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# Rigid dualizing complex for quantum enveloping algebras and algebras of generalized differential operators

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## Abstract

In the first part of this article, we compute the rigid dualizing complex of a quantum enveloping algebra. We consider the generic case and the case of a specialization at a non-root of unity. This answers a question of Yekutieli [J. Pure Appl. Algebra 150 (2000) 85]. In [Bull. Soc. Math. France 122 (1994) 371] and [Math. Z. 232 (1999) 367], we generalized  $\mathcal{D}$ -module theory to Lie algebroids. Using these results, we compute explicitly the rigid dualizing complex of the algebra of differential operators defined by an affine Lie algebroid. This generalizes results of Yekutieli [J. Pure Appl. Algebra 150 (2000) 85].

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## 1. Introduction

Grothendieck duality involves dualizing complexes which were introduced in [16]. The extension of the definition of dualizing complexes to the non-commutative setting is due to Yekutieli [31]. Let  $k$  be a field and let  $A$  be a noetherian associative unital  $k$ -algebra. Denote by  $A^{\text{op}}$  the opposite algebra and put  $A^e = A \otimes_k A^{\text{op}}$ . Let  $D_f^b(A)$  be the bounded derived category of complexes of left  $A$ -modules with finitely generated cohomologies. A dualizing complex over  $A$  is roughly speaking a complex of bimodules  $R \in D^b(A^e)$  such that the functors  $\text{RHom}_A(-, R)$  and  $\text{RHom}_{A^{\text{op}}}(-, R)$  induce a duality between  $D_f^b(A)$  and  $D_f^b(A^{\text{op}})$ . Dualizing complexes are not unique. To rigidify the definition, van den Bergh

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introduced the notion of rigid dualizing complex. A dualizing complex is rigid if there is an isomorphism

$$R \simeq \mathrm{RHom}_{A^e}(A, R \otimes R)$$

in  $D(A^e)$ . A rigid dualizing complex, if it exists, is unique up to isomorphism.

In the first part of this article, we compute the rigid dualizing complex of the quantum enveloping algebra of a complex finite-dimensional semi-simple Lie algebra for  $q$  generic and also in the case of a specialization at a complex number satisfying suitable conditions. This answers a question of Yekutieli [34]. Let  $\mathfrak{g}$  be a finite-dimensional semi-simple Lie algebra over  $\mathbb{C}$  and  $A$  its Cartan matrix. Let  $(d_1, \dots, d_n)$  be relatively prime entries such that  $(d_i a_{i,j})$  is symmetric and positive definite. Put

$$\mathbb{C}_{\mathfrak{g}} = \{\epsilon \in \mathbb{C}^* \mid \epsilon^{2d_i} \neq 1, \forall i \in [1, n]\}.$$

Denote by  $U_q(\mathfrak{g})$  the quantum enveloping algebra for  $q$  generic and  $U_{\epsilon}(\mathfrak{g})$  its specialization at  $\epsilon \in \mathbb{C}^*$ .

**Theorem.** *Let  $\epsilon \in \mathbb{C}_{\mathfrak{g}}$ . The rigid dualizing complex of  $U_q(\mathfrak{g})$  and  $U_{\epsilon}(\mathfrak{g})$  are  $U_q(\mathfrak{g})[\dim \mathfrak{g}]$  and  $U_{\epsilon}(\mathfrak{g})[\dim \mathfrak{g}]$ , respectively.*

The theorem is a consequence of the following proposition which is interesting by itself.

**Proposition.** *We have the following isomorphism of right  $U_q(\mathfrak{g})$ -modules:*

$$\mathrm{Ext}_{U_q(\mathfrak{g})}^i(\mathbb{C}(q), U_q(\mathfrak{g})) = 0 \quad \text{if } i \neq \dim \mathfrak{g},$$

$$\mathrm{Ext}_{U_q(\mathfrak{g})}^{\dim \mathfrak{g}}(\mathbb{C}(q), U_q(\mathfrak{g})) = \mathbb{C}(q).$$

*Replacing  $\mathbb{C}(q)$  by  $\mathbb{C}$ , a similar result holds for  $U_{\epsilon}(\mathfrak{g})$  if  $\epsilon \in \mathbb{C}_{\mathfrak{g}}$ .*

To prove the proposition, we use a filtration on  $U_q(\mathfrak{g})$  and  $U_{\epsilon}(\mathfrak{g})$  such that the graded algebras are  $q$ -commutative algebras [12]. As corollaries of this proposition, we obtain duality properties (already known for Lie algebras) in the quantum group setting.

In the second part of this article, we compute the rigid dualizing complex defined by a Lie Rinehart algebra over a smooth integral domain. Our computation uses the generalization of  $\mathcal{D}$ -module theory to Lie algebroids that we developed in previous articles [8,9]. Let  $G$  be an integral domain which is a smooth commutative algebra of dimension  $n$ . Let  $L_G$  be a Lie Rinehart algebra [23] over  $G$  which is a projective  $G$ -module of constant rank  $d_{L_G}$  and let  $D(L_G)$  be the algebra of generalized differential operators defined by  $L_G$ . We know from previous work that  $\Lambda^{d_{L_G}}(L_G^*)$  and  $\omega_G$  (the  $G$ -module of differential form of maximal degree) are right  $D(L_G)$ -modules so that  $\mathrm{Hom}_G(\Lambda^{d_{L_G}}(L_G^*), \omega_G)$  is a left  $D(L_G)$ -module.

**Theorem.** *The rigid dualizing complex of  $D(L_G)$  is*

$$D(L_G) \otimes_G \mathrm{Hom}_G(\Lambda^{d_{L_G}}(L_G^*), \omega_G)[d_{L_G} + n].$$

If  $L_G$  is the module of vector fields over  $G$ , then  $D(L_G)$  is the ring of differential operators over  $G$ ,  $D_G$ . From our theorem, we deduce that the rigid dualizing complex of  $D_G$  is  $D_G[2n]$ . If  $G$  is a point,  $L_G$  is nothing but a finite-dimensional Lie algebra  $\mathfrak{g}$ . As a particular case of our theorem, we get that the rigid dualizing complex of the enveloping algebra  $U(\mathfrak{g})$  is  $U(\mathfrak{g}) \otimes \Lambda^{\dim \mathfrak{g}}(\mathfrak{g})[\dim \mathfrak{g}]$ . These two particular cases were already treated in [34].

**Notation.** If  $k$  is the base field, we write  $\otimes$  for  $\otimes_k$ . If  $A$  is a  $k$ -algebra, we put  $A^e = A \otimes A^{\mathrm{op}}$ .

Given a commutative totally ordered semi-group  $S$ , an  $S$  filtration of an algebra  $A$  is a collection of subspaces  $(A^s)_{s \in S}$  such that  $\bigcup_{s \in S} A^s = A$ ,  $A^s \subset A^{s'}$  if  $s < s'$  and  $A^s \cdot A^{s'} \subset A^{s+s'}$ . The associated graded algebra is

$$\mathrm{Gr} A = \bigoplus_s \left( A^s / \sum_{s' < s} A^{s'} \right).$$

Let  $(H, i, \mu, \Delta, S, \epsilon)$  be a Hopf algebra. One defines the adjoint representation from  $H$  with values in  $\mathrm{End} H$  by  $\mathrm{ad}(x)u = \sum_i a_i u S(b_i)$  where  $\Delta(x) = \sum_i a_i \otimes b_i$ . Recall the following well-known result [13, p. 387].

**Lemma 1.1.** *Let  $(H, i, \mu, \Delta, S, \epsilon)$  be a Hopf algebra and  $V$  an  $H$ -module.*

(a) *The map*

$$\Theta : H \otimes V \rightarrow H \otimes V, \quad a \otimes v \mapsto \sum_{i=1}^n a'_i \otimes a''_i v \quad \text{with } \Delta(a) = \sum_{i=1}^n a'_i \otimes a''_i$$

*is an isomorphism between  $H \otimes V$  endowed with the  $H$  module structure given by left multiplication and  $H \otimes V$  endowed with the  $H$ -module structure given by the coproduct. One checks that  $\Theta^{-1}$  is given by*

$$\Theta^{-1}(a \otimes v) = \sum_{i=1}^n a'_i \otimes S(a''_i)v.$$

(b) *The map*

$$\Psi : V \otimes H \rightarrow V \otimes H, \quad v \otimes a \mapsto \sum_{i=1}^n v a'_i \otimes a''_i \quad \text{with } \Delta(a) = \sum_{i=1}^n a'_i \otimes a''_i$$

is an isomorphism between  $V \otimes H$  endowed with the right  $H$ -module structure given by right multiplication and  $V \otimes H$  endowed with the right  $H$ -module structure given by the coproduct. One checks that  $\Psi^{-1}$  is given by

$$\Psi^{-1}(v \otimes a) = \sum_{i=1}^n vS(a'_i) \otimes a''_i.$$

For homological algebra and sheaves, we will use the same notation as in [20].

## 2. Dualizing complexes

The next definition is due to Yekutieli [31].

**Definition 2.0.1.** Assume that  $A$  is a left and right noetherian ring. An object  $R$  of  $D^b(A^e)$  is called a dualizing complex if it satisfies the following conditions:

- (a)  $R$  has finite injective dimension over  $A$  and  $A^{\text{op}}$ .
- (b) The cohomology of  $R$  is given by bimodules which are finitely generated on both sides.
- (c) The natural morphisms  $\Phi : A \rightarrow \text{RHom}_A(R, R)$  and  $\Phi : A \rightarrow \text{RHom}_{A^{\text{op}}}(R, R)$  are isomorphisms in  $D^b(A^e)$ .

**Remarks** [31,33].

- (1) A dualizing complex is only determined up to derived tensor product with a tilting module [33, Theorem 4.5].
- (2) If  $A$  has finite injective dimension over  $A$  as a left and right  $A$ -module, then  $A$  is a dualizing complex.
- (3) If  $R$  is a dualizing complex, then  $\text{RHom}_A(-, R)$  defines a duality between  $D_f^b(A)$  and  $D_f^b(A^{\text{op}})$ , the full subcategories of  $D^b(A)$  and  $D^b(A^{\text{op}})$  consisting of complexes with finitely generated cohomology.

The next definition is due to M. van den Bergh [28].

**Definition 2.0.2.** Let  $A$  be a left and right noetherian ring. A dualizing complex  $R$  is rigid if

$$R \simeq \text{RHom}_{A^e}(A, {}_A R \otimes R_A)$$

in  $D(\text{Mod } A^e)$ . The notations  ${}_A R$  and  $R_A$  mean that we take the  $\text{RHom}$  over the left and the right  $A$ -structures of  $R$ , respectively.

**Remark.** The rigid dualizing complex, if it exists, is unique up to isomorphism.

### 3. Rigid dualizing complex for quantum enveloping algebras

#### 3.1. The algebras $U_q(\mathfrak{g})$ and $U_\epsilon(\mathfrak{g})$

For basic results on quantum groups, we refer the reader to [7].

Let  $q$  be an indeterminate and let  $\mathcal{A} = \mathbb{C}[q, q^{-1}]$ . We will use the usual notation:

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \in \mathcal{A}, \quad [n]_q! = [n]_q [n-1]_q \cdots [1]_q,$$

$$\binom{n}{j}_q = [n]_q [n-1]_q \cdots [n-j+1]_q / [j]_q! \quad \forall j \in \mathbb{N}.$$

One knows that  $\binom{j}{n}_q$  is in  $\mathcal{A}$ .

Let  $\mathfrak{g}$  be a finite-dimensional semi-simple Lie algebra over  $\mathbb{C}$  and  $A = (a_{i,j})_{(i,j) \in [1,n]^2}$  its Cartan matrix. The matrix  $A$  is not always symmetric but it is always symmetrizable [19, Chapter 4, Proposition 4.9]. This implies that there exists a unique  $n$ -uple of integers with relatively prime entries  $(d_1, \dots, d_n)$  such that  $(d_i a_{i,j})$  is symmetric and positive definite. Put  $q_i = q^{d_i}$ .

We set

$$\mathbb{C}_{\mathfrak{g}} = \{\epsilon \in \mathbb{C}^* \mid \epsilon^{2d_i} \neq 1, \forall i \in [1, n]\}.$$

Following Jimbo, we consider the  $\mathbb{C}(q)$ -algebra  $U_q(\mathfrak{g})$  defined by the generators  $E_i, F_i, K_i, K_i^{-1}$  for  $i$  in  $[1, n]$  and the relations

- (1)  $K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad K_i K_j = K_j K_i,$
- (2)  $K_i E_j K_i^{-1} = q_i^{a_{i,j}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-a_{i,j}} F_j,$
- (3)  $E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$
- (4)  $\sum_{s=0}^{1-a_{i,j}} (-1)^s \binom{1-a_{i,j}}{s}_{q_i} E_i^{1-a_{i,j}-s} E_j E_i^s = 0 \quad \text{if } i \neq j,$
- (5)  $\sum_{s=0}^{1-a_{i,j}} (-1)^s \binom{1-a_{i,j}}{s}_{q_i} F_i^{1-a_{i,j}-s} F_j F_i^s = 0 \quad \text{if } i \neq j.$

$U_q(\mathfrak{g})$ , endowed with the comultiplication  $\Delta$ , the antipode  $S$  and the counit  $\epsilon$  defined below, is a Hopf algebra.

$$(6) \quad \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i,$$

$$(7) \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,$$

- (8)  $\Delta(K_i) = K_i \otimes K_i,$
- (9)  $S(E_i) = -K_i^{-1} E_i,$
- (10)  $S(F_i) = -F_i K_i,$
- (11)  $S(K_i) = K_i^{-1},$
- (12)  $\epsilon(E_i) = 0, \quad \epsilon(F_i) = 0, \quad \epsilon(K_i) = 1.$

$U_q(\mathfrak{g})$  is called the quantum group associated to  $\mathfrak{g}$  with  $q$  generic. Let  $P$  be the free abelian group with basis  $(\omega_i)_{i \in [1, n]}$ . Define the following element  $\alpha_j = \sum_{i=1}^n a_{i, j} \omega_i$  and put

$$Q = \sum_{i=1}^n \mathbb{Z} \alpha_i \quad \text{and} \quad Q^+ = \sum_{i=1}^n \mathbb{Z}_+ \alpha_i.$$

For  $\beta = \sum_{i=1}^n k_i \alpha_i \in Q$  let  $\text{ht}(\beta) = \sum_{i=1}^n k_i$ . Denote by  $(, )$  the  $\mathbb{Z}$ -valued symmetric bilinear form on  $Q$  such that  $(\alpha_i, \alpha_j) = d_i a_{i, j}$ . Let  $r_i$  be the automorphism of  $P$  defined by

$$r_i(\omega_j) = \omega_j - \delta_{i, j} \alpha_i \quad \forall (i, j) \in [1, n].$$

Let  $W$  be the finite subgroup of  $GL(P)$  generated by  $r_1, \dots, r_n$ . We put

$$\Pi = (\alpha_1, \dots, \alpha_n), \quad R = W\Pi, \quad \text{and} \quad R^+ = R \cap Q^+.$$

Then  $R$  is the root system corresponding to the Cartan matrix  $A$ ,  $W$  is its Weyl group,  $R^+$  is the set of positive roots.

Let  $U_{\mathcal{A}}(\mathfrak{g})$  be the  $\mathcal{A}$ -subalgebra of  $U_q(\mathfrak{g})$  generated by elements  $E_i, F_i, K_i, K_i^{-1}, [K_i, 0] = (K_i - K_i^{-1}) / (q_i - q_i^{-1})$ .

Note that relations (1), (2), (4)–(12) together with relations

$$\begin{aligned} E_i F_j - F_j E_i &= \delta_{i, j} [K_i, 0], & (q_i - q_i^{-1}) [K_i, 0] &= K_i - K_i^{-1}, \\ \Delta([K_i, 0]) &= [K_i, 0] \otimes K_i + K_i^{-1} \otimes [K_i, 0], & S([K_i, 0]) &= -[K_i, 0], \\ \epsilon([K_i, 0]) &= 0 \end{aligned}$$

define a Hopf algebra over  $\mathcal{A}$ .

Given  $\epsilon$  in  $\mathbb{C}^*$ , we consider the specialization

$$U_{\epsilon}(\mathfrak{g}) = U_{\mathcal{A}}(\mathfrak{g}) / (q - \epsilon) U_{\mathcal{A}}(\mathfrak{g}).$$

One defines the operators  $T_i$  (for  $i \in [1, n]$ ) over  $U_q(\mathfrak{g})$  by

$$\begin{aligned} T_i E_j &= \text{ad}(-E_i^{-a_{i, j}}) E_j \quad \text{if } i \neq j, & T_i E_i &= -F_i K_i, \\ T_i F_j &= \text{ad}(-F_i^{-a_{i, j}}) F_j \quad \text{if } i \neq j, & T_i F_i &= -K_i^{-1} E_i, & T_i K_j &= K_j K_i^{-a_{i, j}}. \end{aligned}$$

The operators  $T_i$  are also defined over  $U_\epsilon(\mathfrak{g})$ .

We will now recall a filtration defined on  $U_q(\mathfrak{g})$  and  $U_\epsilon(\mathfrak{g})$  by de Concini and Kac [12, Proposition 1.7].

Fix a reduced expression  $w_0 = r_{i_1} r_{i_2} \dots r_{i_N}$  of the longest element of  $W$ . This gives an ordering of the positive roots

$$\beta_1 = \alpha_{i_1}, \quad \beta_2 = r_{i_1}(\alpha_{i_2}), \quad \dots, \quad \beta_N = r_{i_1} \dots r_{i_{N-1}}(\alpha_{i_N}).$$

One sets

$$E_{\beta_j} = T_{i_1} \dots T_{i_{j-1}}(E_{i_j}) \quad \text{and} \quad F_{\beta_j} = T_{i_1} \dots T_{i_{j-1}}(F_{i_j}).$$

For  $(k_1, \dots, k_N) \in \mathbb{Z}_+^N$  and  $(l_1, \dots, l_N) \in \mathbb{Z}_+^N$  and  $u \in U_q(\mathfrak{g})^0$ , define

$$M_{k,l,u} = F_{\beta_N}^{k_N} \dots F_{\beta_1}^{k_1} u E_{\beta_1}^{l_1} \dots E_{\beta_N}^{l_N}.$$

Define the height of this monomial by

$$\text{ht}(M_{k,l,u}) = \sum_{i=1}^N (k_i + l_i) \text{ht}(\beta_i).$$

Consider  $\mathbb{Z}_+^{2N+1}$  as a semi-group totally ordered with the lexicographic order  $<$  such that  $u_1 < \dots < u_{2N+1}$  where

$$u_i = (\delta_{i,1}, \dots, \delta_{i,2N+1})$$

is the standard basis of  $\mathbb{Z}_+^{2N+1}$ . Define the degree of the monomial  $M_{k,l,u}$  as being

$$d(M_{k,l,u}) = (k_N, k_{N-1}, \dots, k_1, l_1, \dots, l_N, \text{ht}(M_{k,l,u})).$$

Given  $s \in \mathbb{Z}_+^{2N+1}$ , denote by  $\Sigma_s U_q(\mathfrak{g})$  the linear span over  $\mathbb{C}(q)$  of the monomials  $M_{k,l,u}$  such that  $d(M_{k,l,u}) \leq s$ . Define  $\Sigma_s U_\epsilon(\mathfrak{g}) \subset U_\epsilon(\mathfrak{g})$  similarly.

**Proposition 3.1.1.**

- (a) The  $(\Sigma_s U_q(\mathfrak{g}))_{s \in \mathbb{Z}_+^{2N+1}}$  form a filtration of  $U_q(\mathfrak{g})$  (similarly  $U_\epsilon(\mathfrak{g})$ ).
- (b) The associated graded algebra  $\text{Gr } U_q(\mathfrak{g})$  (respectively  $\text{Gr } U_\epsilon(\mathfrak{g})$ ) is an associative algebra over  $\mathbb{C}(q)$  (respectively  $\mathbb{C}$ ) on generators  $E_\alpha, F_\alpha (\alpha \in R_+), K_i, K_i^{-1} (i \in [1, n])$  subject to the following relations:

$$\begin{aligned} K_i K_j &= K_j K_i, & K_i K_i^{-1} &= 1, & E_\alpha F_\beta &= F_\beta E_\alpha, & K_i E_\alpha &= q^{(\alpha, \alpha_i)} E_\alpha K_i, \\ K_i F_\alpha &= q^{-(\alpha, \alpha_i)} F_\alpha K_i, & E_\alpha E_\beta &= q^{(\alpha, \beta)} E_\beta E_\alpha & \text{if } \alpha > \beta, \\ F_\alpha F_\beta &= q^{(\alpha, \beta)} F_\beta F_\alpha & \text{if } \alpha > \beta \end{aligned}$$

(respectively same relations with  $q = \epsilon$  provided that  $\epsilon \in \mathbb{C}_q$ ).

**Proof.** See [12, Proposition 1.7].  $\square$

**Corollary 3.1.2.** *Let  $\mathfrak{g}$  be a finite-dimensional semi-simple complex Lie algebra. Assume that  $\epsilon$  is in  $\mathbb{C}_{\mathfrak{g}}$ . Then  $U_q(\mathfrak{g})$  and  $U_\epsilon(\mathfrak{g})$  are noetherian algebras of finite homological dimension.*

This follows from the filtration above (see [6]).

$U_q(\mathfrak{g})$  is a right (respectively left) module via right (respectively left) multiplication. Using the antipode, one can transform right multiplication into a left  $U_q(\mathfrak{g})$ -action on  $U_q(\mathfrak{g})$  denoted  $\cdot_S$ ,

$$\forall \alpha, u \in U_q(\mathfrak{g}), \quad \alpha \cdot_S u = uS(\alpha).$$

Thus  $U_q(\mathfrak{g})$  becomes a  $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ -module. The antipode  $S$  provides an isomorphism between the left modules  $(U_q(\mathfrak{g}), \cdot_S)$  and  $U_q(\mathfrak{g})$  endowed with left multiplication. More precisely, it exchanges the two left module structures. Introduce the functor  $D_{U_q(\mathfrak{g})}$  from  $D_f^b(U_q(\mathfrak{g}))$  to  $D_f^b(U_q(\mathfrak{g}))$  defined by

$$\forall M^\bullet \in D^b(U_q(\mathfrak{g})), \quad D_{U_q(\mathfrak{g})}(M^\bullet) = \text{RHom}_{U_q(\mathfrak{g})}(M^\bullet, U_q(\mathfrak{g}))[-\dim \mathfrak{g}].$$

One introduces similarly the functor  $D_{U_\epsilon(\mathfrak{g})}$  from  $D_f^b(U_\epsilon(\mathfrak{g}))$  to  $D_f^b(U_\epsilon(\mathfrak{g}))$ . The canonical arrow  $M^\bullet \rightarrow D_{U_q(\mathfrak{g})}^2(M^\bullet)$  is an isomorphism for  $M^\bullet = U_q(\mathfrak{g})$ . Hence, by standard homological algebra arguments, one proves that for any  $M^\bullet$  in  $D_f^b(U_q(\mathfrak{g}))$ ,  $D_{U_q(\mathfrak{g})}^2(M^\bullet) = M^\bullet$ . In other words,  $D_{U_q(\mathfrak{g})}$  is a duality functor. Similarly, if  $\epsilon \in \mathbb{C}_{\mathfrak{g}}$ ,  $D_{U_\epsilon(\mathfrak{g})}$  is a duality functor.

### 3.2. Computation of $\text{Ext}_{U_q(\mathfrak{g})}^i(\mathbb{C}(q), U_q(\mathfrak{g}))$ and $\text{Ext}_{U_\epsilon(\mathfrak{g})}^i(\mathbb{C}, U_\epsilon(\mathfrak{g}))$

**Proposition 3.2.1.** *Let  $\mathfrak{g}$  be a complex finite-dimensional semi-simple Lie algebra. We have the following isomorphisms:*

$$\text{Ext}_{U_q(\mathfrak{g})}^i(\mathbb{C}(q), U_q(\mathfrak{g})) = 0 \quad \text{for } i \neq \dim \mathfrak{g}, \quad \text{Ext}_{U_q(\mathfrak{g})}^{\dim \mathfrak{g}}(\mathbb{C}(q), U_q(\mathfrak{g})) = \mathbb{C}(q).$$

*If we endow  $\mathbb{C}(q)$  with the trivial representation and  $\text{Ext}_{U_q(\mathfrak{g})}^{\dim \mathfrak{g}}(\mathbb{C}(q), U_q(\mathfrak{g}))$  with right multiplication, the last isomorphism is an isomorphism of right  $U_q(\mathfrak{g})$ -modules. If  $\epsilon$  is in  $\mathbb{C}_{\mathfrak{g}}$ , the same result holds replacing  $U_q(\mathfrak{g})$  by  $U_\epsilon(\mathfrak{g})$  and  $\mathbb{C}(q)$  by  $\mathbb{C}$ .*

Proposition 3.2.1 is already known for Lie algebras [8, Proposition 5.4.1].

**Proof of Proposition 3.2.1.** We do the proof in the case of  $U_q(\mathfrak{g})$  and put  $U = U_q(\mathfrak{g})$ . We endow  $U_q(\mathfrak{g})$  with the filtration of Proposition 3.1.1 and we put

$$\text{Gr } U = \mathcal{C}, \quad \mathcal{B} = \mathcal{C} / \langle E_{\beta_1}, \dots, E_{\beta_N}, F_{\beta_1}, \dots, F_{\beta_N} \rangle.$$



Considering  $\mathcal{B}$  as a  $\mathcal{C} \otimes \mathcal{B}^{\text{op}}$ -bimodule, one has the following isomorphisms:

$$\text{RHom}_{\mathcal{C}}(\mathbb{C}(q), \mathcal{C}) \simeq \text{RHom}_{\mathcal{C}}\left(\mathcal{B} \otimes_{\mathcal{B}} \mathbb{C}(q), \mathcal{C}\right) \simeq \text{RHom}_{\mathcal{B}}(\mathbb{C}(q), \text{RHom}_{\mathcal{C}}(\mathcal{B}, \mathcal{C})).$$

The canonical arrow

$$\text{RHom}_{\mathcal{B}}(\mathbb{C}(q), \mathcal{B}) \overset{L}{\otimes}_{\mathcal{B}} \text{RHom}_{\mathcal{C}}(\mathcal{B}, \mathcal{C}) \rightarrow \text{RHom}_{\mathcal{B}}(\mathbb{C}(q), \text{RHom}_{\mathcal{C}}(\mathcal{B}, \mathcal{C}))$$

is an isomorphism because  $\mathbb{C}(q)$  is a finitely presented  $\mathcal{B}$ -module. Hence, we get the following isomorphism:

$$\text{RHom}_{\mathcal{C}}(\mathbb{C}(q), \mathcal{C}) \simeq \text{RHom}_{\mathcal{B}}(\mathbb{C}(q), \mathcal{B}) \overset{L}{\otimes}_{\mathcal{B}} \text{RHom}_{\mathcal{C}}(\mathcal{B}, \mathcal{C}).$$

But  $\text{RHom}_{\mathcal{C}}(\mathcal{B}, \mathcal{C})$  is a complex concentrated in one degree namely the degree  $2N$  where it is a  $\mathcal{B}$ -module of rank one (see Proposition 5.0.7 of Appendix A) and  $\text{RHom}_{\mathcal{B}}(\mathbb{C}(q), \mathcal{B})$  is a one-dimensional  $\mathbb{C}(q)$ -module concentrated in degree  $n$  where it is  $\mathcal{B}/\langle K_1 - 1, \dots, K_n - 1 \rangle$ . From this computation, we deduce the following isomorphisms:

$$\text{Ext}_{\mathcal{C}}^i(\mathbb{C}(q), \mathcal{C}) = 0 \quad \text{for } i \neq \dim \mathfrak{g}, \quad \text{Ext}_{\mathcal{C}}^{\dim \mathfrak{g}}(\mathbb{C}(q), \mathcal{C}) = \mathbb{C}(q).$$

Endow  $\mathbb{C}(q)$  with the obvious good filtration. Consider a resolution of  $\mathbb{C}(q)$  [3, p. 45],

$$\dots \rightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \rightarrow \mathbb{C}(q) \rightarrow 0$$

by free  $U$ -modules equipped with a filtration  $(\Gamma_v F_j)_{v \in \mathbb{Z}^{2N+1}}$  such that

- $\text{Gr } F_j$  are free  $\mathcal{C}$ -modules,
- $d$  preserves the filtration,
- $\dots \rightarrow \text{Gr}^{\Gamma} F_2 \xrightarrow{\text{Gr } d_2} \text{Gr}^{\Gamma} F_1 \xrightarrow{\text{Gr } d_1} \text{Gr}^{\Gamma} F_0 \rightarrow \mathbb{C}(q) \rightarrow 0$  is a resolution of the  $\mathcal{C}$ -module  $\mathbb{C}(q)$ .

Put  $F_j^* = \text{Hom}_U(F_j, U)$ . Endow  $F_j^*$  with the following filtration:

$$\phi \in \Gamma_v^* F_j^* \Leftrightarrow \phi(\Gamma_k F_j) \subset \Sigma_{k+v} U.$$

Put  $d_{-j}^* = \text{Hom}_U(d_j, U)$ . The cohomology of the complex  $(F_{\bullet}^*, d^*)$  computes the  $\text{Ext}_U^j(\mathbb{C}(q), U)$ 's. The cohomology of the associated graded complex  $(\text{Gr } F_{\bullet}^*, \text{Gr } d^*)$  computes the  $\text{Ext}_{\mathcal{C}}^j(\mathbb{C}(q), \mathcal{C})$ 's (see [3, p. 72]). We filter  $\text{Ext}_U^j(\mathbb{C}(q), U)$  by

$$\Gamma_v^* \text{Ext}_U^j(\mathbb{C}(q), U) = \frac{\text{Ker } d_j^* \cap \Gamma_v^*(F_j^*) + d^*(F_{j-1}^*)}{d^*(F_{j-1}^*)}.$$

We know (see [3, p. 72] and the proof of [10, Proposition 2.1.7] applied to the complex  $(F_\bullet^*, d^*)$ ) that the  $\mathcal{C}$ -module  $\text{Gr}^{\Gamma^*} \text{Ext}_U^j(\mathbb{C}(q), U)$  is a subfactor of the  $\mathcal{C}$ -module  $\text{Ext}_{\mathcal{C}}^j(\mathbb{C}(q), \mathcal{C})$ . It follows from the previous computation that

$$\text{Ext}_U^i(\mathbb{C}(q), U) = 0 \quad \text{for } i \neq \dim \mathfrak{g}, \quad \text{Ext}_U^{\dim \mathfrak{g}}(\mathbb{C}(q), U) = \mathbb{C}(q) \quad \text{or } 0.$$

As  $D_U^2(\mathbb{C}(q)) = \mathbb{C}(q)$ , one has  $\text{Ext}_U^{\dim \mathfrak{g}}(\mathbb{C}(q), U) = \mathbb{C}(q)$ . Note that as  $K_i$  acts by 1 on  $\text{Ext}_{\mathcal{C}}^{\dim \mathfrak{g}}(\mathbb{C}(q), \mathcal{C}) = \mathbb{C}(q)$ , it also acts by 1 on  $\text{Ext}_U^{\dim \mathfrak{g}}(\mathbb{C}(q), U)$ . Hence, if we endow  $\mathbb{C}(q)$  with the trivial representation and  $\text{Ext}_U^{\dim \mathfrak{g}}(\mathbb{C}(q), U)$  with right multiplication, the last isomorphism is an isomorphism of right  $U$ -modules.  $\square$

Proposition 3.2.1 allows to prove Poincaré duality for quantum groups.

**Corollary 3.2.2.** *Let  $\mathfrak{g}$  be a semi-simple Lie algebra of finite dimension. Let  $M$  be an  $U_q(\mathfrak{g})$ -module.  $\mathbb{C}(q)$  can be considered as a left and as a right  $U_q(\mathfrak{g})$ -module. For all  $i$  in  $\mathbb{N}$ , we have an isomorphism*

$$\text{Tor}_i^{U_q(\mathfrak{g})}(\mathbb{C}(q), M) \simeq \text{Ext}_{U_q(\mathfrak{g})}^{\dim \mathfrak{g} - i}(\mathbb{C}(q), M).$$

The same result holds for  $U_\epsilon(\mathfrak{g})$  with  $\epsilon$  in  $\mathbb{C}_\mathfrak{g}$ .

**Proof.** We prove the proposition for  $U_q(\mathfrak{g})$ . The  $U_q(\mathfrak{g})$ -module  $\mathbb{C}(q)$  admitting a finite presentation, the canonical arrow

$$\text{RHom}_{U_q(\mathfrak{g})}(\mathbb{C}(q), U_q(\mathfrak{g})) \overset{L}{\otimes}_{U_q(\mathfrak{g})} M \rightarrow \text{RHom}_{U_q(\mathfrak{g})}(\mathbb{C}(q), M)$$

is an isomorphism. The proposition follows from Proposition 3.2.1.  $\square$

Proposition 3.2.1 allows also to prove duality properties existing for Lie algebras [8] in the quantum groups context.

**Corollary 3.2.3.** *Let  $M$  be a finite-dimensional  $U_q(\mathfrak{g})$ -module. Then  $D_{U_q(\mathfrak{g})}(M)$  and  $M^* = \text{Hom}_{\mathbb{C}(q)}(M, \mathbb{C}(q))$  are isomorphic as left  $U_q(\mathfrak{g})$ -modules. A similar result holds for  $U_\epsilon(\mathfrak{g})$  if  $\epsilon$  is in  $\mathbb{C}_\mathfrak{g}$ .*

**Proof.** We have the following sequence of isomorphisms: let  $L_\bullet$  be a free resolution of  $\mathbb{C}(q)$ ,

$$\begin{aligned} \text{RHom}_{U_q(\mathfrak{g})}(M, U_q(\mathfrak{g})) &\simeq \text{Hom}_{U_q(\mathfrak{g})}(L_\bullet \otimes M, U_q(\mathfrak{g})) \\ &\simeq \text{Hom}_{\mathbb{C}(q)}(M, \text{Hom}_{U_q(\mathfrak{g})}(L_\bullet, U_q(\mathfrak{g}))) \\ &\simeq M^* \otimes \text{RHom}_{U_q(\mathfrak{g})}(\mathbb{C}(q), U_q(\mathfrak{g})) \\ &\simeq M^*. \quad \square \end{aligned}$$

**Corollary 3.2.4.** *Let  $\mathfrak{h}$  and  $\mathfrak{g}$  be two finite-dimensional semi-simple Lie algebras. Denote by  $U_q(\mathfrak{h})$  and  $U_q(\mathfrak{g})$  the quantum groups constructed from  $\mathfrak{h}$  and  $\mathfrak{g}$  for  $q$  generic. Let  $\phi$  be an algebra morphism from  $U_q(\mathfrak{h})$  to  $U_q(\mathfrak{g})$ . Let  $M$  be a finite-dimensional  $U_q(\mathfrak{h})$ -module. Put  $M^* = \text{Hom}_{\mathbb{C}(q)}(M, \mathbb{C}(q))$ . One has an isomorphism*

$$D_{U_q(\mathfrak{g})} \left( U_q(\mathfrak{g}) \underset{U_q(\mathfrak{h})}{\overset{L}{\otimes}} M \right) \simeq U_q(\mathfrak{g}) \underset{U_q(\mathfrak{h})}{\overset{L}{\otimes}} M^*[\dim \mathfrak{g} - \dim \mathfrak{h}].$$

We have a similar result for  $U_\epsilon(\mathfrak{h})$  and  $U_\epsilon(\mathfrak{g})$  the specialization of  $U_q(\mathfrak{h})$  and  $U_q(\mathfrak{g})$  respectively at  $\epsilon$  in  $\mathbb{C}_\mathfrak{h}$ .

**Proof.** We have the following isomorphism in  $D^b(U_q(\mathfrak{g}))$ :

$$\begin{aligned} \text{RHom}_{U_q(\mathfrak{g})} \left( U_q(\mathfrak{g}) \underset{U_q(\mathfrak{h})}{\overset{L}{\otimes}} M, U_q(\mathfrak{g}) \right) &\simeq U_q(\mathfrak{g}) \underset{U_q(\mathfrak{h})}{\overset{L}{\otimes}} D_{U_q(\mathfrak{h})}(M)[- \dim \mathfrak{h}] \\ &\simeq U_q(\mathfrak{g}) \underset{U_q(\mathfrak{h})}{\overset{L}{\otimes}} M^*[- \dim \mathfrak{h}]. \quad \square \end{aligned}$$

**Corollary 3.2.5.** *Let  $\mathfrak{h}$ ,  $\mathfrak{t}$ , and  $\mathfrak{g}$  be three finite-dimensional semi-simple Lie algebras. Denote by  $U_q(\mathfrak{h})$ ,  $U_q(\mathfrak{t})$ , and  $U_q(\mathfrak{g})$  the quantum groups constructed from  $\mathfrak{h}$ ,  $\mathfrak{t}$ , and  $\mathfrak{g}$ , respectively, for  $q$  generic. Let  $\phi$  be an algebra morphism from  $U_q(\mathfrak{h})$  to  $U_q(\mathfrak{g})$  and  $\psi$  an algebra morphism from  $U_q(\mathfrak{t})$  to  $U_q(\mathfrak{g})$ . Let  $M$  be a finite-dimensional  $U_q(\mathfrak{h})$ -module and let  $N$  be a finite-dimensional  $U_q(\mathfrak{t})$ -module. Put  $M^* = \text{Hom}_{\mathbb{C}(q)}(M, \mathbb{C}(q))$  and  $N^* = \text{Hom}_{\mathbb{C}(q)}(N, \mathbb{C}(q))$ . For all  $n$  in  $\mathbb{Z}$ , there is an isomorphism*

$$\begin{aligned} \text{Ext}_{U_q(\mathfrak{g})}^{n+\dim \mathfrak{h}} \left( U_q(\mathfrak{g}) \underset{U_q(\mathfrak{h})}{\overset{L}{\otimes}} M, U_q(\mathfrak{g}) \underset{U_q(\mathfrak{t})}{\overset{L}{\otimes}} N \right) \\ \simeq \text{Ext}_{U_q(\mathfrak{g})}^{n+\dim \mathfrak{t}} \left( U_q(\mathfrak{g}) \underset{U_q(\mathfrak{t})}{\overset{L}{\otimes}} N^*, U_q(\mathfrak{g}) \underset{U_q(\mathfrak{h})}{\overset{L}{\otimes}} M^* \right). \end{aligned}$$

We have a similar result for  $U_\epsilon(\mathfrak{h})$ ,  $U_\epsilon(\mathfrak{t})$ , and  $U_\epsilon(\mathfrak{g})$  the specialization of  $U_q(\mathfrak{h})$  and  $U_q(\mathfrak{g})$ , respectively, at  $\epsilon$  in  $\mathbb{C}_\mathfrak{h} \cap \mathbb{C}_\mathfrak{t}$ .

**Proof.** We have the following sequence of isomorphisms:

$$\begin{aligned} \text{Hom}_{D(U_q(\mathfrak{g}))} \left( U_q(\mathfrak{g}) \underset{U_q(\mathfrak{h})}{\overset{L}{\otimes}} M, U_q(\mathfrak{g}) \underset{U_q(\mathfrak{t})}{\overset{L}{\otimes}} N[n] \right) \\ \simeq \text{Hom}_{D(U_q(\mathfrak{g}))} \left( D_{U_q(\mathfrak{g})} \left( U_q(\mathfrak{g}) \underset{U_q(\mathfrak{h})}{\overset{L}{\otimes}} N \right), D_{U_q(\mathfrak{g})} \left( U_q(\mathfrak{g}) \underset{U_q(\mathfrak{t})}{\overset{L}{\otimes}} M \right)[n] \right) \\ \simeq \text{Hom}_{D(U_q(\mathfrak{g}))} \left( U_q(\mathfrak{g}) \underset{U_q(\mathfrak{h})}{\overset{L}{\otimes}} N^*, U_q(\mathfrak{g}) \underset{U_q(\mathfrak{t})}{\overset{L}{\otimes}} M^*[n + \dim \mathfrak{t} - \dim \mathfrak{h}] \right). \end{aligned}$$

Note that Corollary 3.2.4 is used to get the last isomorphisms. The corollary follows.  $\square$

Corollary 3.2.5 was proved for any Lie algebra in [8]. Particular cases of Corollary 3.2.5 (for Lie algebras) can be found in [4,11,14,15].

3.3. Computation of the rigid dualizing complex of  $U_q(\mathfrak{g})$  and  $U_\epsilon(\mathfrak{g})$

Let  $\mathfrak{g}$  be a finite-dimensional semi-simple complex Lie algebra. We identify  $U_q(\mathfrak{g} \oplus \mathfrak{g})$  with  $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ . It has a structure of  $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g} \oplus \mathfrak{g})^{\text{op}}$ -module described as follows: for all  $\alpha, \beta, \gamma, a, b$  in  $U_q(\mathfrak{g})$ ,

$$\alpha \cdot (a \otimes b) = (\Delta\alpha)(a \otimes b), \quad (a \otimes b)(\beta \otimes \gamma) = a\beta \otimes b\gamma.$$

We have the same structure on  $U_\epsilon(\mathfrak{g})$ .

**Lemma 3.3.1.**

(a) The  $U_q(\mathfrak{g})^{\text{op}} \otimes U_q(\mathfrak{g})^{\text{op}}$ -module  $\mathbb{C}(q) \otimes_{U_q(\mathfrak{g})} U_q(\mathfrak{g} \oplus \mathfrak{g})$  is isomorphic to  $U_q(\mathfrak{g})$  endowed with the following  $U_q(\mathfrak{g})^{\text{op}} \otimes U_q(\mathfrak{g})^{\text{op}}$ -module structure:

$$u \cdot (\alpha \otimes \beta) = S(\alpha)u\beta.$$

(b)  $U_q(\mathfrak{g} \oplus \mathfrak{g}) \otimes_{U_q(\mathfrak{g})} \mathbb{C}(q) \simeq (U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})) \otimes_{U_q(\mathfrak{g})} \mathbb{C}(q)$  has a  $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})^{\text{op}}$ -module structure defined by

$$\forall \alpha, \beta, u, v \in U_q(\mathfrak{g}), \quad \alpha \cdot (u \otimes v \otimes 1) \cdot \beta = \alpha u \otimes S(\beta)v \otimes 1$$

and so is isomorphic to the  $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})^{\text{op}}$ -module  $U_q(\mathfrak{g})$ .

The same result holds for  $U_\epsilon(\mathfrak{g})$  if  $\epsilon$  is in  $\mathbb{C}_{\mathfrak{g}}$ .

Lemma 3.3.1(a) follows from Lemma 1.1(a). Lemma 3.3.1(b) follows from Lemma 1.1(b).

**Theorem 3.3.2.** Let  $\mathfrak{g}$  be a complex finite-dimensional semi-simple Lie algebra. Assume that  $\epsilon$  is in  $\mathbb{C}_{\mathfrak{g}}$ . The rigid dualizing complex of  $U_q(\mathfrak{g})$  and  $U_\epsilon(\mathfrak{g})$  are  $U_q(\mathfrak{g})[\dim \mathfrak{g}]$  and  $U_\epsilon(\mathfrak{g})[\dim \mathfrak{g}]$ , respectively.

**Proof.** We prove the theorem for  $U_q(\mathfrak{g})$ . We write  $U$  for  $U_q(\mathfrak{g})$ . As  $U$  has finite homological dimension,  $U$  and  $U[\dim \mathfrak{g}]$  are dualizing complexes. We have the following sequence of isomorphisms in  $D^b(U \otimes U^{\text{op}})$ :

$$\begin{aligned} \text{RHom}_{U \otimes U^{\text{op}}}(U_U, U \otimes U_U) &\simeq \text{RHom}_{U^{\text{op}} \otimes U^{\text{op}}}(U_{U \otimes U}, U_U \otimes U_U) \\ &\simeq \text{RHom}_{U^{\text{op}} \otimes U^{\text{op}}}\left(\mathbb{C}(q) \otimes_U (U \otimes U), U_U \otimes U_U\right) \\ &\simeq (U \otimes U) \otimes_U \text{RHom}_{U^{\text{op}}}(\mathbb{C}(q), U) \end{aligned}$$

$$\begin{aligned} &\simeq (U \otimes U) \otimes_U \mathbb{C}(q)[- \dim \mathfrak{g}] \\ &\simeq U[- \dim \mathfrak{g}]. \end{aligned}$$

The first isomorphism follows from the identification of  $U$  and  $U^{\text{op}}$  through  $S$  in the first summand of  $U \otimes U^{\text{op}}$  and  $U \otimes U$ , respectively.  $U \otimes U$  is then endowed with the right  $U \otimes U$ -module structure given by right multiplication and with the  $U \otimes U^{\text{op}}$ -module structure defined by

$$\forall \alpha, \beta, u, v \in U, \quad \alpha \cdot (u \otimes v \otimes 1) \cdot \beta = \alpha u \otimes S(\beta)v \otimes 1.$$

The second isomorphism follows from Lemma 3.3.1(a). The fourth isomorphism follows from Proposition 3.2.1. The last isomorphism follows from Lemma 3.3.1(b).

Shifting our arrows by  $[2 \dim \mathfrak{g}]$ , one gets that  $U[\dim \mathfrak{g}]$  is the rigid dualizing complex of  $U$ . This finishes the proof of the theorem.  $\square$

As in [34], using a result of van den Bergh [29], we get a corollary of Theorem 3.3.2 linking Hochschild cohomology and homology in a kind of Poincaré duality. Let  $M$  be any  $U_q(\mathfrak{g})^e$ -module. Denote by  $H^p(U_q(\mathfrak{g}), M)$  and  $H_p(U_q(\mathfrak{g}), M)$  the Hochschild cohomology and homology, respectively. We adopt the same notation for  $U_\epsilon(\mathfrak{g})$ .

### Corollary 3.3.3.

(a) *There are  $U_q(\mathfrak{g})^e$ -module isomorphisms*

$$H^{\dim \mathfrak{g}}(U_q(\mathfrak{g}), U_q(\mathfrak{g})^e) = U_q(\mathfrak{g}), \quad H^p(U_q(\mathfrak{g}), U_q(\mathfrak{g})^e) = 0 \quad \text{if } p \neq \dim \mathfrak{g}.$$

(b) *Let  $M$  be any  $U_q(\mathfrak{g})^e$ -module, we have an isomorphism*

$$H^p(U_q(\mathfrak{g}), M) \simeq H_{\dim \mathfrak{g} - p}(U_q(\mathfrak{g}), M).$$

*The same results hold for  $U_\epsilon(\mathfrak{g})$  if  $\epsilon$  is in  $\mathbb{C}_{\mathfrak{g}}$ .*

(a) is a direct consequence of the proof of Theorem 3.3.2. (b) is a corollary of [29, Theorem 1].

## 4. Rigid dualizing complex for the algebra of differential operators defined by an affine Lie algebroid

### 4.1. Definitions

Let  $X$  be a complex smooth affine variety and let  $\mathcal{O}_X$  be the sheaf of regular functions on  $X$ . Let  $\Theta_X$  be the  $\mathcal{O}_X$ -module of regular vector fields on  $X$ . We put  $G_X = \mathcal{O}_X(X)$  and  $L_X = \mathcal{L}_X(X)$ .

**Definition 4.1.1.** A sheaf of Lie algebras,  $\mathcal{L}_X$ , is a sheaf of  $\mathbb{C}$ -vector spaces such that for any open subset  $U$ ,  $\mathcal{L}_X(U)$  is equipped with a Lie bracket compatible with the restriction morphisms.

A morphism between two sheaves of Lie algebras  $\mathcal{L}_X$  and  $\mathcal{M}_X$  is a  $\mathcal{O}_X$ -module morphism which is a Lie algebra morphism on each open subset.

**Definition 4.1.2.** A Lie algebroid over  $X$  is a pair  $(\mathcal{L}_X, \omega)$  where

- $\mathcal{L}_X$  is a locally free  $\mathcal{O}_X$ -module,
- $\mathcal{L}_X$  is a sheaf of Lie algebras,
- $\omega$  is a  $\mathcal{O}_X$ -linear morphism of sheaves of Lie algebras from  $\mathcal{L}_X$  to  $\Theta_X$  such that the following compatibility relation holds:

$$\forall(\xi, \zeta) \in \mathcal{L}_X^2, \forall f \in \mathcal{O}_X, \quad [\xi, f\zeta] = \omega(\xi)(f)\zeta + f[\xi, \zeta].$$

One calls  $\omega$  the anchor map. When there is no ambiguity, we will drop the anchor map in the notation of the Lie algebroid. Note that  $(G_X, L_X)$  is a Lie–Rinehart algebra [23].

A Lie algebroid  $(\mathcal{L}_X, \omega)$  gives rise to a sheaf of generalized differential operators. We will denote by  $\mathcal{D}(\mathcal{L}_X)$  the sheaf associated to the presheaf,

$$U \mapsto T_{\mathbb{C}}^+(\mathcal{O}_X(U) \oplus \mathcal{L}_X(U))/J_U,$$

where  $J_U$  is the two sided ideal generated by the relations

$$\forall(f, g) \in \mathcal{O}_X(U), \forall(\xi, \zeta) \in \mathcal{L}_X(U)^2,$$

- (1)  $f \otimes g = fg,$
- (2)  $f \otimes \xi = f\xi,$
- (3)  $\xi \otimes \zeta - \zeta \otimes \xi = [\xi, \zeta],$
- (4)  $\xi \otimes f - f \otimes \xi = \omega(\xi)(f).$

**Definition 4.1.3.** Let  $(\mathcal{L}_X, \omega_X)$  and  $(\mathcal{L}_Y, \omega_Y)$  be Lie algebroids over  $X$  and  $Y$ , respectively. A morphism  $\Phi$  from  $(\mathcal{L}_X, \omega_X)$  to  $(\mathcal{L}_Y, \omega_Y)$  is a pair  $(f, F)$  such that

- $f: X \rightarrow Y$  is an algebraic map,
- $F: \mathcal{L}_X \rightarrow f^*\mathcal{L}_Y = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{L}_Y$  is an  $\mathcal{O}_X$ -module morphism such that the two following conditions are satisfied:

(1) The diagram

$$\begin{array}{ccc} \mathcal{L}_X & \xrightarrow{F} & f^*\mathcal{L}_Y \\ \omega_X \downarrow & & \downarrow f^*\omega_Y \\ \Theta_X & \xrightarrow{Tf} & f^*\Theta_Y \end{array}$$

commutes (where  $Tf$  is the differential of  $f$ ).

(2)  $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} (f^{-1}\mathcal{D}(\mathcal{L}_Y))$  endowed with the two following operations:

$$\begin{aligned} \forall (a, b) \in \mathcal{O}_X^2, \forall \xi \in \mathcal{L}_X, \forall v \in f^{-1}\mathcal{D}(\mathcal{L}_Y), \\ a \cdot (b \otimes v) = ab \otimes v, \quad \xi \cdot (b \otimes v) = \omega_{\mathcal{L}_X}(\xi)(b) \otimes v + \sum_i ba_i \otimes \xi_i v \end{aligned}$$

(where  $F(\xi) = \sum_i a_i \otimes \xi_i$  with  $a_i$  in  $\mathcal{O}_X$  and  $\xi_i$  in  $f^{-1}\mathcal{L}_Y$ ) is a left  $\mathcal{D}(\mathcal{L}_X)$ -module.

Note that condition (2) is equivalent to the following more explicit condition. Let  $\xi$  and  $\eta$  be two elements of  $\mathcal{L}_X^2$ . Put  $F(\xi) = \sum_{i=1}^m a_i \otimes \xi_i$  and  $F(\eta) = \sum_{j=1}^m b_j \otimes \eta_j$ , then

$$F([\xi, \eta]) = \sum_{j=1}^n \omega_X(\xi)(b_j) \otimes \eta_j - \sum_{i=1}^n \omega_X(\eta)(a_i) \otimes \xi_i + \sum_{i,j} a_i b_j \otimes [\xi_i, \eta_j].$$

Our definition coincides with that of Almeida and Kumpera [1]. For examples of Lie algebroids and of Lie algebroid morphisms, see [8–10].

Let  $(\mathcal{L}_X, \omega_X)$  and  $(\mathcal{L}_Y, \omega_Y)$  be Lie algebroids over the smooth algebraic varieties  $X$  and  $Y$ , respectively. Let  $p_1$  and  $p_2$  be the projections from  $X \times Y$  over  $X$  and  $Y$ , respectively. The  $\mathcal{O}_{X \times Y}$ -module

$$\mathcal{L}_{X \times Y} = \mathcal{O}_{X \times Y} \otimes_{p_1^{-1}\mathcal{O}_X} p_1^{-1}\mathcal{L}_X \oplus \mathcal{O}_{X \times Y} \otimes_{p_2^{-1}\mathcal{O}_Y} p_2^{-1}\mathcal{L}_Y$$

is endowed with a natural Lie algebroid structure over  $X \times Y$ . The anchor map of  $\mathcal{L}_{X \times Y}$  is  $\omega_{X \times Y} = \mathcal{L}_{X \times Y} \rightarrow \Theta_{X \times Y}$  determined by

$$\begin{aligned} \forall (f, g) \in \mathcal{O}_X \times \mathcal{O}_Y, \forall (\xi, \zeta) \in \mathcal{L}_X \times \mathcal{L}_Y, \\ \omega_{X \times Y}(\xi)(f \otimes g) = \omega_X(\xi)(f)g, \quad \omega_{X \times Y}(\zeta)(f \otimes g) = f\omega_Y(\zeta)(g). \end{aligned}$$

On  $\mathcal{L}_{X \times Y}$ , we put the Lie bracket extending that of  $\mathcal{L}_X$  and  $\mathcal{L}_Y$  and satisfying the following relations  $[p_1^{-1}\mathcal{L}_X, p_2^{-1}\mathcal{L}_Y] = 0$ .

Let  $(\mathcal{L}_X, \omega_X)$  be a Lie algebroid. In the following we will make use of the diagonal embedding  $\mathcal{V}_{\mathcal{L}_X} = (v, V)$  from  $(X, \mathcal{L}_X)$  to  $(X \times X, \mathcal{L}_{X \times X})$ . One has

$$v: X \rightarrow X \times X, \quad x \mapsto (x, x),$$

$$V: \mathcal{L}_X \rightarrow \mathcal{O}_X \otimes_{v^{-1}\mathcal{O}_{X \times X}} v^{-1}\mathcal{L}_{X \times X} \simeq \mathcal{L}_X \oplus \mathcal{L}_X, \quad D \mapsto D + D.$$

4.2. Left and right-modules

The following proposition is classical for  $\mathcal{D}$ -modules and is easy to generalize to Lie algebroids.

**Proposition 4.2.1.**

(a) If  $\mathcal{M}$  (respectively  $\mathcal{N}$ ) is a right (respectively a left)  $\mathcal{D}(\mathcal{L}_X)$ -module, then  $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$  endowed with the two following operations:

$$\forall a \in \mathcal{O}_X, \forall m \in \mathcal{M}, \forall n \in \mathcal{N}, \forall D \in \mathcal{L}_X,$$

$$(m \otimes n) \cdot a = m \otimes a \cdot n = m \cdot a \otimes n, \quad (m \otimes n) \cdot D = m \cdot D \otimes n - m \otimes D \cdot n$$

is a right  $\mathcal{D}(\mathcal{L}_X)$ -module.

(b) If  $\mathcal{M}$  and  $\mathcal{M}'$  are two right  $\mathcal{D}(\mathcal{L}_X)$ -modules, then  $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{M}')$  endowed with the two following operations:

$$\forall \phi \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{M}'), \forall m \in \mathcal{M}, \forall a \in \mathcal{O}_X, \forall D \in \mathcal{L}_X,$$

$$(\phi \cdot a)(m) = \phi(m) \cdot a, \quad (\phi \cdot D)(m) = \phi(m) \cdot D - \phi(m \cdot D)$$

is a left  $\mathcal{D}(\mathcal{L}_X)$ -module.

The following theorem is now a consequence of the previous proposition.

**Theorem 4.2.2.** Let  $\mathcal{E}$  be right  $\mathcal{D}(\mathcal{L}_X)$ -module which is a locally free  $\mathcal{O}_X$ -module of rank one. The functor  $\mathcal{N}^\bullet \mapsto \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{N}^\bullet$  establishes an equivalence of categories between complexes of left and complexes of right  $\mathcal{D}(\mathcal{L}_X)$ -modules. Its inverse functor is given by  $\mathcal{M} \mapsto \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{M})$ .

It is a well-known fact that  $\Omega_X$  is endowed with a right  $\mathcal{D}_X$ -module structure (see [24, p. 9], [5, p. 226]).

Consider

$$\mathcal{K}_{\mathcal{L}_X} = \Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}(\mathcal{L}_X).$$

It is endowed with a natural  $\mathcal{D}(\mathcal{L}_X)^{op} \otimes \mathcal{D}(\mathcal{L}_X)^{op}$ -module (the first right  $\mathcal{D}(\mathcal{L}_X)$ -module structure is given by right multiplication, the second one is obtained from left multiplication by the theorem above).



Let  $\mathcal{N}^\bullet$  be an element of  $D_{\text{coh}}^b(\mathcal{D}(\mathcal{L}_X)^{\text{op}})$ , then we set

$$\underline{\Delta}_{\mathcal{L}_X}(\mathcal{N}^\bullet) = R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)^{\text{op}}}(\mathcal{N}^\bullet, \mathcal{K}_{\mathcal{L}_X})[x].$$

The natural arrow  $\mathcal{N}^\bullet \mapsto \underline{\Delta}_{\mathcal{L}_X}(\underline{\Delta}_{\mathcal{L}_X}(\mathcal{N}^\bullet))$  is an isomorphism. That is why  $\underline{\Delta}_{\mathcal{L}_X}$  is called duality functor.

**Proposition 4.2.3.** *If  $\mathcal{N}$  is a right  $\mathcal{D}(\mathcal{L}_X)$ -module which is locally free of finite rank as  $\mathcal{O}_X$ -module, then*

$$\underline{\Delta}_{\mathcal{L}_X}(\mathcal{N}) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \Omega_X) \otimes_{\mathcal{O}_X} \Lambda^{d_{\mathcal{L}_X}} \mathcal{L}_X[x - d_{\mathcal{L}_X}]$$

#### 4.3. Direct image

Let  $(X, \mathcal{L}_X)$  and  $(Y, \mathcal{L}_Y)$  be two Lie algebroids over two smooth algebraic varieties  $X$  and  $Y$ , respectively. Let  $\Phi = (f, F)$  be a Lie algebroid morphism from  $(X, \mathcal{L}_X)$  to  $(Y, \mathcal{L}_Y)$ . Then  $\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}(\mathcal{L}_Y)$  has a  $\mathcal{D}(\mathcal{L}_X) \otimes f^{-1}\mathcal{D}(\mathcal{L}_Y)^{\text{op}}$  module structure. It is called the transfer module of  $\Phi$  and is denoted  $\mathcal{D}_{\mathcal{L}_X \rightarrow \mathcal{L}_Y}$ . One can also define the following  $f^{-1}\mathcal{D}(\mathcal{L}_Y) \otimes \mathcal{D}(\mathcal{L}_X)^{\text{op}}$ -module:

$$\mathcal{D}_{\mathcal{L}_Y \leftarrow \mathcal{L}_X} = \Lambda^{d_{\mathcal{L}_X}}(\mathcal{L}_X^*) \otimes_{\mathcal{O}_X} \mathcal{D}_{\mathcal{L}_X \rightarrow \mathcal{L}_Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Lambda^{d_{\mathcal{L}_Y}}(\mathcal{L}_Y).$$

Let  $\mathcal{N}^\bullet$  be an object of  $D^b(\mathcal{D}(\mathcal{L}_X)^{\text{op}})$ . Set

$$\underline{\Phi}_!(\mathcal{N}^\bullet) = Rf_! \left( \mathcal{N}^\bullet \otimes_{\mathcal{D}(\mathcal{L}_X)}^L \mathcal{D}_{\mathcal{L}_X \rightarrow \mathcal{L}_Y} \right).$$

If  $\Phi = (f, Tf)$ , we recover the  $\mathcal{D}$ -module construction (see [24] for example). Then  $\mathcal{D}_{\mathcal{O}_X \rightarrow \mathcal{O}_Y}$  and  $\mathcal{D}_{\mathcal{O}_Y \leftarrow \mathcal{O}_X}$  are denoted  $\mathcal{D}_{X \rightarrow Y}$  and  $\mathcal{D}_{Y \leftarrow X}$ , respectively and  $\underline{\Phi}_!$  is denoted  $\underline{f}_!$ .

**Theorem 4.3.1.** *Let  $X$  and  $Y$  be two complex algebraic smooth manifolds of dimension  $x$  and  $y$ , respectively. Let  $(\mathcal{L}_X, \omega_X)$  and  $(\mathcal{L}_Y, \omega_Y)$  be Lie algebroids over  $X$  and  $Y$ , respectively. Let  $\Phi = (f, F)$  be a Lie algebroid morphism from  $(\mathcal{L}_X, \omega_X)$  to  $(\mathcal{L}_Y, \omega_Y)$ . Let  $\mathcal{N}^\bullet$  be an element of  $D_{\text{coh}}^b(\mathcal{D}(\mathcal{L}_X)^{\text{op}})$  such that  $f$  is proper on the support of  $\mathcal{N}^\bullet$ . Then there is a functorial isomorphism from  $\underline{\Phi}_! \Delta_{\mathcal{L}_X}(\mathcal{N}^\bullet)$  to  $\Delta_{\mathcal{L}_Y} \underline{\Phi}_!(\mathcal{N}^\bullet)$  in  $D^b(\mathcal{D}(\mathcal{L}_Y)^{\text{op}})$ .*

#### Remarks.

- (1) Theorem 4.3.1 is proved in [9] and generalizes Schneiders' thesis [25–27] where the case of relative differential operators is treated. In the  $\mathcal{D}$ -modules context, the case of a proper morphism was previously treated by Bernstein [2,5,17]. Moreover, Mebkhout treated the absolute case (i.e.,  $Y$  consists in a point) in [21,22].
- (2) If  $\mathcal{L}_X = \mathcal{L}_Y = \{0\}$ , we recover Verdier duality.

4.4. Rigid dualizing complex for the algebra of differential operators defined by an affine Lie algebroid

By Proposition 4.2.1,  $\mu_{\mathcal{L}_X} = \mathcal{H}om_{\mathcal{O}_X}(\Lambda^{d_{\mathcal{L}_X}} \mathcal{L}_X^*, \Omega_X)$  has a left module structure. Hence  $\mathcal{D}(\mathcal{L}_X) \otimes_{\mathcal{O}_X} \mu_{\mathcal{L}_X}$  is a  $\mathcal{D}(\mathcal{L}_X) \otimes \mathcal{D}(\mathcal{L}_X)^{op}$ -module.

**Theorem 4.4.1.** *Let  $(X, \mathcal{L}_X)$  be a Lie algebroid over a smooth affine variety. Put  $x = \dim X$ ,  $d_{\mathcal{L}_X} = \text{rank}(\mathcal{L}_X)$ ,  $G_X = \mathcal{O}_X(X)$ ,  $L_X = \mathcal{L}_X(X)$ ,  $\omega_X = \Omega_X(X)$ ,  $D(L_X) = \mathcal{D}(\mathcal{L}_X)(X)$ . The rigid dualizing complex of  $D(L_X)$  is*

$$R_{L_X} = D(L_X) \otimes_{G_X} \text{Hom}_{G_X}(\Lambda^{d_{\mathcal{L}_X}} L_X^*, \omega_X)[x + d_{L_X}].$$

This theorem was proved in the case of enveloping algebras and  $\mathcal{D}$ -modules in [34].

**Proof of Theorem 4.4.1.** The proof of the theorem is analogous to the  $\mathcal{D}$ -module case (see [34]). We will make use of the two following lemmas.

**Lemma 4.4.2.** *Let  $\mathcal{E}$  be a right  $\mathcal{D}(\mathcal{L}_X)$  which is a locally free  $\mathcal{O}_X$ -module. There are two right  $\mathcal{D}(\mathcal{L}_X)$ -module structures on  $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}(\mathcal{L}_X)$ . The first one is given by right multiplication. The second one is obtained from left multiplication using Proposition 4.2.1(a). There is an involution exchanging these two structures.*

The proof of Lemma 4.4.2 can be found in [27, Lemma 2.3].

**Lemma 4.4.3.**  $\mathcal{D}(\mathcal{L}_X) \otimes_{\mathcal{O}_X} \mu_{\mathcal{L}_X}$  is endowed with a  $\mathcal{D}(\mathcal{L}_X) \otimes \mathcal{D}(\mathcal{L}_X)^{op}$ -module structure defined by: for all  $P, Q \in \mathcal{D}(\mathcal{L}_X)$ , all  $D \in \mathcal{L}_X$  and all  $a \in \mathcal{O}_X$ ,

$$\begin{aligned} Q \cdot (P \otimes \mu) &= QP \otimes \mu, \\ (P \otimes \mu) \cdot D &= PD \otimes \mu - P \otimes D \cdot \mu, \\ (P \otimes \mu) \cdot a &= Pa \otimes \mu. \end{aligned}$$

$\mu_{\mathcal{L}_X} \otimes_{\mathcal{O}_X} \mathcal{D}(\mathcal{L}_X)$  is endowed with a  $\mathcal{D}(\mathcal{L}_X) \otimes \mathcal{D}(\mathcal{L}_X)^{op}$ -module structure defined by: for all  $P, Q \in \mathcal{D}(\mathcal{L}_X)$ , all  $D \in \mathcal{L}_X$  and all  $a \in \mathcal{O}_X$ ,

$$\begin{aligned} a \cdot (\mu \otimes P) &= a\mu \otimes P = \mu \otimes aP, \\ D \cdot (\mu \otimes P) &= D \cdot \mu \otimes P + \mu \otimes DP, \\ (\mu \otimes P) \cdot Q &= \mu \otimes PQ. \end{aligned}$$

There is an involution between the  $\mathcal{D}(\mathcal{L}_X) \otimes \mathcal{D}(\mathcal{L}_X)^{op}$ -modules  $\mathcal{D}(\mathcal{L}_X) \otimes_{\mathcal{O}_X} \mu_{\mathcal{L}_X}$  and  $\mu_{\mathcal{L}_X} \otimes_{\mathcal{O}_X} \mathcal{D}(\mathcal{L}_X)$ .

**Proof.** Let  $\mu$  be a local basis of  $\mu_{\mathcal{L}_X}$ . The morphism

$$\mathcal{D}(\mathcal{L}_X) \otimes_{\mathcal{O}_X} \mu_{\mathcal{L}_X} \rightarrow \mu_{\mathcal{L}_X} \otimes_{\mathcal{O}_X} \mathcal{D}(\mathcal{L}_X), \quad P \otimes \mu \mapsto P \cdot_2 (\mu \otimes 1)$$

is well-defined and provides an isomorphism between the  $\mathcal{D}(\mathcal{L}_X) \otimes \mathcal{D}(\mathcal{L}_X)^{\text{op}}$ -module structures.  $\square$

For short, we will write  $\mathcal{V}$  for  $\mathcal{V}_{\mathcal{L}_X}$ .

One sees that

$$\underline{\mathcal{V}}_! (\Lambda^{d_{\mathcal{L}_X}} \mathcal{L}_X^*) = v_! \left( \Lambda^{d_{\mathcal{L}_X}} (\mathcal{L}_X^*) \otimes_{\mathcal{O}_X} \mathcal{D}(\mathcal{L}_X) \right).$$

Moreover, as  $\Delta_{\mathcal{L}_X}(\Omega_X) = \Lambda^{d_{\mathcal{L}_X}} (\mathcal{L}_X^*) [x - d_{\mathcal{L}_X}]$  (see [9, Theorem 3.2.1]) and  $\Delta_{\mathcal{L}_X \times X} \circ \underline{\mathcal{V}}_! \simeq \underline{\mathcal{V}}_! \circ \Delta_{\mathcal{L}_X}$  (see [9, Theorem 4.3.1]), we have an isomorphism of  $\mathcal{D}(\mathcal{L}_X \times X)^{\text{op}}$ -modules

$$\begin{aligned} & v_! \left( \Lambda^{d_{\mathcal{L}_X}} (\mathcal{L}_X^*) \otimes_{\mathcal{O}_X} \mathcal{D}(\mathcal{L}_X) \right) \\ & \simeq R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X \times X)^{\text{op}}} \left( v_! \left( \Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}(\mathcal{L}_X) \right), \Omega_{X \times X} \otimes_{\mathcal{O}_{X \times X}} \mathcal{D}(\mathcal{L}_X \times X) \right) [x + d_{\mathcal{L}_X}]. \end{aligned}$$

Hence the isomorphism

$$\begin{aligned} & \Lambda^{d_{\mathcal{L}_X}} (\mathcal{L}_X^*) \otimes_{\mathcal{O}_X} \mathcal{D}(\mathcal{L}_X) \\ & \simeq R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X) \otimes \mathcal{D}(\mathcal{L}_X)^{\text{op}}} \left( \mathcal{D}(\mathcal{L}_X), \mathcal{D}(\mathcal{L}_X) \otimes \left( \Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}(\mathcal{L}_X) \right) \right) [x + d_{\mathcal{L}_X}]. \end{aligned}$$

Using Lemma 4.4.2, we get

$$\begin{aligned} & \mu_{\mathcal{L}_X}^{-1} \otimes_{\mathcal{O}_X} \mathcal{D}(\mathcal{L}_X) \\ & \simeq R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X) \otimes \mathcal{D}(\mathcal{L}_X)^{\text{op}}} \left( \mathcal{D}(\mathcal{L}_X), {}_{\mathcal{D}(\mathcal{L}_X)} \mathcal{D}(\mathcal{L}_X) \otimes {}_{\mathcal{D}(\mathcal{L}_X)} \mathcal{D}(\mathcal{L}_X) \right) [x + d_{\mathcal{L}_X}]. \end{aligned}$$

Hence an isomorphism

$$\begin{aligned} & \mathcal{D}(\mathcal{L}_X) \otimes_{\mathcal{O}_X} \mu_{\mathcal{L}_X} \\ & \simeq R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X) \otimes \mathcal{D}(\mathcal{L}_X)^{\text{op}}} \left( \mathcal{D}(\mathcal{L}_X), \left( \mathcal{D}(\mathcal{L}_X) \otimes_{\mathcal{O}_X} \mu_{\mathcal{L}_X} \right) \otimes \left( \mu_{\mathcal{L}_X} \otimes_{\mathcal{O}_X} \mathcal{D}(\mathcal{L}_X) \right) \right) [x + d_{\mathcal{L}_X}]. \end{aligned}$$

Set

$$\mathcal{R}_{\mathcal{L}_X} = \mathcal{D}(\mathcal{L}_X) \otimes_{\mathcal{O}_X} \mu_{\mathcal{L}_X} [x + d_{\mathcal{L}_X}].$$

Using Lemma 4.4.3, we get easily the following isomorphism in  $D^b(\mathcal{D}(\mathcal{L}_X))^e$ :

$$\mathcal{R}_{\mathcal{L}_X} \simeq R\mathcal{H}om_{\mathcal{D}(\mathcal{L}_X)^e}(\mathcal{D}(\mathcal{L}_X), \mathcal{R}_{\mathcal{L}_X} \otimes \mathcal{R}_{\mathcal{L}_X}).$$

Applying the exact functor “global section,” we get

$$R_{L_X} \simeq R\mathcal{H}om_{D(L_X)^e}(D(L_X), R_{L_X} \otimes R_{L_X}).$$

Analogues of Corollaries 3.2.2–3.2.5 are already known for Lie algebroids (see [8,9]). As in [34], Theorem 4.4.1 implies a kind of Poincaré duality between Hochschild cohomology and homology.  $\square$

**Corollary 4.4.4.** *Let  $D(L_X)$  be as in Theorem 4.4.1.*

(a) *We have the following isomorphisms of  $D(L_X)^e$ -modules:*

$$H^{x+d_{L_X}}(D(L_X), D(L_X)^e) = \text{Hom}_{G_X}(\omega_X, \Lambda^{d_{L_X}} L_X^*) \otimes_{G_X} D(L_X),$$

$$H^i(D(L_X), D(L_X)^e) = 0 \quad \text{if } i \neq x + d_{L_X}.$$

(b) *For any  $D(L_X)^e$ -module  $M$ , one has an isomorphism*

$$H^i(D(L_X), M) \simeq H_{x+d_{L_X}-i}(D(L_X), \text{Hom}_{G_X}(\omega_X, \Lambda^{d_{L_X}} L_X^*) \otimes_{G_X} M).$$

(a) follows from the proof of Theorem 4.4.1. (b) is a direct application of [29, Theorem 1].

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**Appendix A**

In this appendix, we review results about  $q$ -commutative algebras and their Koszul complexes. The reference for this appendix is [30] where the Koszul complex is constructed in a more general situation. We assume that  $k$  is a field of characteristic zero. Let  $Q = (q_{i,j})_{(i,j) \in [1,n]^2}$  be an  $n \times n$ -matrix such that  $q_{i,i} = 1$  and  $q_{i,j}q_{j,i} = 1$ . Let  $V = \bigoplus_{i=1}^n ke_i$ . Following [30], we define the following graded algebras:

$$S_Q(V) = \frac{T(V)}{\langle e_i \otimes e_j - q_{i,j}e_j \otimes e_i \rangle}, \quad \Lambda_Q(V) = \frac{T(V)}{\langle e_i \otimes e_j + q_{i,j}e_j \otimes e_i \rangle}.$$

One sees easily,

$$\Lambda_Q(V)^* = \Lambda_{Q^*}(V^*).$$

**Lemma 5.0.5.** *Let  $Q = (q_{i,j})_{(i,j) \in [1,n]^2}$  be an  $n \times n$ -matrix such that  $q_{i,i} = 1$  and  $q_{i,j}q_{j,i} = 1$ . Put*

$$Q = \begin{pmatrix} Q_1 & B \\ C & Q_2 \end{pmatrix},$$

where  $Q_1$  and  $Q_2$  are matrices of order  $n_1$  and  $n_2$ , respectively (with  $n = n_1 + n_2$ ). Put  $V_1 = ke_1 \oplus \cdots \oplus ke_{n_1}$  and  $V_2 = ke_{n_1+1} \oplus \cdots \oplus ke_n$ .  $S_Q(V)$  has a natural left  $S_{Q_1}(V_1)$ -module structure and one has the following isomorphisms of left graded  $S_{Q_1}(V_1)$ -modules:

$$S_Q(V) = S_{Q_1}(V_1) \otimes S_{Q_2}(V_2).$$

The same assertion holds for  $\Lambda_Q(V)$ .

**Theorem 5.0.6.**  *$k$  is a  $S_Q(V)$ -module as follows: for all  $\mu$  in  $k$ ,*

$$\forall \lambda \in k, \quad \lambda \cdot \mu = \lambda\mu, \quad \forall v \in V, \quad v \cdot \mu = 0.$$

The complex

$$K_Q^\bullet = (S_Q(V) \otimes \Lambda_Q^\bullet(V), d),$$

where the differential  $d_r : S_Q(V) \otimes \Lambda_Q^r(V) \rightarrow S_Q(V) \otimes \Lambda_Q^{r-1}(V)$  is defined by

$$d_r(m \otimes e_{i_1} \wedge \cdots \wedge e_{i_r}) = \sum_{i=1}^r (-1)^{k+1} \left( \prod_{h=1}^{i-1} q_{i_h i_k} \right) m e_{i_k} \otimes e_{i_1} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_r},$$

is a resolution of the trivial  $S_Q(V)$ -module  $k$ .

The theorem is due to Wambst [30] who proved it in a much more general situation. For the simplified form we are interested in, one can adjust easily the classical proof [18, p. 243]. The resolution  $K_Q^\bullet$  is called the Koszul resolution of  $k$ .

**Proposition 5.0.7.** *One has*

- (a)  $\text{Ext}_{S_Q(V)}^i(k, S_Q(V)) = 0$  if  $i \neq n$ ,
- (b)  $\text{Ext}_{S_Q(V)}^n(k, S_Q(V)) = \Lambda_Q^n(V^*)$ .

Proposition 5.0.7(b) was already proved in [32, Corollary 1.2.2].

**Proof of Proposition 5.0.7.** We study the complex  $C_Q(V) = \text{Hom}_{S_Q(V)}(K_Q^\bullet, S_Q(V))$ . Denote by  $(\lambda_1, \dots, \lambda_n)$  the dual basis of  $(e_1, \dots, e_n)$ .

A computation shows that  $C_Q(V)$  is isomorphic to the complex  $(\Lambda_{r_Q}^\bullet(V^*) \otimes S_Q(V), \delta)$  with the differential

$$\delta(\lambda_{i_1} \wedge \dots \wedge \lambda_{i_r} \otimes P) = \sum_{j=1}^n (-1)^r q_{i_1 j} \dots q_{i_r j} \lambda_{i_1} \wedge \dots \wedge \lambda_{i_r} \wedge \lambda_j \otimes e_j P.$$

We show the proposition by induction on  $n$ . For  $n = 1$ , the cohomology of the complex  $C_Q(V)$  is easy to compute and the proposition follows. Assume now that  $n > 1$ . Put

$$Q = \begin{pmatrix} Q' & R \\ S & 1 \end{pmatrix},$$

where  $Q'$  is a  $(n - 1) \times (n - 1)$ -matrix. Set  $V' = ke_1 \oplus \dots \oplus ke_{n-1}$ . Consider the morphisms of complexes

$$\begin{aligned} \alpha : C_{Q'}(V') \otimes_{S_{Q'}(V')} S_Q(V)[-1] &\rightarrow C_Q(V) \quad \text{and} \\ \beta : C_Q(V) &\rightarrow C_{Q'}(V') \otimes_{S_{Q'}(V')} S_Q(V) \end{aligned}$$

defined by

For  $\{i_1, \dots, i_t\} \subset [1, n - 1]$ ,

$$\alpha(\lambda_{i_1} \wedge \dots \wedge \lambda_{i_t} \otimes P') \otimes P = \lambda_{i_1} \wedge \dots \wedge \lambda_{i_t} \wedge \lambda_n \otimes P' P.$$

For  $\{i_1, \dots, i_t\} \subset [1, n - 1]$ ,

$$\beta(\lambda_{i_1} \wedge \dots \wedge \lambda_{i_t} \otimes P) = (\lambda_{i_1} \wedge \dots \wedge \lambda_{i_t} \otimes 1) \otimes P.$$

If  $n \in \{i_1, \dots, i_t\}$ ,  $\beta(\lambda_{i_1} \wedge \dots \wedge \lambda_{i_t} \otimes P) = 0$ .

We have the following exact sequence of complexes:

$$0 \rightarrow C_{Q'}(V') \otimes_{S_{Q'}(V')} S_Q(V)[-1] \xrightarrow{\alpha} C_Q(V) \xrightarrow{\beta} C_{Q'}(V') \otimes_{S_{Q'}(V')} S_Q(V) \rightarrow 0.$$

We get a long exact sequence of cohomology. It is then easy to see that  $\text{Ext}_{S_Q(V)}^i(k, S_Q(V)) = 0$  if  $i \neq n, n - 1$ . Moreover, we have the following exact sequence where  $\mu_{e_n}$  denotes left multiplication by  $e_n$  and  $\gamma_n = (-1)^{n-1} q_{1,n} \dots q_{n-1,n}$ :

$$0 \rightarrow H^{n-1}(C_Q(V)) \rightarrow k \otimes_{S_{Q'}(V')} S_Q(V) \xrightarrow{\gamma_n \mu_{e_n}} k \otimes_{S_{Q'}(V')} S_Q(V) \rightarrow H^n(C_Q(V)) \rightarrow 0.$$

The proposition follows.  $\square$

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