A new proof of some identities of Bressoud

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Abstract

We provide a new proof of two identities due to Bressoud:

$$\sum_{m=0}^{N} q^{m^2} \begin{bmatrix} N \\ m \end{bmatrix} = \sum_{m=-\infty}^{\infty} (-1)^m q^{m(5m+1)/2} \begin{bmatrix} 2N \\ N+2m \end{bmatrix}$$

and

$$\sum_{m=0}^{N} q^{m^2+m} \begin{bmatrix} N \\ m \end{bmatrix} = \frac{1}{1-q^{N+1}} \sum_{m=-\infty}^{\infty} (-1)^m q^{m(5m+3)/2} \begin{bmatrix} 2N+2 \\ N+2m+2 \end{bmatrix}$$

which can be considered as finite versions of the Rogers-Ramanujan identities.

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In [1] Bressoud proves the following theorem, from which the Rogers-Ramanujan identities follow on letting $N \to \infty$.

Theorem 1 *For each integer $N \geq 0$,*

$$\sum_{m=0}^{N} q^{m^2} \begin{bmatrix} N \\ m \end{bmatrix} = \sum_{m=-\infty}^{\infty} (-1)^m q^{m(5m+1)/2} \begin{bmatrix} 2N \\ N+2m \end{bmatrix} \tag{1}$$

and

$$\sum_{m=0}^{N} q^{m^2+m} \begin{bmatrix} N \\ m \end{bmatrix} = \frac{1}{1-q^{N+1}} \sum_{m=-\infty}^{\infty} (-1)^m q^{m(5m+3)/2} \begin{bmatrix} 2N+2 \\ N+2m+2 \end{bmatrix}. \tag{2}$$

□

Here

$$\begin{bmatrix} N \\ m \end{bmatrix} = \begin{cases} \dfrac{(q)_N}{(q)_m (q)_{N-m}} & \text{if } 0 \leq m \leq N; \\ 0 & \text{otherwise.} \end{cases}$$

denotes a Gaussian binomial coefficient, where we adopt the standard q-series notation:

$$(q)_n = \prod_{j=1}^{n} (1-q^j).$$

We give an alternative proof of Theorem 1 by showing that the left and right sides of (1) and (2) satisfy the same recurrence relations.

Define, for integers a and $N \geq 0$,

$$S_a(N) = \sum_{n=0}^{N} q^{n^2+an} \begin{bmatrix} N \\ n \end{bmatrix}.$$

Lemma 1 *For each integer $N \geq 1$ and each a we have*

$$S_a(N) = S_a(N-1) + q^{N+a} S_{a+1}(N-1) \tag{3}$$

and

$$S_a(N) = S_{a+1}(N-1) + q^{a+1} S_{a+2}(N-1). \tag{4}$$

Proof Using the identity

$$\begin{bmatrix} N \\ n \end{bmatrix} = q^{N-n} \begin{bmatrix} N-1 \\ n-1 \end{bmatrix} + \begin{bmatrix} N-1 \\ n \end{bmatrix}$$

gives

$$\begin{aligned} S_a(N) &= q^N \sum_{n=1}^{N} q^{n^2+(a-1)n} \begin{bmatrix} N-1 \\ n-1 \end{bmatrix} + \sum_{n=0}^{N-1} q^{n^2+an} \begin{bmatrix} N-1 \\ n \end{bmatrix} \\ &= q^N \sum_{n=0}^{N-1} q^{(n+1)^2+(a-1)(n+1)} \begin{bmatrix} N-1 \\ n \end{bmatrix} + S_a(N-1) \\ &= q^{N+a} S_{a+1}(N-1) + S_a(N-1). \end{aligned}$$

On the other hand, using the identity

$$\begin{bmatrix} N \\ n \end{bmatrix} = \begin{bmatrix} N-1 \\ n-1 \end{bmatrix} + q^n \begin{bmatrix} N-1 \\ n \end{bmatrix}$$

gives

$$\begin{aligned} S_a(N) &= \sum_{n=1}^{N} q^{n^2+an} \begin{bmatrix} N-1 \\ n-1 \end{bmatrix} + \sum_{n=0}^{N-1} q^{n^2+(a+1)n} \begin{bmatrix} N-1 \\ n \end{bmatrix} \\ &= \sum_{n=0}^{N-1} q^{(n+1)^2+a(n+1)} \begin{bmatrix} N-1 \\ n \end{bmatrix} + S_{a+1}(N-1) \\ &= q^{a+1} S_{a+2}(N-1) + S_{a+1}(N-1) \end{aligned}$$

□

We now equate (3) and (4).

Lemma 2 *For integers $N \geq 0$ and each a we have*

$$S_a(N) + (q^{N+a+1} - 1)S_{a+1}(N) - q^{a+1} S_{a+2}(N) = 0.$$

Proof Equating (3) and (4) gives

$$S_a(N-1) + (q^{N+a} - 1)S_{a+1}(N-1) - q^{a+1} S_{a+2}(N-1) = 0$$

for $N \geq 1$. Replacing N by $N+1$ gives

$$S_a(N) + (q^{N+a+1} - 1)S_{a+1}(N) - q^{a+1} S_{a+2}(N) = 0.$$

□

We shall use the $a = 0$ case of Lemma 2 which is

$$S_0(N) + (q^{N+1} - 1)S_1(N) - qS_2(N) = 0. \tag{5}$$

Clearly $S_a(0) = 1$ for all a. Also for $N > 0$, (3) gives

$$S_0(N) = S_0(N-1) + q^N S_1(N-1) \tag{6}$$

and together with (5) gives

$$\begin{aligned} S_1(N) &= S_1(N-1) + q^{N+1}S_2(N-1) \\ &= S_1(N-1) + q^N[S_0(N-1) + (q^N-1)S_1(N-1)] \\ &= q^N S_0(N-1) + (q^{2N} - q^N + 1)S_1(N-1). \end{aligned} \tag{7}$$

Together with the initial conditions $S_0(0) = S_1(0) = 1$, (6) and (7) completely define $S_0(N)$ and $S_1(N)$ for $N \geq 0$.

We now gather some consequences of these recurrences which will be used later.

Lemma 3 *For $N \geq 2$ we have*

$$S_0(N) = (1 + q^{2N-1})S_0(N-1) + q^N(1-q^N)S_1(N-2). \tag{8}$$

and for $N \geq 1$ we have

$$S_1(N) = q^N S_0(N) + (1-q^N)S_1(N-1). \tag{9}$$

Proof First of all from (6) and (7) we have

$$S_1(N) - q^N S_0(N) = (1-q^N)S_1(N-1)$$

and so for $N \geq 2$,

$$S_1(N-1) - q^{N-1}S_0(N-1) = (1-q^{N-1})S_1(N-1)$$

Hence by (6) again,

$$\begin{aligned} S_0(N) &= S_0(N-1) + q^N S_1(N-1) \\ &= S_0(N-1) + q^N[q^{N-1}S_0(N-1) + (1-q^N)S_1(N-2)] \\ &= (1+q^{2N-1})S_0(N-1) + q^N(1-q^N)S_1(N-2) \end{aligned}$$

and by also using (7),

$$\begin{aligned} S_1(N) &= q^N S_0(N-1) + (1-q^N+q^{2N})S_1(N-1) \\ &= q^N[S_0(N) - q^N S_1(N-1)] + (1-q^N+q^{2N})S_1(N-1) \\ &= q^N S_0(N) + (1-q^N)S_1(N-1). \end{aligned}$$

□

The recurrences (8) and (9) with the initial conditions $S_0(0) = S_1(0) = 1$, $S_0(1) = 1 + q$ define $S_0(N)$ and $S_1(N)$ uniquely for $N \geq 0$.

Let

$$B_0(N) = \sum_m (-1)^m q^{m(5m+1)/2} \begin{bmatrix} 2N \\ N+2m \end{bmatrix}$$

and

$$B_1(N) = \sum_m (-1)^m q^{m(5m+3)/2} \begin{bmatrix} 2N+2 \\ N+2m+2 \end{bmatrix}$$

denote the sums appearing on the right sides of the identities in Theorem 1. Setting $r = N + 2m$ in the definition of $B_0(N)$ gives

$$\begin{aligned} B_0(N) &= \sum_{r \equiv N\ (4)} q^{\frac{5}{8}(r-N)^2 + \frac{1}{4}(r-N)} \begin{bmatrix} 2N \\ r \end{bmatrix} - \sum_{r \equiv N+2\ (4)} q^{\frac{5}{8}(r-N)^2 + \frac{1}{4}(r-N)} \begin{bmatrix} 2N \\ r \end{bmatrix} \\ &= q^{-1/40} \left[\sum_{r \equiv N\ (4)} q^{\frac{5}{8}(r-N+\frac{1}{5})^2} \begin{bmatrix} 2N \\ r \end{bmatrix} - \sum_{r \equiv N+2\ (4)} q^{\frac{5}{8}(r-N+\frac{1}{5})^2} \begin{bmatrix} 2N \\ r \end{bmatrix} \right]. \end{aligned}$$

This suggests the notation

$$A(M, k, b) = \sum_{2r \equiv M+k\ (8)} q^{\frac{5}{8}(r-M/2+b)^2} \begin{bmatrix} M \\ r \end{bmatrix}$$

so that

$$q^{1/40} B_0(N) = A(2N, 0, 1/5) - A(2N, 4, 1/5).$$

Of course, $A(M, k, b) = 0$ if $M + k$ is odd and $A(M, k, b)$ depends only on M, b and the congruence class of k modulo 8. A similar computation yields

$$q^{9/40} B_1(N) = A(2N + 2, 2, -2/5) - A(2N + 2, -2, -2/5).$$

We aim to show that $B_0(N)$ and $(1 - q^{N+1})B_1(N)$ satisfy the same system of recurrences as $S_0(N)$ and $S_1(N)$.

Lemma 4 *We have*

$$A(M, k, b) = A(M, -k, -b)$$

for each M, k and b.

Proof Replacing r by $M - r$ in the sum for $A(M, k, b)$ yields

$$\begin{aligned} A(M, k, b) &= \sum_{2M-2r \equiv M+k\ (8)} q^{\frac{5}{8}(M/2-r+b)^2} \begin{bmatrix} M \\ M-r \end{bmatrix} \\ &= \sum_{2r \equiv M-k\ (8)} q^{\frac{5}{8}(r-M/2-b)^2} \begin{bmatrix} M \\ r \end{bmatrix} \\ &= A(M, -k, -b). \end{aligned}$$

□

We now wish to produce recurrences for the $A(M, k, b)$.

Lemma 5 *We have*

$$A(M+1,k,b) = A(M,k-1,b+1/2) + q^{M/2+1/10-b}A(M,k+1,b+3/10)$$

and

$$A(M+1,k,b) = A(M,k+1,b-1/2) + q^{M/2+1/10+b}A(M,k-1,b-3/10)$$

for each M, k and b.

Proof Using the formula

$$\begin{bmatrix} M+1 \\ r \end{bmatrix} = \begin{bmatrix} M \\ r-1 \end{bmatrix} + q^r \begin{bmatrix} M \\ r \end{bmatrix}$$

in the definition of $A(M+1,k,b)$ gives $A(M+1,k,b) = S_1 + S_2$ where

$$\begin{aligned} S_1 &= \sum_{2r\equiv M+k+1\,(8)} q^{\frac{5}{8}(r-M/2-\frac{1}{2}+b)^2} \begin{bmatrix} M \\ r-1 \end{bmatrix} \\ &= \sum_{2s\equiv M+k-1\,(8)} q^{\frac{5}{8}(s-M/2+\frac{1}{2}+b)^2} \begin{bmatrix} M \\ s \end{bmatrix} \\ &= A(M,k-1,b+1/2) \end{aligned}$$

and

$$S_2 = \sum_{2r\equiv M+k+1\,(8)} q^{r+\frac{5}{8}(r-M/2-\frac{1}{2}+b)^2} \begin{bmatrix} M \\ r \end{bmatrix}.$$

But

$$r + \frac{5(r-M/2-\frac{1}{2}+b)^2}{8} = \frac{5(r-M/2+3/10+b)^2}{8} + \frac{M}{2} + \frac{1}{10} - b.$$

Hence

$$A(M+1,k,b) = A(M,k-1,b+1/2) + q^{M/2+1/10-b}A(M,k+1,b+3/10).$$

Consequently, by Lemma 4 also

$$\begin{aligned} & A(M+1,k,b) \\ =\; & A(M+1,-k,-b) \\ =\; & A(M,-k-1,-b+1/2) + q^{M/2+1/10+b}A(M,-k+1,-b+3/10) \\ =\; & A(M,k+1,b-1/2) + q^{M/2+1/10+b}A(M,k-1,b-3/10). \end{aligned}$$

□

It is convenient to note that replacing M by $M-1$ in these identities gives

$$\begin{aligned} & A(M,k,b) \\ = \; & A(M-1,k-1,b+1/2) + q^{M/2-2/5-b} A(M-1,k+1,b+3/10) \\ = \; & A(M-1,k+1,b-1/2) + q^{M/2-2/5+b} A(M-1,k-1,b-3/10). \end{aligned}$$

Lemma 6 *The sums $B_0(N)$ and $B_1(N)$ obey the recurrences*

$$B_0(N) = (1+q^{2N-1})B_0(N-1) + q^N B_1(N-2)$$

for $N \geq 2$ and

$$B_1(N) = (1-q^{N+1})B_1(N-1) + q^N(1-q^{N+1})B_0(N)$$

for $N \geq 1$.

Proof We compute

$$\begin{aligned} & A(2N,k,1/5) \\ = \; & A(2N-1,k+1,-3/10) + q^{N-1/5} A(2N-1,k-1,-1/10) \\ = \; & A(2N-2,k,1/5) + q^{N-3/5} A(2N-2,k+2,0) \\ & + q^{N-1/5} A(2N-2,k-2,2/5) + q^{2N-1} A(2N-2,k,1/5) \\ = \; & (1+q^{2N-1}) A(2N-2,k,1/5) + q^{N-3/5} A(2N-2,k+2,0) \\ & + q^{N-1/5} A(2N-2,k-2,2/5). \end{aligned}$$

In particular

$$\begin{aligned} A(2N,0,1/5) \; = \; & (1+q^{2N-1}) A(2N-2,0,1/5) \\ & + q^{N-3/5} A(2N-2,2,0) + q^{N-1/5} A(2N-2,-2,2/5). \end{aligned}$$

and

$$\begin{aligned} A(2N,4,1/5) \; = \; & (1+q^{2N-1}) A(2N-2,4,1/5) \\ & + q^{N-3/5} A(2N-2,6,0) + q^{N-1/5} A(2N-2,2,2/5) \\ & + q^{N-3/5} A(2N-2,-2,0) + q^{N-1/5} A(2N-2,2,2/5). \end{aligned}$$

Noting that

$$A(2N-2,2,0) = A(2N-2,-2,0)$$

and

$$A(2N-2,2,2/5) = A(2N-2,-2,-2/5)$$

subtracting gives

$$\begin{aligned} q^{1/40}B_0(N) &= A(2N,0,1/5)-A(2N,4,1/5) \\ &= (1+q^{2N-1})[A(2N-2,0,1/5)-A(2N-2,4,1/5)] \\ &\quad + q^{N-1/5}[A(2N-2,2,-2/5)-A(2N-2,-2,-2/5)] \\ &= (1+q^{2N-1})q^{1/40}B_0(N-1)+q^{N-1/5}q^{9/40}B_1(N-2) \end{aligned}$$

and so

$$B_0(N)=(1+q^{2N-1})B_0(N-1)+q^N B_1(N-2).$$

Also

$$\begin{aligned} & A(2N+2,k,-2/5) \\ = {} & A(2N+1,k-1,1/10)+q^{N+1}A(2N+1,k+1,-1/10) \\ = {} & A(2N,k,-2/5)+q^{N+1/5}A(2N,k-2,-1/5) \\ & + q^{N+1}A(2N,k,2/5)+q^{2N+6/5}A(2N,k+2,1/5) \\ = {} & A(2N,k,-2/5)+q^{N+1}A(2N,-k,-2/5) \\ & + q^{N+1/5}A(2N,2-k,1/5)+q^{2N+6/5}A(2N,k+2,1/5). \end{aligned}$$

Consequently

$$\begin{aligned} q^{9/40}B_1(N) &= A(2N+2,2,-2/5)-(2N+2,-2,-2/5) \\ &= A(2N,2,-2/5)+q^{N+1}A(2N,-2-2/5) \\ &\quad - A(2N,-2,-2/5)-q^{N+1}A(2N,2,-2/5) \\ &\quad + q^{N+1/5}[A(2N,0,1/5)-A(2N,4,1/5)] \\ &\quad + q^{2N+6/5}[A(2N,4,1/5)-A(2N,0,1/5)] \\ &= (1-q^{N+1})[q^{9/40}B_1(N-1)+q^{N+1/5}q^{1/40}B_0(N)] \end{aligned}$$

and so

$$B_1(N)=(1-q^{N+1})B_1(N-1)+q^N(1-q^{N+1})B_0(N).$$

□

By Lemma 3 $S_0(N)$ and $(1-q^{N+1})S_1(N)$ satisfy the same recurrences as $B_0(N)$ and $B_1(N)$. Also $S_0(0)=1=B_0(0)$, $S_0(1)=1+q=B_0(1)$ and $(1-q)S_1(0)=1-q=B_1(0)$. Consequently we deduce Theorem 1: $S_0(N)=B_0(N)$ and $(1-q^{N+1})S_1(N)=B_1(N)$.

References

[1] D. M. Bressoud, 'Some identities for terminating q-series', *Math. Proc. Cambridge Philos. Soc.* **89** (1981), 211–223.