Cubic identities for theta series in three variables

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1 Introduction

In [1] (see also [2]) Borwein and Borwein proved the identity

$$a(q)^3 = b(q)^3 + c(q)^3 \tag{1}$$

where

$$a(q) = \sum_{m,n\in\mathbf{Z}} q^{m^2+mn+n^2},$$

$$b(q) = \sum_{m,n\in\mathbf{Z}} \omega^{m-n} q^{m^2+mn+n^2}$$

and

$$c(q) = \sum_{m,n\in\mathbf{Z}} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}$$

where $\omega = \exp(2\pi i/3)$. We call these functions theta series for convenience. Subsequently Hirschhorn, Garvan and J. Borwein [3] proved the corresponding identity for two-variable analogues of these theta series. Solé [4] (see also [5]) gave a new proof of (1) using a lattice having the structure of a $\mathbf{Z}[\omega]$-module. Here we introduce three-variable analogues of the theta series $a(q)$, $b(q)$ and $c(q)$, and adapt Solé's method to prove corresponding identities for them.

2 Theta series

We introduce our three-variable theta series as sums over elements of the Eisenstein field $\mathbf{Q}(\sqrt{-3})$.

Let $K = \mathbf{Q}(\sqrt{-3})$ and let $\mathcal{O} = \mathbf{Z}[\omega]$ be its ring of integers, where $\omega = \frac{1}{2}(-1+\sqrt{-3}) = \exp(2\pi i/3)$. Write $\lambda = \omega - \omega^2 = \sqrt{-3}$. For $\alpha \in K$ define $T(\alpha) = \alpha + \overline{\alpha}$, the trace of α. The element λ generates a prime ideal of $\mathcal{O}$ of norm 3; the inclusion $\mathbf{Z} \to \mathcal{O}$ induces an isomorphism $\mathbf{Z}/3\mathbf{Z} \cong \mathcal{O}/\lambda\mathcal{O}$. Hence we can unambiguously define, for $\alpha \in \mathcal{O}$, $\chi(\alpha) = \omega^a$ where $a \in \mathbf{Z}$ and $\alpha \equiv a \pmod{\lambda\mathcal{O}}$.

We now define our theta series. We start with

$$a(q,z,w) = \sum_{\alpha\in\mathcal{O}} q^{|\alpha|^2} z^{T(\alpha)} w^{T(\alpha/\lambda)}.$$

Next for any integer k define

$$b_k(q,z,w) = \sum_{\alpha\in\mathcal{O}} \chi(\alpha)^k q^{|\alpha|^2} z^{T(\alpha)} w^{T(\alpha/\lambda)}.$$

It is apparent that $b_k(q,z,w)$ depends only on the congruence class of k modulo 3 and that $b_0(q,z,w) = a(q,z,w)$. We also define

$$c_k(q,z,w) = \sum_{\alpha\in\mathcal{O}+k/\lambda} q^{|\alpha|^2} z^{T(\alpha)} w^{T(\alpha/\lambda)}.$$

Again $c_k(q,z,w)$ depends only on the congruence class of k modulo 3 and $c_0(q,z,w) = a(q,z,w)$.

We observe some symmetry properties of these functions.

Lemma 1 *We have*

$$a(q,z,w) = a(q,z,w^{-1}) = a(q,z^{-1},w^{-1}) = a(q,z^{-1},w), \tag{2}$$

$$b_k(q,z,w) = b_k(q,z,w^{-1}) = b_{-k}(q,z^{-1},w^{-1}) = b_{-k}(q,z^{-1},w) \tag{3}$$

and

$$c_k(q,z,w) = c_{-k}(q,z,w^{-1}) = c_{-k}(q,z^{-1},w^{-1}) = c_k(q,z^{-1},w). \tag{4}$$

Proof We replace α in the definition of each series in turn by $\overline{\alpha}$, $-\alpha$ and $-\overline{\alpha}$. It helps to note that $T(\overline{\alpha}) = T(\alpha)$, $T(\overline{\alpha}/\lambda) = -T(\alpha/\lambda)$, $\chi(\overline{\alpha}) = \chi(\alpha)$, $\chi(-\alpha) = \chi(\alpha)^{-1}$ and $\overline{k/\lambda} = -k/\lambda$. Of course (2) is a special case of both (3) and (4). $\square$

From (3) and (4) we see that $b_1(q,1,1) = b_{-1}(q,1,1)$ and $c_1(q,1,1) = c_{-1}(q,1,1)$. We write

$$a(q) = a(q,1,1), \quad b(q) = b_1(q,1,1) \quad \text{and} \quad c(q) = c_1(q,1,1).$$

We shall soon see that this agrees with our previous definition.

We show that these functions specialize to the two-variable functions introduced in [3]. First of all, each element $\alpha \in \mathcal{O}$ can be uniquely written as $\alpha = n\omega - m\omega^2$. Then $T(\alpha) = m - n$ and

$$|\alpha|^2 = (n\omega - m\omega^2)(n\omega^2 - m\omega) = m^2 + mn + n$$

and so

$$a(q,z,1) = \sum_{m,n\in\mathbf{Z}} q^{m^2+mn+n^2} z^{m-n}$$

which is denoted as $a(q,z)$ in [3]. In particular

$$a(q,1,1) = \sum_{m,n\in\mathbf{Z}} q^{m^2+mn+n^2}$$

in agreement with the original definition. Also $|-\omega\alpha|^2 = |\alpha|^2$ and $T(-\omega\alpha) = T((m - n\omega^2)/\lambda) = n$. Hence

$$a(q,1,z) = \sum_{\alpha\in\mathcal{O}} q^{|-\omega\alpha|^2} z^{T(-\omega\alpha/\lambda)} = \sum_{m,n\in\mathbf{Z}} q^{m^2+mn+n^2} z^n$$

which is denoted as $a'(q,z)$ in [3]. Now $\chi(-\omega\alpha) = \omega^{m-n}$ and similarly

$$b_1(q,1,z) = \sum_{m,n\in\mathbf{Z}} \omega^{m-n} q^{m^2+mn+n^2} z^n$$

which is denoted as $b(q,z)$ in [3]. In particular

$$b_1(q,1,1) = \sum_{m,n\in\mathbf{Z}} \omega^{m-n} q^{m^2+mn+n^2}.$$

Note that $b_1(q,1,z) = b_{-1}(q,1,z)$ by (3). Finally $\frac{1}{3}(\omega - \omega^2) = -1/\lambda$ and so

$$c_{-1}(q,z,1) = \sum_{m,n\in\mathbf{Z}} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2} z^{m-n}$$

and this is denoted by $q^{1/3}c(q,z)$ in [3]. Again note that $c_{-1}(q,z,1) = c_1(q,z,1)$ by (3). In particular

$$c_1(q,1,1) = c_{-1}(q,1,1) = \sum_{m,n\in\mathbf{Z}} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}.$$

3 Identities

Our main result is a generalization of (1.25) in [3].

Theorem 1 *For each integer k,*

$$\begin{aligned} 3c_k(q,z,w)^3 &= a(q,w,z^{-3})a(q)^2 \\ &\quad + \omega^k b_1(q,w,z^{-3})b(q)^2 + \omega^{-k} b_{-1}(q,w,z^{-3})b(q)^2 \\ &\quad + c_1(q,w,z^{-3})c(q)^2 + c_{-1}(q,w,z^{-3})c(q)^2. \end{aligned} \tag{5}$$

In particular

$$\begin{aligned} 3a(q,z,w)^3 &= a(q,w,z^{-3})a(q)^2 + b_1(q,w,z^{-3})b(q)^2 + b_{-1}(q,w,z^{-3})b(q)^2 \\ &\quad + c_1(q,w,z^{-3})c(q)^2 + c_{-1}(q,w,z^{-3})c(q)^2. \end{aligned} \tag{6}$$

Proof Cubing the definition of $c_k(q,z,w)$ gives

$$c_k(q,z,w)^3 = \sum_{\alpha_0,\alpha_1,\alpha_2 \in \mathcal{O}+k/\lambda} q^{|\alpha_0|^2+|\alpha_1|^2+|\alpha_2|^2} z^{T(\alpha_0+\alpha_1+\alpha_2)} w^{T((\alpha_0+\alpha_1+\alpha_2)/\lambda)}. \tag{7}$$

This is a sum over triples $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ where α runs through a certain subset of

$$\Lambda = \mathcal{O}^3 + \mathbf{Z}(1/\lambda, 1/\lambda, 1/\lambda).$$

We partition the group Λ into various cosets. If $\alpha \in \Lambda$ then $\alpha_0+\alpha_1+\alpha_2 \in \mathcal{O}$. For integers j and k let

$$\Lambda_{j,k} = \{\alpha \in \mathcal{O}^3 + k(1/\lambda, 1/\lambda, 1/\lambda) : \alpha_0+\alpha_1+\alpha_2 \equiv j \pmod{\lambda}\}.$$

Then $\Lambda_{j,k}$ depends only on the integers j and k modulo 3. They are the nine cosets of the subgroup $\Lambda_{0,0}$ of Λ. Define, for $\alpha \in K^3$,

$$|\alpha|^2 = |\alpha_0|^2 + |\alpha_1|^2 + |\alpha_2|^2$$

and

$$\Phi(\alpha) = q^{|\alpha|^2} z^{T(\alpha_0+\alpha_1+\alpha_2)} w^{T((\alpha_0+\alpha_1+\alpha_2)/\lambda)}.$$

Then

$$c_k(q,z,w)^3 = \sum_{\alpha \in \Lambda_{0,k} \cup \Lambda_{1,k} \cup \Lambda_{-1,k}} \Phi(\alpha). \tag{8}$$

We now consider the matrix

$$M = \frac{1}{\lambda} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}.$$

It is straightforward to check that $\Lambda_{j,k}M = \Lambda_{-k,j}$. Also M is a unitary matrix so that if $\beta = \alpha M$ then $|\beta|^2 = |\alpha|^2$. Thus

$$\Phi(\alpha) = q^{|\beta|^2} z^{T(\lambda\beta_0)} w^{T(\beta_0)} = q^{|\beta|^2} z^{-3T(\beta_0/\lambda)} w^{T(\beta_0)}.$$

From (8) we get

$$c_k(q,z,w)^3 = \sum_{\beta \in \Lambda_{-k,0} \cup \Lambda_{-k,1} \cup \Lambda_{-k,-1}} q^{|\beta|^2} z^{-3T(\beta_0/\lambda)} w^{T(\beta_0)}. \tag{9}$$

We split this sum into sums over each of the three cosets $\Lambda_{-k,j}$.

Consider $\Lambda_{-k,0}$. This can be written as

$$\Lambda_{-k,0} = \{\beta \in \mathcal{O}^3 : \chi(\beta_0)\chi(\beta_1)\chi(\beta_2) = \omega^{-k}\}.$$

Hence

$$\begin{aligned}
& 3 \sum_{\beta \in \Lambda_{-k,0}} q^{|\beta|^2} z^{T(-3\beta_0/\lambda)} w^{T(\beta_0)} \\
= & \sum_{\beta \in \mathcal{O}^3} q^{|\beta|^2} z^{T(-3\beta_0/\lambda)} w^{T(\beta_0)} \\
& + \omega^k \sum_{\beta \in \mathcal{O}^3} \chi(\beta_0)\chi(\beta_1)\chi(\beta_2) q^{|\beta|^2} z^{T(-3\beta_0/\lambda)} w^{T(\beta_0)} \\
& + \omega^{-k} \sum_{\beta \in \mathcal{O}^3} \chi(\beta_0)^{-1}\chi(\beta_1)^{-1}\chi(\beta_2)^{-1} q^{|\beta|^2} z^{T(-3\beta_0/\lambda)} w^{T(\beta_0)} \\
= & a(q,w,z^{-3})a(q)^2 + \omega^k b_1(q,w,z^{-3})b(q)^2 + \omega^{-k} b_{-1}(q,w,z^{-3})b(q)^2 \quad (10)
\end{aligned}$$

To aid with the remaining cosets consider the matrix

$$N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix}.$$

Then N is unitary and one may easily check that $\Lambda_{j,k}N = \Lambda_{j+k,k}$. As N does not alter the first coordinate of a triple $\beta \in K^3$ then for $k = \pm 1$

$$\sum_{\beta \in \Lambda_{j,k}} q^{|\beta|^2} z^{-3T(\beta_0/\lambda)} w^{T(\beta_0)}$$

is independent of j. Hence for $k = \pm 1$

$$\begin{aligned}
3 \sum_{\beta \in \Lambda_{j,k}} q^{|\beta|^2} z^{-3T(\beta_0/\lambda)} w^{T(\beta_0)} &= \sum_{\beta \in (\mathcal{O}+k/\lambda)^3} q^{|\beta|^2} z^{-3T(\beta_0/\lambda)} w^{T(\beta_0)} \\
&= c_k(q,w,z^{-3})c(q)^2. \quad (11)
\end{aligned}$$

From (9), (10) and (11) we obtain (5). The $k = 0$ case of (5) is (6). $\square$

Corollary 1 *We have*

$$
\begin{aligned}
2a(q,z,w)^3 &= b_1(q,w,z^{-3})b(q)^2 + b_{-1}(q,w,z^{-3})b(q)^2 \\
&\quad + c_1(q,z,w)^3 + c_2(q,z,w)^3. \qquad (12)
\end{aligned}
$$

Also

$$a(q)^3 = b(q)^3 + c(q)^3. \qquad (13)$$

Proof To obtain (12) subtract the sum the $k = 1$ and $k = -1$ cases of (5) from twice (6). To obtain (13), either substitute $z = w = 1$ in (12), or make this substitution in either (6) or (5). □

Another particular case is obtained by setting $w = 1$ to give

$$a(q,z,1)^3 = b_1(q,1,z^3)b(q)^2 + c_1(q,z,1)^3$$

(using (4)) which is (1.25) in [3].

A variant of the argument of Theorem 1 gives the following result.

Theorem 2 *For each* k*,*

$$
\begin{aligned}
3c_k(q,z,w)c_k(q^2,z^2,w^2) &= a(q,w,z^{-3})a(q^2) \\
&\quad + \omega^k b_1(q,w,z^{-3})b(q^2) + \omega^{-k} b_{-1}(q,w,z^{-3})b(q^2) \\
&\quad + c_1(q,w,z^{-3})c(q^2) + c_{-1}(q,w,z^{-3})c(q^2). \qquad (14)
\end{aligned}
$$

In particular

$$
\begin{aligned}
3a(q,z,w)a(q^2,z^2,w^2) &= a(q,w,z^{-3})a(q^2) \\
&\quad + b_1(q,w,z^{-3})b(q^2) + b_{-1}(q,w,z^{-3})b(q^2) \\
&\quad + c_1(q,w,z^{-3})c(q^2) + c_{-1}(q,w,z^{-3})c(q^2). \qquad (15)
\end{aligned}
$$

Proof As the proof follows closely the proof of Theorem 1, we shall suppress most of the details.

Let $V = \{(\alpha_0, \alpha_1, \alpha_2) \in K^3 : \alpha_1 = \alpha_2\}$. The space V is stable under the action of the matrices M and N. The key is to rewrite the proof of Theorem 1 restricting the summations to triples in V. We start by noting that

$$c_k(q,z,w)c_k(q^2,z^2,w^2) = \sum_{\alpha \in (\mathcal{O}+1/\lambda)^3 \cap V} \Phi(\alpha).$$

This gives

$$c_k(q,z,w)c_k(q^2,z^2,w^2) = \sum_{\beta \in (\Lambda_{-k,0} \cap V) \cup (\Lambda_{-k,1} \cap V) \cup (\Lambda_{-k,-1} \cap V)} q^{|\beta|^2} z^{-3T(\beta_0/\lambda)} w^{T(\beta_0)}.$$

We then get

$$\begin{aligned} & 3\sum_{\beta\in\Lambda_{-k,0}\cap V} q^{|\beta|^2} z^{T(-3\beta_0/\lambda)} w^{T(\beta_0)} \\ = & \; a(q,w,z^{-3})a(q^2)+\omega^k b_1(q,w,z^{-3})b(q^2)+\omega^{-k}b_{-1}(q,w,z^{-3})b(q^2) \end{aligned}$$

and for $j=\pm 1$

$$3\sum_{\beta\in\Lambda_{j,k}\cap V} q^{|\beta|^2} z^{-3T(\beta_0/\lambda)} w^{T(\beta_0)} = c_k(q,w,z^{-3})c(q^2).$$

The theorem then follows. □

Corollary 2 *We have*

$$\begin{aligned} 2a(q,z,w)a(q^2,z^2,w^2) &= b_1(q,w,z^{-3})b(q)^2+b_{-1}(q,w,z^{-3})b(q)^2 \\ &\quad + c_1(q,z,w)^3+c_2(q,z,w)^3. \end{aligned}$$

Also

$$a(q)a(q^2)=b(q)b(q^2)+c(q)c(q^2).$$

Proof This follows from Theorem 2 in exactly the same way that Corollary 1 follows from Theorem 1. □

Another special case is

$$a(q,z,1)a(q^2,z^2,1)=b_1(q,1,z^3)b(q^2)+c_1(q,z,1)c_1(q^2,z^2,1)$$

which is (1.26) in [3].

References

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