Journal of Number Theory 97, 186–198 (2002)
doi:10.1006/jnth.2002.2801

Generalizations of Knopp's Identity

Beifang Chen

Department of Mathematics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong
E-mail: mabfchen@ust.hk

and

Zhi-Wei Sun[1,2]

Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China
E-mail: zwsun@nju.edu.cn

Communicated by D. Goss

Received November 28, 2001; revised February 9, 2002

For integers a, b and $n > 0$ we define

$$S_\Gamma(a,b,n) = \sum_{\substack{r=0 \\ n \nmid br}}^{n-1} \left(\left(\frac{ar}{n}\right)\right) \ln \Gamma\left(\left\{\frac{br}{n}\right\}\right)$$

and

$$T_\Gamma(a,b,n) = \sum_{\substack{r=0 \\ n \nmid br}}^{n-1} \left(\left(\frac{ar}{n}\right)\right) \frac{\Gamma'(\{br/n\})}{\Gamma(\{br/n\})},$$

which are similar to the homogeneous Dedekind sum $S(a,b,n)$. In this paper we establish functional equations for S_Γ and T_Γ. Moreover, by means of uniform function (introduced by Sun in 1989) we are able to extend Knopp's identity on Dedekind sums vastly. © 2002 Elsevier Science (USA)

Key Words: Dedekind sum; Knopp identity; Gamma function.

[1] To whom correspondence should be addressed.
[2] Supported by the Teaching and Research Award Program for Outstanding Young Teachers in Higher Education Institutions of MOE, and the National Natural Science Foundation of the People's Republic of China.

0022-314X/02 $35.00

1. INTRODUCTION

For a real number x, we use $\{x\}$ to denote its fractional part, and define

$$((x)) = \begin{cases} \{x\} - 1/2 & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

Given $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, in 1892 R. Dedekind introduced the classical *Dedekind sum*

$$S(m, n) = \sum_{r=0}^{n-1} \left(\left(\frac{r}{n}\right)\right)\left(\left(\frac{mr}{n}\right)\right) \tag{1.2}$$

in his study of the functional equation of Dedekind's eta function

$$\eta(\tau) = e^{\pi i\tau/12} \prod_{k=1}^{\infty} (1 - e^{2\pi ik\tau}) \qquad (\tau \text{ is in the upper half plane}).$$

When $m, n \in \mathbb{Z}^+$ are relatively prime, Dedekind determined $S(m, n) + S(n, m)$ explicitly. The result is now known as the reciprocity law for Dedekind sums (see, e.g. [A, S5, Z]).

By means of η's functional equation and Hecke operators, in 1980 Knopp [K] established the following arithmetic identity:

$$\sum_{cd=m} \sum_{r=0}^{d-1} S(ac + rn, dn) = \sigma(m)S(a, n), \tag{1.3}$$

where $a \in \mathbb{Z}$, $m, n \in \mathbb{Z}^+$ and $\sigma(m)$ denotes the sum $\sum_{d|m} d$ of (positive) divisors of m. We can view this identity as a functional equation of Dedekind sums. For $a, b \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$, the sum

$$S(a, b, n) = \sum_{r=0}^{n-1} \left(\left(\frac{ar}{n}\right)\right)\left(\left(\frac{br}{n}\right)\right) \tag{1.4}$$

is called a *homogeneous Dedekind sum*. In 1996 Zheng [Zh] proved the following extension of Knopp's identity:

$$\sum_{cd=m} \sum_{r_1, r_2 \in d_*} S(ac + r_1 n, bc + r_2 n, dn) = m\sigma(m)S(a, b, n), \tag{1.5}$$

where $a, b \in \mathbb{Z}$, $m, n \in \mathbb{Z}^+$ and

$$d_* = \{r \in \mathbb{Z}\colon\ 0 \leqslant r < d\}.$$

In this paper we will make a further generalization.

DEFINITION 1.1. For a function F of two complex variables into the complex field $\mathbb{C}$, if for any ordered pair $\langle x, y \rangle$ in the domain $\mathrm{Dom}(F)$ of F we have

$$\left\{\left\langle \frac{x+r}{n}, ny \right\rangle\colon\ r \in n_*\right\} \subseteq \mathrm{Dom}(F) \quad \text{and} \quad \sum_{r=0}^{n-1} F\left(\frac{x+r}{n}, ny\right) = F(x, y) \quad (1.6)$$

for every $n = 1, 2, 3, \ldots$, then we call F a *uniform function* (into $\mathbb{C}$).

The concept of uniform function was first introduced by the second author in [S1] where he showed that, among functions F of two complex variables into $\mathbb{C}$ with $\{\langle \frac{x+r}{n}, ny \rangle\colon\ r \in n_*\} \subseteq \mathrm{Dom}(F)$ for all $\langle x, y \rangle \in \mathrm{Dom}(F)$ and $n \in \mathbb{Z}^+$, uniform functions are those F such that whenever

$$\sum_{\substack{1 \leqslant s \leqslant k \\ x \equiv a_s \pmod{n_s}}} \lambda_s = \sum_{\substack{1 \leqslant t \leqslant l \\ x \equiv b_t \pmod{m_t}}} \mu_t \qquad \text{for all } x \in \mathbb{Z}$$

(where $0 \leqslant a_s < n_s$, $0 \leqslant b_t < m_t$ and $\lambda_s, \mu_t \in \mathbb{C}$) we have

$$\sum_{s=1}^{k} \lambda_s F\left(\frac{x + a_s}{n_s}, n_s y\right) = \sum_{t=1}^{l} \mu_t F\left(\frac{x + b_t}{m_t}, m_t y\right) \quad (1.7)$$

for all $\langle x, y \rangle \in \mathrm{Dom}(F)$. See also [S3, Theorem 2.1; S4, Corollary 2].

A uniform function F is said to be *periodic* if

$$\langle x, y \rangle \in \mathrm{Dom}(F) \Rightarrow \langle x \pm 1, y \rangle \in \mathrm{Dom}(F) \ \&\ F(x \pm 1, y) = F(x, y). \quad (1.8)$$

We use PUF to denote the class of all periodic uniform functions. It is easy to see that the function D on $\mathbb{R} \times \mathbb{R}$ given by

$$D(x, y) = ((x)) \quad (1.9)$$

belongs to PUF where $\mathbb{R}$ is the field of real numbers. Generalizing the homogeneous Dedekind sums, we introduce the following definition.

DEFINITION 1.2. Let $F, G \in \mathrm{PUF}$, $\langle x, y\rangle \in \mathrm{Dom}(F)$ and $\langle u, v\rangle \in \mathrm{Dom}(G)$. For $a, b \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ we set

$$\begin{bmatrix} F; x, y \\ G; u, v \end{bmatrix}(a, b, n) = \sum_{r=0}^{n-1} F\Big(\frac{x+ar}{n}, ny\Big) G\Big(\frac{u+br}{n}, nv\Big). \tag{1.10}$$

Now we give our extension of Knopp's identity.

THEOREM 1.1. *Let* $a, b \in \mathbb{Z}$, $m, n \in \mathbb{Z}^+$, $F, G \in \mathrm{PUF}$, $\langle x, y\rangle \in \mathrm{Dom}(F)$ *and* $\langle u, v\rangle \in \mathrm{Dom}(G)$. *Then we have the identity*

$$\sum_{cd=m} \sum_{r_1, r_2 \in d_*} \begin{bmatrix} F; x, y \\ G; u, v \end{bmatrix}(ac + r_1 n, bc + r_2 n, dn) = m \sum_{d \mid m} d \begin{bmatrix} F; x/d, dy \\ G; u/d, dv \end{bmatrix}(a, b, n). \tag{1.11}$$

DEFINITION 1.3. For $a, b \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ we define

$$S_\Gamma(a, b, n) = \sum_{\substack{r=0 \\ n \nmid br}}^{n-1} \Big(\Big(\frac{ar}{n}\Big)\Big) \ln \Gamma\Big(\Big\{\frac{br}{n}\Big\}\Big) \tag{1.12}$$

and

$$T_\Gamma(a, b, n) = \sum_{\substack{r=0 \\ n \nmid br}}^{n-1} \Big(\Big(\frac{ar}{n}\Big)\Big) \frac{\Gamma'(\{br/n\})}{\Gamma(\{br/n\})}, \tag{1.13}$$

where $\Gamma(x)$ is the well-known gamma function.

By applying Theorem 1.1 to certain periodic uniform functions involving $\Gamma(x)$, we can deduce the following result.

THEOREM 1.2. *Let* $a, b \in \mathbb{Z}$ *and* $m, n \in \mathbb{Z}^+$.

(i) *For the function* T_Γ *we have the following functional equation*:

$$\sum_{cd=m} \frac{1}{d} \sum_{r_1, r_2 \in d_*} T_\Gamma(ac + r_1 n, bc + r_2 n, dn) = m d(m) T_\Gamma(a, b, n) \tag{1.14}$$

where $d(m)$ *is the number of positive divisors of* m.

(ii) *For* S_Γ *we have*

$$\sum_{cd=m}\sum_{r_1,r_2\in d_*} S_\Gamma(ac+r_1n, bc+r_2n, dn) - m\sigma(m)S_\Gamma(a,b,n)$$

$$= m\sum_{d\mid m}\Lambda(d)\sigma\Big(\frac{m}{d}\Big)\bigg(\frac{S(a,b,n/(d,n))}{d/(d,n)} - S(a,b,n)\bigg), \tag{1.15}$$

where (d,n) *is the greatest common divisor of* d *and* n, *and the Mangoldt function* Λ *is given by*

$$\Lambda(d) = \begin{cases} \ln p & \text{if } d = p^\alpha \text{ for some prime } p \text{ and } \alpha\in\mathbb{Z}^+,\\ 0 & \text{otherwise.}\end{cases} \tag{1.16}$$

Actually there are lots of examples of periodic uniform functions (see [S2, S3, S4]), so we can apply Theorem 1.1 to obtain many other results.

2. PROOF OF THEOREM 1.1

LEMMA 2.1. *Let* $a\in\mathbb{Z}$, $n\in\mathbb{Z}^+$, $F\in\mathrm{PUF}$ *and* $\langle x,y\rangle\in\mathrm{Dom}(F)$. *Then*

$$\sum_{r=0}^{n-1} F\Big(\frac{x+ar}{n}, ny\Big) = (a,n)F\bigg(\frac{x}{(a,n)}, (a,n)y\bigg). \tag{2.1}$$

Proof. Let $d=(a,n)$. Then $a'=a/d$ is relatively prime to $n'=n/d$. Each $r\in n_*$ can be written uniquely in the form $sn'+t$ where $s\in d_*$ and $t\in n'_*$. Thus

$$\sum_{r=0}^{n-1} F\Big(\frac{x+ar}{n}, ny\Big) = \sum_{s=0}^{d-1}\sum_{t=0}^{n'-1} F\bigg(\frac{x}{n}+\frac{a'}{n'}(sn'+t), ny\bigg)$$

$$= d\sum_{t=0}^{n'-1} F\bigg(\frac{x}{n}+\bigg\{\frac{a't}{n'}\bigg\}, ny\bigg) = d\sum_{r=0}^{n'-1} F\Big(\frac{x}{dn'}+\frac{r}{n'}, n'(dy)\Big) = dF\Big(\frac{x}{d}, dy\Big).$$

We are done. ∎

Proof of Theorem 1.1. Suppose that $c, d \in \mathbb{Z}^+$ and $cd = m$. Then

$$W(c,d) := \sum_{r_1,r_2 \in d_*} \begin{bmatrix} F; x, y \\ G; u, v \end{bmatrix} (ac + r_1 n, bc + r_2 n, dn)$$

$$= \sum_{r=0}^{dn-1} \sum_{r_1,r_2 \in d_*} F\left(\frac{x + (ac + r_1 n)r}{dn}, dny\right) G\left(\frac{u + (bc + r_2 n)r}{dn}, dnv\right)$$

$$= \sum_{r=0}^{dn-1} \sum_{r_1=0}^{d-1} F\left(\frac{(x + acr)/n + rr_1}{d}, dny\right) \sum_{r_2=0}^{d-1} F\left(\frac{(u + bcr)/n + rr_2}{d}, dnv\right)$$

$$= \sum_{r=0}^{dn-1} (d,r)^2 F\left(\frac{(x + acr)/n}{(d,r)}, (d,r)ny\right) G\left(\frac{(u + bcr)/n}{(d,r)}, (d,r)nv\right),$$

where we apply Lemma 2.1 in the last step. For the Möbius function μ, it is well known that $\sum_{r|k} \mu(r)$ equals 1 for $k = 1$, and 0 for $k = 2, 3, \ldots$. Thus

$$W(c,d) = \sum_{t|d} t^2 \sum_{\substack{0 \leqslant s < nd/t \\ (d,st)=t}} F\left(\frac{x + acst}{nt}, nty\right) G\left(\frac{u + bcst}{nt}, ntv\right)$$

$$= \sum_{t|d} t^2 \sum_{0 \leqslant s < nd/t} \sum_{r|(s,d/t)} \mu(r) F\left(\frac{x + acst}{nt}, nty\right) G\left(\frac{u + bcst}{nt}, ntv\right)$$

$$= \sum_{t|d} t^2 \sum_{r|\frac{d}{t}} \mu(r) \sum_{0 \leqslant s' < nd/(rt)} F\left(\frac{x + acrs't}{nt}, nty\right) G\left(\frac{u + bcrs't}{nt}, ntv\right)$$

and hence

$$W(c,d) = \sum_{rt|d} \mu(r) t^2 \sum_{k=0}^{\frac{d}{rt}-1} \sum_{l=0}^{n-1} F\left(\frac{x + acrt(kn + l)}{nt}, nty\right) G\left(\frac{u + bcrt(kn + l)}{nt}, ntv\right)$$

$$= \sum_{rt|d} \mu(r) t^2 \frac{d}{rt} \sum_{l=0}^{n-1} F\left(\frac{x/t + acrl}{n}, n(ty)\right) G\left(\frac{u/t + bcrl}{n}, n(tv)\right)$$

$$= \sum_{rst=d} \mu(r) st^2 \begin{bmatrix} F; x/t, ty \\ G; u/t, tv \end{bmatrix} (acr, bcr, n).$$

In view of the above, we have

$$\sum_{cd=m} W(c,d) = \sum_{rst|m} \mu(r)st^2 \begin{bmatrix} F; x/t, ty \\ G; u/t, tv \end{bmatrix} \left(ar\frac{m}{rst}, br\frac{m}{rst}, n\right)$$

$$= \sum_{d|m} d \sum_{st=d} t \sum_{r|\frac{m}{d}} \mu(r) \begin{bmatrix} F; x/t, ty \\ G; u/t, tv \end{bmatrix} \left(a\frac{m}{d}, b\frac{m}{d}, n\right)$$

$$= m \sum_{st=m} t \begin{bmatrix} F; x/t, ty \\ G; u/t, tv \end{bmatrix} (a,b,n) = m \sum_{d|m} d \begin{bmatrix} F; x/d, dy \\ G; u/d, dv \end{bmatrix} (a,b,n).$$

This concludes the proof. ∎

3. PROOF OF THEOREM 1.2

For a uniform function F with $\mathrm{Dom}(F) \subseteq \mathbb{R} \times \mathbb{R}$, the function $\tilde{F}(x,y) = F(\{x\},y)$ obviously lies in PUF.

Let $\mathbb{R}^+ = \{x \in \mathbb{R}\colon\ x > 0\}$. Define Γ^*: $(\mathbb{R}^+ \cup \{0\}) \times \mathbb{R}^+ \to \mathbb{R}^+$ as follows:

$$\Gamma^*(x,y) = \begin{cases} \Gamma(x)y^x/\sqrt{2\pi y} & \text{if } x > 0, \\ \sqrt{2\pi y} & \text{if } x = 0. \end{cases} \tag{3.1}$$

By Example 2.2 of Sun [S2], the function $\ln \Gamma^*(x,y)$ is a uniform function. The reader can verify this by means of the following famous multiplication formula of Gauss (cf. [E]):

$$\prod_{r=0}^{n-1} \Gamma\left(z + \frac{r}{n}\right) = (2\pi)^{(n-1)/2} n^{1/2-nz} \Gamma(nz) \ \ (n \in \mathbb{Z}^+ \text{ and } nz \neq 0, -1, \ldots).$$

So the function $\Gamma_*(x,y) = \ln \Gamma^*(\{x\},y)$ belongs to PUF.

Let $f(0) = -\gamma$ (where γ is the Euler constant $0.577\ldots$), and $f(x) = \Gamma'(x)/\Gamma(x)$ for $x > 0$. By Sections 1.7 and 1.7.1 of [E], we have

$$f(x) = \frac{1}{n} \sum_{r=0}^{n-1} f\left(\frac{x+r}{n}\right) + \ln n = -\gamma + (x-1) \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+x)}$$

for any $x \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$. Observe that

$$f(x) - f(0) - \frac{1}{n}\left(f\left(\frac{x}{n}\right) - f\left(\frac{0}{n}\right)\right)$$

$$= \sum_{k=0}^{\infty}\left(\frac{1}{k+1} - \frac{1}{k+x}\right) - \frac{1}{n}\sum_{k=0}^{\infty}\left(\frac{1}{k+1} - \frac{1}{k+x/n}\right)$$

$$= 1 - \frac{1}{n} + \sum_{k=1}^{\infty}\left(\left(1-\frac{1}{n}\right)\frac{1}{k+1} + \frac{1}{kn+x} - \frac{1}{k+x}\right)$$

tends to zero as $x \to 0$. So $f(x) = \frac{1}{n}\sum_{r=0}^{n-1} f(\frac{x+r}{n}) + \ln n$ for all $x \geqslant 0$. For $x \geqslant 0$ and $y > 0$ let

$$\Psi(x,y) = \begin{cases} (\Gamma'(x)/\Gamma(x) + \ln y)/y & \text{if } x > 0, \\ (-\gamma + \ln y)/y & \text{if } x = 0. \end{cases} \tag{3.2}$$

Then Ψ is a uniform function. (In fact, $\Psi(x,y) + \gamma/y$ is just the uniform function $G(x,y)$ given in Example 2.3 of [S2].) Thus the function $\psi(x,y) = \tilde{\Psi}(x,y)$ also lies in PUF.

Lemma 3.1. *Let $a, b \in \mathbb{Z}$ and $m, n \in \mathbb{Z}^+$. Then*

$$\begin{bmatrix} D; 0, m \\ \Gamma_*; 0, m \end{bmatrix}(a,b,n) = S_\Gamma(a,b,n) + \ln(mn)S(a,b,n) \tag{3.3}$$

and

$$\begin{bmatrix} D; 0, m \\ \psi; 0, m \end{bmatrix}(a,b,n) = \frac{T_\Gamma(a,b,n)}{mn}. \tag{3.4}$$

Proof. We first claim that both

$$R_1 = \sum_{\substack{r \in n_* \\ n \mid br}} \left(\left(\frac{ar}{n}\right)\right) \quad \text{and} \quad R_2 = \sum_{\substack{r \in n_* \\ n \nmid br}} \left(\left(\frac{ar}{n}\right)\right)$$

vanish. In fact, by Lemma 2.1 we have

$$\sum_{r=0}^{k-1}\left(\left(\frac{ar}{k}\right)\right) = (a,k)((0)) = 0 \quad \text{for every } k = 1, 2, 3, \ldots .$$

So $R_1 + R_2 = 0$. Let $d = (b, n)$, $b' = b/d$ and $n' = n/d$. Then

$$R_1 = \sum_{\substack{0 \leqslant r < n \\ n' | b'r}} \left(\left(\frac{ar}{n}\right)\right) = \sum_{r'=0}^{d-1} \left(\left(\frac{ar'n'}{n}\right)\right) = \sum_{r'=0}^{d-1} \left(\left(\frac{ar'}{d}\right)\right) = 0.$$

In view of the above,

$$\begin{aligned}
\begin{bmatrix} D; 0, m \\ \Gamma_*; 0, m \end{bmatrix}(a, b, n) &= \sum_{r=0}^{n-1} \left(\left(\frac{ar}{n}\right)\right) \Gamma_*\left(\frac{br}{n}, nm\right) \\
&= \sum_{\substack{r \in n_* \\ n \nmid br}} \left(\left(\frac{ar}{n}\right)\right) \left(\ln \Gamma\left(\left\{\frac{br}{n}\right\}\right)\right. \\
&\quad \left. + \left(\left\{\frac{br}{n}\right\} - \frac{1}{2}\right) \ln(mn) - \frac{\ln(2\pi)}{2}\right) \\
&\quad + \sum_{\substack{r \in n_* \\ n | br}} \left(\left(\frac{ar}{n}\right)\right) \frac{\ln(2\pi mn)}{2} \\
&= S_\Gamma(a, b, n) + \ln(mn) S(a, b, n)
\end{aligned}$$

and

$$\begin{aligned}
\begin{bmatrix} D; 0, m \\ \psi; 0, m \end{bmatrix}(a, b, n) &= \sum_{r=0}^{n-1} \left(\left(\frac{ar}{n}\right)\right) \psi\left(\frac{br}{n}, nm\right) \\
&= \sum_{\substack{r \in n_* \\ n \nmid br}} \left(\left(\frac{ar}{n}\right)\right) \frac{1}{mn} \left(\frac{\Gamma'(\{br/n\})}{\Gamma(\{br/n\})} + \ln(mn)\right) \\
&\quad + \sum_{\substack{r \in n_* \\ n | br}} \left(\left(\frac{ar}{n}\right)\right) \frac{1}{mn} (-\gamma + \ln(mn)) \\
&= \frac{T_\Gamma(a, b, n)}{mn}.
\end{aligned}$$

We are done. ∎

Proof of Theorem 1.2. (i) By Theorem 1.1,

$$\sum_{cd=m}\sum_{r_1,r_2\in d_*}\begin{bmatrix} D;0,1 \\ \psi;0,1\end{bmatrix}(ac+r_1n,bc+r_2n,dn)=m\sum_{d|m}d\begin{bmatrix} D;0,d \\ \psi;0,d\end{bmatrix}(a,b,n).$$

In light of Lemma 3.1, this says that

$$\sum_{cd=m}\sum_{r_1,r_2\in d_*}\frac{1}{dn}T_\Gamma(ac+r_1n,bc+r_2n,dn)=m\sum_{d|m}d\,\frac{T_\Gamma(a,b,n)}{dn},$$

which is equivalent to (1.14).

(ii) If $c,d\in\mathbb{Z}^+$ and $cd=m$, then by Lemma 3.1 we have

$$\begin{aligned}\sum_{r_1,r_2\in d_*}&\begin{bmatrix} D;0,1 \\ \Gamma_*;0,1\end{bmatrix}(ac+r_1n,bc+r_2n,dn)\\ &=\sum_{r_1,r_2\in d_*}S_\Gamma(ac+r_1n,bc+r_2n,dn)\\ &\quad+\ln(dn)\sum_{r_1,r_2\in d_*}S(ac+r_1n,bc+r_2n,dn).\end{aligned}$$

This, together with (1.5), yields that

$$\begin{aligned}&\sum_{cd=m}\sum_{r_1,r_2\in d_*}\begin{bmatrix} D;0,1 \\ \Gamma_*;0,1\end{bmatrix}(ac+r_1n,bc+r_2n,dn)\\ &\quad-\sum_{cd=m}\sum_{r_1,r_2\in d_*}S_\Gamma(ac+r_1n,bc+r_2n,dn)\\ &=\sum_{cd=m}(\ln d+\ln n)\sum_{r_1,r_2\in d_*}S(ac+r_1n,bc+r_2n,dn)\\ &=(\ln n)m\sigma(m)S(a,b,n)+\Sigma,\end{aligned}$$

where

$$\Sigma=\sum_{cd=m}\ln d\sum_{r_1,r_2\in d_*}\begin{bmatrix} D;0,1 \\ D;0,1\end{bmatrix}(ac+r_1n,bc+r_2n,dn).$$

It is well known that $\sum_{d|k} \Lambda(d) = \ln k$ for $k \in \mathbb{Z}^+$ (see, e.g. [A]). By Möbius' theorem, this implies that

$$\Lambda(k) = \sum_{d|k} \mu(d) \ln \frac{k}{d} = -\sum_{d|k} \mu(d) \ln d \qquad \text{for any } k \in \mathbb{Z}^+.$$

By the proof of Theorem 1.1,

$$\begin{aligned} \Sigma &= \sum_{rst|m} \ln(rst)\mu(r)st^2 \begin{bmatrix} D;0,t \\ D;0,t \end{bmatrix} \left(ar\frac{m}{rst}, br\frac{m}{rst}, n\right) \\ &= \sum_{d|m} d \sum_{st=d} t \sum_{r|\frac{m}{d}} \mu(r) \ln(rd) \begin{bmatrix} D;0,t \\ D;0,t \end{bmatrix} \left(a\frac{m}{d}, b\frac{m}{d}, n\right) \end{aligned}$$

and hence

$$\begin{aligned} \Sigma &= \sum_{d|m} d\left(\sum_{r\mid\frac{m}{d}} \mu(r)\ln d + \sum_{r\mid\frac{m}{d}} \mu(r)\ln r\right) \sum_{t|d} tS\left(a\frac{m}{d}, b\frac{m}{d}, n\right) \\ &= \sum_{d=m} d(\ln d)\sigma(d)S\left(a\frac{m}{d}, b\frac{m}{d}, n\right) \\ &\quad -\sum_{d|m} d\Lambda\left(\frac{m}{d}\right)\sigma(d)S\left(a\frac{m}{d}, b\frac{m}{b}, n\right) \\ &= m\sigma(m)(\ln m)S(a,b,n) - \sum_{d|m} d\sigma(d)\Lambda\left(\frac{m}{d}\right)S\left(a\frac{m}{d}, b\frac{m}{d}, n\right). \end{aligned}$$

In view of Lemma 3.1,

$$\begin{aligned} \sum_{d|m} d \begin{bmatrix} D;0/d,d \\ \Gamma_*;0/d,d \end{bmatrix}(a,b,n) &= \sum_{d|m} d(S_\Gamma(a,b,n) + \ln(dn)S(a,b,n)) \\ &= \sigma(m)S_\Gamma(a,b,n) + S(a,b,n) \\ &\quad \times \left(\sigma(m)\ln n + \sum_{d|m} d\ln d\right). \end{aligned}$$

Thus, by Theorem 1.1 and the above,

$$m\sigma(m)S_\Gamma(a,b,n) + mS(a,b,n)\left(\sigma(m)\ln n + \sum_{d|m} d\ln d\right)$$
$$- \sum_{cd=m}\ \sum_{r_1,r_2\in d_*} S_\Gamma(ac+r_1n, bc+r_2n, dn)$$
$$= m\sigma(m)(\ln n + \ln m)S(a,b,n) - \sum_{d|m} d\sigma(d)\Lambda\left(\frac{m}{d}\right)S\left(a\frac{m}{d}, b\frac{m}{d}, n\right).$$

Therefore

$$m\sigma(m)S_\Gamma(a,b,n) - \sum_{cd=m}\ \sum_{r_1,r_2\in d_*} S_\Gamma(ac+r_1n, bc+r_2n, dn)$$
$$= m\left(\sum_{d|m} d\ln m - \sum_{d|m} d\ln d\right)S(a,b,n) - \sum_{d|m}\frac{m}{d}\sigma\left(\frac{m}{d}\right)\Lambda(d)S(ad,bd,n)$$
$$= m\sum_{d|m} d\ln\frac{m}{d}S(a,b,n) - m\sum_{d|m}\Lambda(d)\frac{S(ad,bd,n)}{d}\sigma\left(\frac{m}{d}\right),$$
$$= m\sum_{d|m}\Lambda(d)\sigma\left(\frac{m}{d}\right)\left(S(a,b,n) - \frac{S(ad,bd,n)}{d}\right),$$

where in the last step we note that

$$\sum_{d|m} d\ln\frac{m}{d} = \sum_{d|m} d\sum_{t|\frac{m}{d}}\Lambda(t) = \sum_{t|m}\Lambda(t)\sum_{d|\frac{m}{t}} d = \sum_{t|m}\Lambda(t)\sigma\left(\frac{m}{t}\right).$$

Fix $d \in \mathbb{Z}^+$ and let $d' = d/(d,n)$ and $n' = n/(d,n)$. Evidently,

$$\frac{S(ad,bd,n)}{(d,n)} = \frac{1}{(d,n)}\sum_{s=0}^{(d,n)-1}\sum_{t=0}^{n'-1}\left(\left(\frac{ad'(sn'+t)}{n'}\right)\right)\left(\left(\frac{bd'(sn'+t)}{n'}\right)\right)$$
$$= \sum_{t=0}^{n'-1}\left(\left(\frac{ad't}{n'}\right)\right)\left(\left(\frac{bd't}{n'}\right)\right) = \sum_{r=0}^{n'-1}\left(\left(\frac{ar}{n'}\right)\right)\left(\left(\frac{br}{n'}\right)\right) = S(a,b,n').$$

By the above we finally get (1.15). This concludes the proof. ∎

ACKNOWLEDGMENT

The authors are grateful to the referee for his/her comments.

REFERENCES

[A] W. S. Anglin, "The Queen of Mathematics—An Introduction to Number Theory," Kluwer, Dordrecht, 1995, pp. 336–349.

[E] Erdélyi et al., "Higher Transcendental Functions," Vol. I, Chap. 1, McGraw–Hill, New York, 1953.

[K] M. I. Knopp, Hecke operators and an identity for Dedekind sums, *J. Number Theory* **12** (1980), 2–9.

[S1] Z. W. Sun, Systems of congruences with multipliers, *Nanjing Univ. J. Math. Biquarterly* **6** (1989), 124–133. MR 90m:11006; Zbl. M. 703.11002.

[S2] Z. W. Sun, On covering equivalence, *in*: "Analytic Number Theory," (Beijing/Kyoto, 1999), pp. 277–302, *Dev. Math*. **6**, Kluwer Academic, Dordrecht, 2002.

[S3] Z. W. Sun, Products of binomial coefficients modulo p^2, *Acta Arith.* **97** (2001), 87–98.

[S4] Z. W. Sun, Algebraic approaches to periodic arithmetical maps, *J. Algebra* **240** (2001), 723–743.

[S5] Z. W. Sun, Sums of minima and maxima, *Discrete Math.* **257** (2002), 143–159.

[Z] D. Zagier, Higher dimensional Dedekind sums, *Math. Ann.* **202** (1973), 149–172.

[Zh] Z. Y. Zheng, The Petersson-Knopp identity for the homogeneous Dedekind sums, *J. Number Theory* **57** (1996), 223–230.