (s_1, s_2, s_3, s_4)$ with respect to their research abilities, and constructs interval fuzzy preference relation. Suppose that the interval fuzzy preference relations $\widetilde{R}^{(k)},\ k = 1, 2, 3, 4$ are given by the four decision makers respectively. They are listed as follows:

$$\widetilde{R}^{(1)} = \begin{bmatrix} [0.5, 0.5] & [0.5, 0.6] & [0.4, 0.7] & [0.4, 0.5] \\ [0.4, 0.5] & [0.5, 0.5] & [0.5, 0.6] & [0.3, 0.4] \\ [0.3, 0.6] & [0.4, 0.5] & [0.5, 0.5] & [0.6, 0.7] \\ [0.5, 0.6] & [0.6, 0.7] & [0.3, 0.4] & [0.5, 0.5] \end{bmatrix}$$

$$\widetilde{R}^{(2)} = \begin{bmatrix} [0.5, 0.5] & [0.4, 0.5] & [0.5, 0.8] & [0.3, 0.5] \\ [0.5, 0.6] & [0.5, 0.5] & [0.4, 0.6] & [0.4, 0.5] \\ [0.2, 0.5] & [0.4, 0.6] & [0.5, 0.5] & [0.6, 0.8] \\ [0.5, 0.7] & [0.5, 0.6] & [0.2, 0.4] & [0.5, 0.5] \end{bmatrix}$$

$$\widetilde{R}^{(3)} = \begin{bmatrix} [0.5, 0.5] & [0.5, 0.7] & [0.4, 0.7] & [0.3, 0.5] \\ [0.3, 0.5] & [0.5, 0.5] & [0.4, 0.6] & [0.4, 0.5] \\ [0.3, 0.6] & [0.4, 0.6] & [0.5, 0.5] & [0.5, 0.7] \\ [0.5, 0.7] & [0.5, 0.6] & [0.3, 0.5] & [0.5, 0.5] \end{bmatrix}$$

$$\widetilde{R}^{(4)} = \begin{bmatrix} [0.5, 0.5] & [0.4, 0.5] & [0.7, 0.8] & [0.4, 0.5] \\ [0.5, 0.6] & [0.5, 0.5] & [0.4, 0.5] & [0.5, 0.7] \\ [0.2, 0.3] & [0.5, 0.6] & [0.5, 0.5] & [0.7, 0.8] \\ [0.5, 0.6] & [0.3, 0.5] & [0.2, 0.3] & [0.5, 0.5] \end{bmatrix}$$

If the BUM function $Q(y) = y^2$, then $\lambda = \int_0^1 Q(y)dy = \int_0^1 y^2 dy = \frac{1}{3}$. Using Eq. (5), we have the expected value fuzzy preference relations $F_Q(\widetilde{R}^{(k)})$ corresponding to the interval fuzzy preference relations $\widetilde{R}^{(k)},\ k = 1, 2, 3, 4$

$$F_Q(\widetilde{R}^{(1)}) = \begin{bmatrix} 0.5 & 0.53333 & 0.5 & 0.43333 \\ 0.46667 & 0.5 & 0.53333 & 0.33333 \\ 0.5 & 0.46667 & 0.5 & 0.63333 \\ 0.56667 & 0.66667 & 0.36667 & 0.5 \end{bmatrix}$$

$$F_Q(\widetilde{R}^{(2)}) = \begin{bmatrix} 0.5 & 0.43333 & 0.6 & 0.36667 \\ 0.56667 & 0.5 & 0.46667 & 0.43333 \\ 0.4 & 0.53333 & 0.5 & 0.66667 \\ 0.63333 & 0.56667 & 0.33333 & 0.5 \end{bmatrix}$$

$$F_Q(\widetilde{R}^{(3)}) = \begin{bmatrix} 0.5 & 0.56667 & 0.5 & 0.36667 \\ 0.43333 & 0.5 & 0.46667 & 0.43333 \\ 0.5 & 0.53333 & 0.5 & 0.56667 \\ 0.63333 & 0.56667 & 0.43333 & 0.5 \end{bmatrix}$$

$$F_Q(\widetilde{R}^{(4)}) = \begin{bmatrix} 0.5 & 0.43333 & 0.73333 & 0.43333 \\ 0.56667 & 0.5 & 0.43333 & 0.56667 \\ 0.26667 & 0.56667 & 0.5 & 0.73333 \\ 0.56667 & 0.43333 & 0.26667 & 0.5 \end{bmatrix}$$

By Eq. (28), we obtain the consistent fuzzy preference relation $\widehat{F}_Q(\widetilde{R}^{(k)})$ corresponding to $F_Q(\widetilde{R}^{(k)}),\ k = 1, 2, 3, 4$:

$$\widehat{F}_Q(\widetilde{R}^{(1)}) = \begin{bmatrix} 0.5 & 0.53333 & 0.56666 & 0.69999 \\ 0.46667 & 0.5 & 0.53333 & 0.66666 \\ 0.43334 & 0.46667 & 0.5 & 0.63333 \\ 0.30001 & 0.33334 & 0.36667 & 0.5 \end{bmatrix}$$

$$\widehat{F}_Q(\widetilde{R}^{(2)}) = \begin{bmatrix} 0.5 & 0.43333 & 0.4 & 0.56667 \\ 0.56667 & 0.5 & 0.46667 & 0.63334 \\ 0.6 & 0.53333 & 0.5 & 0.66667 \\ 0.43333 & 0.36666 & 0.33333 & 0.5 \end{bmatrix}$$

$$\widehat{F}_Q(\widetilde{R}^{(3)}) = \begin{bmatrix} 0.5 & 0.56667 & 0.53334 & 0.60001 \\ 0.43333 & 0.5 & 0.46667 & 0.53334 \\ 0.46666 & 0.53333 & 0.5 & 0.56667 \\ 0.39999 & 0.46666 & 0.43333 & 0.5 \end{bmatrix}$$

$$\widehat{F}_Q(\widetilde{R}^{(4)}) = \begin{bmatrix} 0.5 & 0.43333 & 0.36666 & 0.59999 \\ 0.5 & 6667 & 0.50.43333 & 0.66666 \\ 0.6 & 3334 & 0.566670.5 & 0.73333 \\ 0.4 & 0001 & 0.43333 & 0.333340.5 \end{bmatrix}$$

Then we calculate the consistency indicator $CI^{(k)}$ by Eq. (29), $k = 1, 2, 3, 4$,

$$CI^{(1)} = 1 - \sqrt{\sum_{i=1}^{4}\sum_{j=1}^{4}(F_Q(\tilde{r}_{ij}^{(1)}) - \widehat{F}_Q(\tilde{r}_{ij}^{(1)}))^2} = 0.38900;$$

Similarly,

$$CI^{(2)} = 0.51009; \quad CI^{(3)} = 0.63789; \quad CI^{(4)} = 0.41311$$

By Eq. (31), we get weighting vector:

$$w_1 = \frac{CI^{(1)}}{\sum_{k=1}^{4} CI^{(k)}} = 0.19948$$

Similarly,

$$w_2 = 0.26157; \quad w_3 = 0.32711; \quad w_4 = 0.21184$$

Let parameter $\gamma = 0.5$. By Eq. (30), we utilize the C-IGCOWA operator to aggregate the interval fuzzy preference relations $\widetilde{R}^{(k)}$, for $i \leqslant j$, e.g., $i = 1$, $j = 2$,

$$a_{12} = C-IGCOWA_w(\langle CI^{(1)}, \tilde{r}_{12}^{(1)}\rangle, \langle CI^{(2)}, \tilde{r}_{12}^{(2)}\rangle, \langle CI^{(3)}, \tilde{r}_{12}^{(3)}\rangle, \langle CI^{(4)}, \tilde{r}_{12}^{(4)}\rangle),$$

$$= \left(\sum_{k=1}^{4} w_j (F_Q(\tilde{r}_{ij}^{(\sigma(k))}))^{0.5}\right)^{1/0.5}$$

$$= (0.19948 \times \sqrt{0.53333} + 0.26157 \times \sqrt{0.43333} + 0.32711 \times \sqrt{0.56667} + 0.21184 \times \sqrt{0.43333})^2$$

$$= 0.49499$$

Similar to the calculation of a_{12}, we can get the collective fuzzy preference relations matrix $A = (a_{ij})_{4\times 4}$ by Eq. (32), i.e.,

$$A = \begin{bmatrix} 0.5 & 0.49499 & 0.5721 & 0.39341 \\ 0.50122 & 0.5 & 0.47235 & 0.43845 \\ 0.41884 & 0.52656 & 0.5 & 0.63995 \\ 0.60546 & 0.55572 & 0.35585 & 0.5 \end{bmatrix}$$

Finally, we can calculate the priority vector $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)$ by Eq. (33) as following

$$\omega_1 = 0.24671; \quad \omega_2 = 0.24267; \quad \omega_3 = 0.25711; \quad \omega_4 = 0.25142$$

Since $\omega_3 = \max_i \omega_i$, then the best school is s_3.

7. Conclusion

In this paper, we have introduced the IGCOWA operator, which can be viewed as an extension of the GOWA operator and ICOWA operator.With the IGCOWA operator, we have been able to obtain a wide range of the ICOWA operator, the ICOWG operator, and the GOWA operator. We also have studied some desirable properties of the IGCOWA operator. Moreover, we introduced the consistency induced generalized continuous ordered weighted averaging (C-IGCOWA) operator and presented a numerical example of the new approaches in the decision making problems under the condition that the interval fuzzy preference relations are given by the experts.

The result shows that the model is effective and feasible.

Acknowledgements

This work was supported by National Natural Science Foundation of China (71071002), Academic Innovation Team of Anhui University (KJTD001B, SKTD007B), Natural Science Foundation of Anhui Provincial Higher School (KJ2010B094) and Foundation for the Young Scholar of Anhui University (2009QN022B).

References

Calvo, T., Mayor, G., & Mesiar, R. (2002). *Aggregation operators: New trends and applications*. New York: Physica-Verlag.

Chen, H. Y., Liu, C. L., & Sheng, Z. H. (2004). Induced ordered weighted harmonic averaging (IOWHA) operator and its application to combination forecasting method. *Chinese Journal of Management Science, 12*, 35–40.

Chen, H. Y., Liu, J. P., & Wang, H. (2008). A class of continuous ordered weighted harmonic (COWH) averaging operators for interval argument and its applications. *Systems Engineering Theory & Practice, 28*, 86–92.

Chiclana, F., Herrera, F., & Herrera-Viedma, E. (1998). Integrating three representation models in fuzzy multipurpose decision making based on fuzzy preference relations. *Fuzzy Sets and Systems, 97*, 33–48.

Chiclana, F., Herrera, F., & Herrera-Viedma, E. (2001). Integrating multiplicative preference relations in a multipurpose decision-making based on fuzzy preference relations. *Fuzzy Sets and Systems, 122*, 277–291.

Chiclana, F., Herrera-Viedma, E., Herrera, F., & Alonso, S. (2007). Some induced ordered weighted averaging operators and their use for solving group decision making problems based on fuzzy preference relations. *European Journal of Operational Research, 182*, 383–399.

Gong, Y. B., & Liang, X. C. (2008). Fuzzy multi-attribute decision making method based on fuzzy COWA operator. *Systems Engineering & Electronics, 30*, 1478–1480.

Herrera, F., Martínez, L., & Sánchez, P. J. (2005). Managing non-homogeneous information in group decision making. *European Journal of Operational Research, 166*, 115–132.

Herrera-Viedma, E., Herrera, F., Chiclana, F., & Luque, M. (2004). Some issues on consistency of fuzzy preference relations. *European Journal of Operational Research, 154*, 98–109.

Kacprzyk, J., & Zadrozny, S. (2001). Computing with words in intelligent database querying: standalone and internet-based applications. *Information Sciences, 134*, 71–109.

Liu, X. (2006). Some properties of the weighted OWA operator. *IEEE Transactions on Systems, Man and Cybernetics, Part B, 36*, 118–127.

Merigó, J. M., & Gil-Lafuente, A. M. (2009). The induced generalized OWA operator. *Information Sciences, 179*, 729–741.

Mitchell, H. B., & Estrakh, D. D. (1997). A modified OWA operator and its use in Lossless DPCM image compression. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 5*, 429–436.

Wu, J., Li, J. C., Li, H., & Duan, W. Q. (2009). The induced continuous ordered weighted geometric operators and their application in group decision making. *Computers & Industrial Engineering, 56*, 1545–1552.

Xu, Z. S. (2001). Algorithm for priority of fuzzy complementary judgment matrix. *Journal of Systems Engineering, 16*, 311–314.

Xu, Z. S. (2002). A practical method for priority of interval number complementary judgment matrix. *Operations Research and Management Science, 10*, 16–19.

Xu, Z. S. (2004). On compatibility of interval fuzzy preference matrices. *Fuzzy Optimization and Decision Making, 3*, 217–225.

Xu, Z. S. (2005). Extended COWA operators and their use in uncertain multi-attribute decision making. *Systems Engineering Theory & Practice, 25*, 7–13.

Xu, Z. S. (2006a). A survey of preference relations. *International Journal of General Systems, 36*, 179–203.

Xu, Z. S. (2006b). Induced uncertain linguistic OWA operators applied to group decision making. *Information Fusion, 7*, 231–238.

Xu, Z. S. (2006c). An approach based on the uncertain LOWG and induced uncertain LOWG operators to group decision making with uncertain multiplicative linguistic preference relations. *Decision Support Systems, 41*, 488–499.

Xu, Z. S. (2006d). On generalized induced linguistic aggregation operators. *International Journal of General Systems, 35*, 17–28.

Xu, Z. S. (2006e). A C-OWA operator-based approach to decision making with interval fuzzy preference relation. *International Journal of Intelligent Systems, 21*, 1289–1298.

Xu, Z. S., & Da, Q. L. (2002). The ordered weighted geometric averaging operators, International. *Journal of Intelligent Systems, 17*, 709–716.

Xu, Z. S., & Da, Q. L. (2003). An overview of operators for aggregating information. *Journal of Intelligent Systems, 18*, 953–969.

Yager, R. R. (1988). On ordered weighted averaging aggregation operators in multicriteria decision making. *IEEE Transactions on Systems, Man and Cybernetics, 18*, 183–190.

Yager, R. R. (2003). Induced aggregation operators. *Fuzzy Sets and Systems, 137*, 59–69.

Yager, R. R. (2004a). OWA aggregation over a continuous interval argument with applications to decision making. *IEEE Transactions on Systems, Man and Cybernetics, Part B, 34*, 1952–1963.

Yager, R. R. (2004b). Generalized OWA aggregation operators. *Fuzzy Optimization and Decision Making, 2*, 93–107.

Yager, R. R., & Filev, D. P. (1999). Induced ordered weighted averaging operators. *IEEE Transactions on Systems, Man and Cybernetics, 29*, 141–150.

Yager, R. R., & Kacprzyk, J. (1997). *The ordered weighted averaging operators: Theory and applications*. MA: Kluwer.

Yager, R. R., & Xu, Z. S. (2006). The continuous ordered weighted geometric operator and its application to decision making. *Fuzzy Sets and Systems, 157*, 1393–1402.