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Some new results on planar harmonic mappings

Dedicated to Professor Yang Lo on the Occasion of his 70th Birthday

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Abstract This survey paper contains some new results on the Landau theorem, Bloch theorem and Schwarz-Pick lemma for planar harmonic mappings.

Keywords harmonic mapping, Landau theorem, Bloch constant, Schwarz-Pick lemma, quasi-regularity

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1 Introduction

A planar harmonic mapping is a complex-valued harmonic function defined on a domain in the complex plane. Harmonic mappings have interesting links with the geometric function theory, minimal surfaces and locally quasiconformal mappings. For a survey of harmonic mappings in the plane, see [3].

For a continuously differentiable function $f(z) = u(z) + iv(z)$, $z = x + iy$, we use the common notations for its formal derivatives:

$$f_z = \frac{1}{2}(f_x - if_y), \quad f_{\overline{z}} = \frac{1}{2}(f_x + if_y);$$

then f is a harmonic mapping if and only if f is twice continuously differentiable and $\Delta f = 4f_{z\overline{z}} = 0$. Let f be a harmonic mapping of a domain G. f is said to be univalent or locally univalent, if f is one-to-one or locally one-to-one on G. If G is simply connected, then f can be written as $f = \overline{g} + h$, where g and h are holomorphic on G. Since $f_z = h'$ and $f_{\overline{z}} = \overline{g'}$, we have

$$J_f = u_x v_y - u_y v_x = |f_z|^2 - |f_{\overline{z}}|^2 = |h'|^2 - |g'|^2.$$

For a continuously differentiable function f, denote

$$\Lambda_f = \max_{0\leqslant\theta\leqslant2\pi} |f_z + e^{-2i\theta} f_{\overline{z}}| = |f_z| + |f_{\overline{z}}|,$$
$$\lambda_f = \min_{0\leqslant\theta\leqslant2\pi} |f_z + e^{-2i\theta} f_{\overline{z}}| = ||f_z| - |f_{\overline{z}}||.$$

Note that $|J_f| = \Lambda_f \lambda_f$. A harmonic mapping f is said to be K-quasiregular ($K \geqslant 1$) on a domain G, if $\Lambda_f \leqslant K\lambda_f$, or equivalently $\Lambda_f^2 \leqslant K|J_f|$ or $\lambda_f^2 \geqslant |J_f|/K$, holds everywhere on G.

We denote the unit disc $\{z : |z| < 1\}$ by D and a disc with center at the origin and radius r by D_r. A harmonic mapping $f = \overline{g} + h$ of the unit disc can be expanded in a series

$$f(re^{i\theta}) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\theta}, \quad 0 \leqslant r < 1,$$

where $g(z) = \sum_{n=1}^{\infty} \overline{c}_{-n} z^n$, $h(z) = \sum_{n=0}^{\infty} c_n z^n$. We call $c_n = c_n(f)$ the coefficients of f. The class of univalent sense preserving harmonic mappings on D normalized by $c_0 = 0$ and $c_1 = 1$ will be denoted by S_H. S_H^0 will denote the subclass with $c_{-1} = 0$. Clunie and Sheil-Small [7] proved a distortion theorem and a Koebe theorem for S_H and S_H^0: If $f \in S_H$, then

$$|f(z)| \geqslant \frac{1}{4}(1 - |c_{-1}(f)|)\frac{|z|}{(1+|z|)^2}, \quad \text{for } z \in D,$$

and, in particular, $D_{R_0} \subset f(D)$ with

$$R_0 = \frac{1}{16}(1 - |c_{-1}(f)|).$$

If $f \in S_H^0$, then $c_{-1}(f) = 0$ and the factor $1 - |c_{-1}(f)|$ in the above will be canceled.

In this survey paper, we introduce some new results on planar harmonic mappings, which generalize the classical Landau theorem, Bloch theorem, and Schwarz-Pick lemma for holomorphic mappings to the harmonic case.

2 The Landau theorem for harmonic mappings

First, we consider bounded harmonic mappings of the unit disc. The following Schwarz lemma for harmonic mappings is known and plays a key role in the proofs of our results. For convenience of readers, we give the proof.

Theorem 1. *Let f be a harmonic mapping of the unit disc D such that $f(0) = 0$ and $f(D) \subset D$. Then*

$$|f(z)| \leqslant \frac{4}{\pi}\arctan|z| \leqslant \frac{4}{\pi}|z|, \quad \textit{for } z \in D. \tag{2.1}$$

As a consequence,

$$\Lambda_f(0) \leqslant \frac{4}{\pi}, \quad \Lambda_f(z) \leqslant \frac{8}{\pi(1-|z|^2)}, \quad \textit{for } z \in D. \tag{2.2}$$

Proof. For $0 \leqslant \theta \leqslant 2\pi$, let $u_\theta = \Re\{e^{i\theta} f\}$ and v_θ be the harmonic conjugate of u_θ with $v_\theta(0) = 0$. Then $F_\theta = u_\theta + iv_\theta$ is a holomorphic function such that $F_\theta(0) = 0$ and $F_\theta(D)$ is contained in the strip $|\Re w| < 1$. A conformal mapping F of D onto the strip with $F(0) = 0$ is written by

$$F(z) = \frac{2i}{\pi}\log\frac{1+z}{1-z}.$$

By the subordination principle, we have $F_\theta(D_r) \subset F(D_r)$ for $r < 1$. This shows that

$$\Re\{e^{i\theta} f(z)\} = \Re\{F_\theta(z)\} \leqslant \frac{4}{\pi}\arctan|z|, \quad \text{for } z \in D.$$

(2.1) is proved now, since θ may be arbitrary. The first one of (2.2) is just a direct consequence of (2.1). For a fixed $z' \in D$, applying the first one of (2.2) to the function $(f((z+z')/(1+\overline{z}'z)) - f(z'))/2$, $z \in D$, we obtain the second one of (2.2) with $z = z'$. The theorem is proved.

We remark that item (2.1) in the above theorem was shown (using the same argument) by Heinz [10].

Recall that the classical Laudau theorem for holomorphic mappings says: Let f be a holomorphic function on the unit disc D with $f(0) = 0$ and $f'(0) = 1$. If $|f(z)| < M$ for $z \in D$, then f is univalent on D_{ρ_0} with $1/(2M) < \rho_0 = (M + \sqrt{M^2-1})^{-1} \leqslant 1/M$, and $f(D\rho_0)$ covers a disc D_{R_0} with $1/(4M) < R_0 = M\left(M + \sqrt{M^2-1}\right)^{-2} \leqslant 1/M$. Moreover, this result is sharp.

The Landau theorem for harmonic mappings was established first by Chen, Gauthier and Hengartner [6], and improved by Dorff and Nowak [8], Grigoryan [9], Huang [11] and Liu [14] later. The following two theorems were proved in [6].

Theorem 2. *Let f be a harmonic mapping of the unit disc D such that $f(0) = 0$, $J_f(0) = 1$ and $|f(z)| < M$ for $z \in D$. Then, f is univalent on a disc D_{ρ_0} with $\rho_0 = 64mM^2/\pi^3$ and $f(D_{\rho_0})$ contains a disc D_{R_0} with $R_0 = \pi\rho_0/(8M) = \pi^4/(512mM^3)$, where $m \approx 6.85$ is the minimum of the function $(3-r^2)/(r(1-r^2))$ for $0 < r < 1$.*

Proof. The function $(3-r^2)/(r(1-r^2))$, $0 < r < 1$, attains its minimum $m \approx 6.85$ at $r_0 \approx 0.63$. For $0 \leqslant \theta \leqslant 2\pi$, the function $\phi_\theta(z) = f_z(z) - f_z(0) + (f_{\overline{z}}(z) - f_{\overline{z}}(0))e^{-2i\theta}$, $z \in D$, is harmonic and satisfies

$$\phi_\theta(0) = 0, \quad |\phi_\theta(z)| \leqslant \Lambda_f(z) + \Lambda_f(0) \leqslant \frac{4M}{\pi}\left(1 + \frac{2}{1-|z|^2}\right) = \frac{4M(3-|z|^2)}{\pi(1-|z|^2)}$$

for $z \in D$, since, by (1.2), $\Lambda_f(0) \leqslant 4M/\pi$ and $\Lambda_f(z) \leqslant 8M/(\pi(1-|z|^2))$ for $z \in D$. In particular,

$$|\phi_\theta(z)| \leqslant m_1 = \frac{4M(3-r_0^2)}{\pi(1-r_0^2)}, \quad \text{for } z \in D_{r_0}.$$

Now, we apply (1.2) to the function $\phi_\theta(r_0 z)/m_1$ and obtain

$$|\phi_\theta(z)| \leqslant \frac{16M|z|}{\pi^2} \cdot \frac{(3-r_0^2)}{r_0(1-r_0^2)} = \frac{16mM|z|}{\pi^2}, \quad \text{for } z \in D_{r_0}.$$

Note that $1 = J_f(0) \leqslant \Lambda_f(0)^2 \leqslant 16M^2/\pi^2$, so $\rho_0 \leqslant \pi/(4m) < r_0$. Thus, $|\phi_\theta(z)| \leqslant 16mM|z|/\pi^2$ for $z \in D_{\rho_0}$.

To prove the univalence, let $z_1, z_2 \in D_{\rho_0}$ be two distinct points, and $z_2 - z_1 = |z_2 - z_1|e^{i\theta}$. We have

$$\begin{aligned} |f(z_2) - f(z_1)| &= \left|\int_{[z_1,z_2]} f_z(z)dz + f_{\overline{z}}(z)\overline{dz}\right| \\ &\geqslant \int_{[z_1,z_2]} |f_z(0) + f_{\overline{z}}(0)e^{-2i\theta}|ds - \int_{[z_1,z_2]} |\phi_\theta(z)|ds, \end{aligned}$$

$$\int_{[z_1,z_2]} |f_z(0) + f_{\overline{z}}(0)e^{-2i\theta}|ds \geqslant \lambda_f(0)|z_2 - z_1| = \frac{J_f(0)}{\Lambda_f(0)}|z_2 - z_1| \geqslant \frac{\pi}{4M}|z_1 - z_2|,$$

$$\int_{[z_1,z_2]} |\phi_\theta(z)|ds < \frac{16mM\rho_0}{\pi^2}|z_2 - z_1| = \frac{\pi}{4M}|z_1 - z_2|.$$

This shows $f(z_1) \neq f(z_2)$. In the same way, for any $z' = \rho_0 e^{i\theta} \in \partial D_{\rho_0}$, we have

$$\begin{aligned} |f(z')| &\geqslant \int_{[0,z']} |f_z(0) + f_{\overline{z}}(0)e^{-2i\theta}|ds - \int_{[0,z']} |\phi_\theta(z)|ds \\ &\geqslant \lambda_f(0)\rho_0 - \frac{16mM}{\pi^2}\int_0^{\rho_0} rdr \geqslant \frac{\pi}{4M}\rho_0 - \frac{8mM\rho_0^2}{\pi^2} = \frac{\pi}{8M}\rho_0. \end{aligned}$$

The theorem is proved.

If we replace the normalization $J_f(0) = 1$ in the above theorem by the stronger one that $f_{\overline{z}}(0) = 0$ and $f_z(0) = 0$, we may obtain a better conclusion.

Theorem 3. *Let f be a harmonic mapping of the unit disc D such that $f(0) = 0$, $f_{\overline{z}}(0) = 0$, $f_z(0) = 1$, and $|f(z)| < M$ for $z \in D$. Then, f is univalent on a disc D_{ρ_0} with $\rho_0 = \pi^2/(16mM)$, and $f(D_{\rho_0})$ contains a disc D_{R_0} with $R_0 = \rho_0/2 = \pi^2/(32mM)$, where $m \approx 6.85$ is the number defined above.*

Under the above normalization, the best value of ρ_0 is $2^{-1/2}\left(1 + 4.55M - \frac{1}{4.55M}\right)^{-1}$, which was obtained by Liu [14]. Using the Schwarz-Pick lemma for harmonic mappings obtained by the author recently, one can improve the result to $\rho_0 = 2^{-1/2}\left(1 + 8M/\pi - \frac{\pi}{8M}\right)^{-1}$.

Remark 1. The following example shows that the powers M^2 and M^3 in Theorem 2 are the best possible. For $M > 4$, define

$$f(z) = \frac{4}{M}x - \frac{M}{4}(x^2 - y^2) + \frac{M}{4}yi, \quad z = x + iy.$$

Then, $f(0) = 0$, $J_f(0) = 1$, and $|f(z)| < M$ for $z \in D$. It is easy to see that f is univalent for $x < 8/M^2$, but not univalent in any neighbourhood of a point z with $x = 8/M^2$. So, the largest disc D_{ρ_0}, in which f is univalent, has radius $\rho_0 \leqslant 8/M^2$ and the largest schlicht disc D_{R_0} centered at the origin and covered by $f(D)$ has radius $R_0 \leqslant 16/M^3$.

Remark 2. Comparing the Bloch theorem for holomorphic functions with Theorem 3, we see that the power of M in the lower bounds of ρ_0 and R_0 is the same for both the holomorphic case and harmonic case, and only the constants differ.

3 The Bloch theorem for *K*-quasiregular harmonic mappings

The Bloch theorem for holomorphic mappings asserts that there exists a positive absolute constant b such that a holomorphic function f normalized by $f(0) = 0$ and $f'(0) = 1$ maps the unit disk D to a covering surface $f(D)$ which contains a schlicht disc of radius b. By a schlicht disc, we mean a disc which is the biholomorphic image of some region in D. The Bloch constant is defined as the "best" such constant, that is, the supremum of such b. The best lower bound of the Bloch constant for holomorphic mappings was obtained by Chen and Gauthier [5].

It turns out that in the harmonic case, some extra assumption is required other than the normalization in order that a Bloch theorem holds. For K-quasiregular harmonic mappings (even in higher dimension), Bochner [2] had already proved the existence of a Bloch constant, but gave no estimate. In [6], the authors gave examples to show that for general harmonic mappings, and, in dimension larger than two, even for univalent harmonic mappings, there is no Bloch theorem. They estimated the Bloch constant for K-quasiregular harmonic mappings in the planar case and obtained a Bloch theorem for *open* planar harmonic mappings. This contains the classical Bloch theorem for general holomorphic functions.

In this section, we consider K-quasiregular harmonic mappings. The following two theorems were proved in [6]. Theorem 4 is a different version of the Landau theorem, in which the boundedness of the Fréchet derivative Λ_f is assumed, and is the basic tool of the proof of Theorem 5.

Theorem 4. *Let f be a harmonic mapping of the unit disc D such that $f(0) = 0$, $\lambda_f(0) = 1$ and $\Lambda_f(z) \leqslant \Lambda$ for $z \in D$. Then, f is univalent on a disc D_{ρ_0} with $\rho_0 = \pi/(4(1 + \Lambda))$, and $f(D_{\rho_0})$ contains a disc D_{R_0} with $R_0 = \rho_0/2$.*

Proof. Let $F(z) = \psi^{-1}(f(z))$ and $\psi(z) = f_z(0)z + f_{\overline{z}}(0)\overline{z}$. Then, $dF(0)$ is the identity mapping and ψ is an expansion, so $\lambda_F(0) = \Lambda_F(0) = 1$ and $\Lambda_F(z) \leqslant \Lambda_f(z) \leqslant \Lambda$ for $z \in D$. As in the proof of Theorem 2, for $0 \leqslant \theta \leqslant \pi$, we introduce the harmonic mappings

$$\phi_\theta(z) = F_z(z) - F_z(0) + (F_{\overline{z}}(z) - F_{\overline{z}}(0))e^{-2i\theta}, \quad z \in D.$$

We have $\phi_\theta(0) = 0$ and $|\phi_\theta(z)| \leqslant \Lambda_F(z) + \Lambda_F(0) \leqslant 1 + \Lambda$ for $z \in D$. Thus, by (1.1), $|\phi_\theta(z)| \leqslant \frac{4}{\pi}(1 + \Lambda)|z|$ for $z \in D$, and the inequality is strict for $z \neq 0$.

To prove the univalence, let z_1, z_2 be distinct points of D_{ρ_0} and $z_2 - z_1 = |z_2 - z_1|e^{i\theta}$. We have

$$\begin{aligned}
|F(z_2) - F(z_1)| &= \left| \int_{[z_1,z_2]} F_z(z)dz + F_{\overline{z}}(z)\overline{dz} \right| \\
&\geqslant \left| \int_{[z_1,z_2]} F_z(0)dz + F_{\overline{z}}(0)\overline{dz} \right| - \left| \int_{[z_1,z_2]} (F_z(z) - F_z(0))dz + (F_{\overline{z}}(z) - F_{\overline{z}}(0))\overline{dz} \right| \\
&= \int_{[z_1,z_2]} |F_z(0) + F_{\overline{z}}(0)e^{-2i\theta}|ds - \int_{[z_1,z_2]} |\phi_\theta(z)|ds,
\end{aligned}$$

$$\int_{[z_1,z_2]} |F_z(0) + F_{\overline{z}}(0)e^{-2i\theta}|ds \geqslant \lambda_F(0)|z_2 - z_1| = |z_2 - z_1|,$$

$$\int_{[z_1,z_2]} |\phi_\theta(z)|ds < \frac{4}{\pi}(1+\Lambda)\rho_0|z_2 - z_1| = |z_2 - z_1|.$$

This shows $F(z_1) \neq F(z_2)$. In the same way, for any $z' = \rho_0 e^{i\theta} \in \partial D_{\rho_0}$, we have

$$\begin{aligned}|F(z')| &\geqslant \int_{[0,z']} |F_z(0) + F_{\overline{z}}(0)e^{-2i\theta}|ds - \int_{[0,z']} |\phi_\theta(z)|ds \\ &> \lambda_F(0)\rho_0 - \frac{4}{\pi}(1+\Lambda)\int_0^{\rho_0} rdr = \rho_0 - \frac{4}{\pi}(1+\Lambda)\frac{\rho_0^2}{2} = \frac{\rho_0}{2}.\end{aligned}$$

Consequently, F is univalent on D_{ρ_0} and $F(D_{\rho_0})$ contains D_{R_0}. The conclusion of the theorem follows since ψ is an affine transformation with $\lambda_\psi = 1$.

Theorem 5. *Let f be a K-quasiregular harmonic mapping on the unit disc D such that $\lambda_f(0) = 1$. Then $f(D)$ contains a schlicht disc of radius at least $R_1 = \pi/(8\sqrt{2}(1+2K))$.*

Proof. Without loss of generality, we assume that f is harmonic on the boundary ∂D also. Then there exists a point $z_0 \in D$ such that $(1-|z|^2)\lambda_f(z) \leqslant (1-|z_0|^2)\lambda_f(z_0) = A \geqslant 1$ for $z \in D$. Let ϕ be the Möbius transformation of D onto itself with $\phi(0) = z_0$. Define $F(\zeta) = f(\phi(\zeta))/A$, for $\zeta \in D$. Since $(1-|\zeta|^2)\lambda_F(\zeta) = (1-|\phi(\zeta)|^2)\lambda_f(\phi(\zeta))/A$, we have $\lambda_F(0) = 1$ and $(1-|\zeta|^2)\lambda_F(\zeta) \leqslant 1$ for $z \in D$. Let $G(\omega) = \sqrt{2}F(\omega/\sqrt{2})$, for $\omega \in D$. Note that G is also K-quasiregular. Thus, $\lambda_G(0) = \lambda_F(0) = 1$ and $\Lambda_G(\omega) \leqslant K\lambda_G(\omega) = K\lambda_F(\omega/\sqrt{2}) < 2K$ for $\omega \in D$. Now, by the preceding theorem, we see that G is univalent on D_{ρ_0} with $\rho_0 = \pi/(4(1+2K))$, and $G(D_{\rho_0})$ contains a disc of radius $\pi/(8(1+2K))$. Consequently, $f(D)$ contains a schlicht disc of radius at least R_1. The theorem is proved.

Theorems 4 and 5 have been improved by Grigoryan [9] and Liu [14].

4 The Bloch theorem for open harmonic mappings

It is easy to give examples to show that neither the normalization $f_z(0) = 1$ nor the normalization $J_f(0) = 1$ yields a Bloch theorem for *general* univalent planar harmonic mappings. The following example shows that there is no Bloch theorem for harmonic mappings even with both of these normalizations, that is, $f_z(0) = 1$ and $f_{\overline{z}}(0) = 0$. Note that imposing both of these normalizations is equivalent to requiring that $df(0)$ be the identity mapping. Let $w = u + iv$ and $z = x + iy$. For $k > 1$, define $f_k(z)$ by

$$u = x, \quad v = \frac{1}{k}e^{ky}\sin kx.$$

It is easy to see that for any integer m, f_k maps the strip $(m-1)\pi/k < x < m\pi/k$ into the strip $(m-1)\pi/k < u < m\pi/k$ injectively, but maps the straight line $x = m\pi/k$ onto the single point $w = m\pi/k$. If we let $F_k(z) = f_k(\pi/(2k) + z)$, then $(F_k)_z(0) = 1$ and $(F_k)_{\overline{z}}(0) = 0$. It is obvious that $F_k(D)$ contains no disc with radius bigger than $\pi/(2k)$.

A mapping of a domain $G \subset \mathbb{C}$ is said to be open, if it maps any open subset of G to an open set in $\mathbb{C}$. A mapping is said to be light, if no continuum (the connected closed set containing more than one point) is mapped to a single point. A continuous mapping of a domain in the plane is called an interior transformation in the sense of Stoïlow, if f is both open and light. The famous Stoïlow theorem [17] says that an interior transformation f is topologically holomorphic, that is, f can be written in the form $f = F \circ \omega$, where F is a non-constant holomorphic mapping and ω is a homeomorphism. It is proved in [6] that for a harmonic mapping, the openness implies the lightness, that is, an open holomorphic mapping is an interior transformation in the sense of Stoïlow.

The mappings F_k in the above examples are neither open nor light. In [6], the authors proved a Bloch theorem for open planar harmonic mappings with the normalization $f_z(0) = 1$ and $f_{\overline{z}}(0) = 0$.

Theorem 6. *Let f be an open harmonic mapping of the unit disc D normalized by $f_z(0) = 1$ and $f_{\overline{z}}(0) = 0$. Then $f(D)$ contains a schlicht disc of radius at least $R_1 \approx 0.02$.*

Proof. According to the above lemma, f is light and, hence, f is an interior transformaton in the sense of Stoïlow. By the Stoïlow theorem, f can be written in the form $f = F \circ \omega$, where F is a non-constant holomorphic mapping F and ω is a homeomorphism. Let E be set of the zeros of F'. Then $E' = \omega^{-1}(E)$ is a closed discrete set in D and f is locally univalent in $D \setminus E'$. It is known [13] that a harmonic mapping is locally univalent if and only if its Jacobian J_f does not vanish anywhere. Now, because of the normalization and the continuity of J_f, J_f is positive and, consequently, $|f_z| > |f_{\overline{z}}|$ on $D \setminus E'$. Then $a(z) = \overline{f_{\overline{z}}(z)}/f_z(z)$ is a holomorpic function on $D \setminus E'$, and $|a(z)| < 1$ for $z \in D \setminus E'$. Hence, the singularities E' are removable and the strict inequality persists on D by the maximum principle. Since $a(0) = 0$ and $|a(z)| < 1$ for $z \in D$, by the classical Schwarz lemma, $|a(z)| < r$ for $z \in D_r$, $0 \leqslant r < 1$. So,

$$\frac{\Lambda_f}{\lambda_f} = \frac{|f_z| + |f_{\overline{z}}|}{|f_z| - |f_{\overline{z}}|} = \frac{1 + |a|}{1 - |a|} \leqslant \frac{1 + r}{1 - r} = K_r$$

holds everywhere on $D_r \setminus E'$. Because of the continuity, $\Lambda_f \leqslant K_r \lambda_f$ holds everywhere on D_r. This shows f is K_r-quasiregular on D_r.

For a fixed $r < 1$, applying Theorem 5 to the function $F(\zeta) = f(r\zeta)/r$, $\zeta \in D$, we see that $f(D_r)$ contains a schlicht disc with radius at least

$$\frac{\pi}{8\sqrt{2}} \cdot \frac{r}{1 + 2(1 + r)/(1 - r)},$$

which has maximum R_1 at $r = 2\sqrt{3} - 3$. This proves the theorem.

Grigoryan [9] and Liu [14] improved Theorem 6. Liu gave the number $R_1 = 0.0277$.

Remark 3. We see from the above proof that if a harmonic mapping is open, then it is topologically holomorphic. So, we may say that in the above Bloch theorem, people essentially put the assumption that the function is topologically holomorphic. If a harmonic mapping is open, then it is light as shown in [6]. On the other hand, there are light harmonic mappings which are not open. The function defined in Remark 1 in Section 2 is such a function. The functions F_k defined in this section, however, are not light. So, one may expect that lightness together with the same normalization also yields a Bloch theorem. This is an open question.

5 The Schwarz-Pick lemma for open harmonic mappings

The classical Schwarz-Pick lemma [1, 15, 16] is formulated as follows.

Schwarz-Pick lemma. *Let f be a holomorphic mapping such that $f(D) \subset D$. Then,*

$$\frac{|f(z_1) - f(z_2)|}{|1 - \overline{f(z_2)}f(z_1)|} \leqslant \frac{|z_1 - z_2|}{|1 - \overline{z_2}z_1|} \tag{4.1}$$

holds for $z_1, z_2 \in D$, and

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leqslant \frac{1}{1 - |z|^2} \tag{4.2}$$

holds for $z \in D$.

If we use the notations

$$d_p(z_1, z_2) = \frac{|z_1 - z_2|}{|1 - \overline{z_2}z_1|}$$

for the pseudo-distance between $z_1, z_2 \in D$, and $\Delta(z, r) = \{\zeta \in D : d_p(\zeta, z) < r\}$, $z \in D$ and $0 < r < 1$, for the pseudo-disk with center at $z \in D$ and pseudo-radius r, (4.1) may be written in the form

$$f(\Delta(z, r)) \subset \Delta(f(z), r). \tag{4.1$'$}$$

For a harmonic mapping F on the unit disk such that $F(D)=D$ and $F(0)=0$, it is known [10] that

$$|F(z)| \leqslant \frac{4}{\pi}\arctan|z| \tag{4.3}$$

holds for $z\in D$, and

$$\Lambda_f(0) \leqslant \frac{4}{\pi}. \tag{4.4}$$

Since the composition $F\circ f$ of a harmonic mapping F and a holomorphic mapping f is harmonic, if the condition $F(0)=0$ is replaced by $F(z)=0$ for some z, as a consequence of (4.3) and (1.4),

$$|F(\zeta)| \leqslant \frac{4}{\pi}\arctan d_p(\zeta,z) \tag{4.5}$$

holds for $\zeta\in D$, and

$$\Lambda_F(z) \leqslant \frac{4}{\pi}\frac{1}{1-|z|^2}. \tag{4.6}$$

Unfortunately, the composition $f\circ F$ of a harmonic mapping F and a holomorphic mapping f do not need to be harmonic, so it is a serious problem to seek the estimates corresponding to $(4.1)'$ and (4.2) for a harmonic mapping F without the assumption $F(z)=0$.

Recently, Chen gives a complete solution of the problem. (I) For any $0<r<1$ and $0<\rho<1$, Chen constructs a closed convex domain $E_{r,\rho}$, which contains ρ and is symmetry to the real axis, with the following properties: Let $z_0\in D$ and $w_0=\rho e^{i\alpha}$ be given. For every harmonic mapping F with $F(D)\subset D$ and $F(z_0)=w_0$, $F(\overline{\Delta}(z_0,r))\subset e^{i\alpha}E_{r,\rho}=\{e^{i\alpha}z: z\in E_{r,\rho}\}$ holds; conversely, for every $w'\in e^{i\alpha}E_{r,\rho}$, there exists a harmonic mapping F such that $F(D)\subset D$, $F(z_0)=w_0$ and $F(z')=w'$ for some $z'\in\partial\Delta(z_0,r)$. This is the Schwarz-Pick lemma for harmonic mappings corresponding to (4.1) or $(4.1)'$. (II) Chen establishes a Finsler metric $\mathcal{H}_z(u)$ on the unit disk D such that for any harmonic mapping F with $F(D)\subset D$,

$$\mathcal{H}_{F(z)}\left(e^{i\theta}F_z(z)+e^{-i\theta}F_{\overline{z}}(z)\right) \leqslant \frac{1}{1-|z|^2}$$

holds for $z\in D$ and $0\leqslant\theta\leqslant 2\pi$. Furthermore, some examples are given to show that the equality can be attained for any values of z, $F(z)$, θ and $\arg\left\{e^{i\theta}F_z(z)+e^{-i\theta}F_{\overline{z}}(z)\right\}$. This is the Schwarz-Pick lemma for harmonic mappings corresponding to (4.2). As a consequence,

$$\frac{\Lambda_F(z)}{h_{|F(z)|}(\pi/2)} \leqslant \frac{1}{1-|z|^2}$$

holds for $z\in D$, where $h_\rho(\pi/2)$ is decreasing from $4/\pi$ to 0 as ρ increasing from 0 to 1, and $h_\rho(\pi/2)\approx \sqrt{2}\sqrt{1-\rho^2}$ as $\rho\to 1$.

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