FINITE DIMENSIONAL POLYNOMIAL ALGEBRAS WITH APPROXIMATING PRODUCT

F. Chaplais *

June 3, 1999

Let $n \in \mathbb{N}$, and denote by $\mathbb{R}_n[t]$ the space of polynomials of degree lower than n, e.g., of finite sequence of maximum length that $n+1$. $\mathbb{R}_n[t]$ is not stable under the classical polynomial product. We can nonetheless give a caracterization *and* a parameterization of all product laws that *do* make an algebra out of $\mathbb{R}_n[t]$, while "doing their best" to approximate the usual product.

Theorem 1 *Assume that $*$ is a product law that makes $\mathbb{R}_n[t]$ an algebra with the additional property*

$$t^i * t^j = t^{i+j} \text{ for all } i,j \text{ such that } i+j \leq n \tag{1}$$

Then $x * y$ can be expressed as the Hermite interpolation of the usual polynomial product xy at a set of point which is entirely caracterized by the product law $*$, and is independent of x and y. Conversely, any product law defined as a Hermite interpolation of degree $\leq n$ makes $\mathbb{R}_n[t]$ an algebra, while satisfying (1).

Proof. Let $T_{n+1} = t^n * t$. There exist a unique family $(a_0, \dots, a_n)$ such that

$$T_{n+1} = \sum_{i=0}^{i=n} a_i t^i \stackrel{\text{def}}{=} q(t) \tag{2}$$

Let $(z_0, \dots, z_n)$ the complex roots of the (usual) polynomial $t^{n+1} - q(t)$. Then T_{n+1} is the Hermite interpolation of t^{n+1} at these points. We are going to show

*Centre Automatique et Systèmes, École Nationale Supérieure des Mines de Paris, 35 rue Saint Honoré, 77305 Fontainebleau Cedex FRANCE, e-mail: chaplais@cas.ensmp.fr

that $*$ is obtained by Hermite interpolation of the usual product at $(z_0, \dots, z_n)$. To do so, we only have to prove it is true for monomials in the sens of $*$.
These are defined recursively by $T_{n+p} = T_{n+p-1} * t$. We assume that T_{n+p-1} is the interpolation of t^{n+p-1} at $(z_0, \dots, z_n)$. There also exist a polynomial $q(t)$ of degree $< n$ and a real b such that $T_{n+p-1} = q + bt^n$. We have then

$$T_{n+p} = qt + bT_{n+1}$$

By assumption, we have $q(z_i) + b(z_i)^n = z_i^{n+p-1}$; hence,

$$T_{n+p}(z_i) = \left(z_i^{n+p-1} - b(z_i)^n\right) z_i + bT_{n+1}(z_i) = z_i^{n+p} - bz_i^{n+1} - bz_i^{n+1} = z_i^{n+p}$$

which proves the interpolation result at the order 0. For higher orders k, the induction becomes:

$$\begin{aligned}
T_{n+p}^{(k)}(z_i) &= (qt)^{(k)}(z_i) + bT_{n+1}^{(k)}(z_i) \\
&= \left(T_{n+p-1}t - bt^{n+1}\right)^{(k)}(z_i) + bT_{n+1}^{(k)}(z_i) \\
&= \left(T_{n+p-1}t\right)^{(k)}(z_i) \\
&= T_{n+p-1}^{(k)}(z_i)z_i + T_{n+p-1}^{(k-1)}(z_i) \\
&= \left(t^{n+p-1}\right)^{(k)}(z_i)z_i + \left(t^{n+p-1}\right)^{(k-1)}(z_i) \\
&= \left(t^{n+p}\right)^{(k)}(z_i)
\end{aligned}$$

Conversely, one verifies that the Hermite interpolation is indeed associative and that, by definition, it preserves the polynomials of degree $\leq n$. □

Observe that the product laws are parameterized by the set of interpolation points.

The important point is that $\mathbb{R}_n[t]$ can be made an algebra (in a wide variety of ways, as we have seen), while satifying a Strang and Fix like condition. The most simple example of such a product law is obtained by truncating the usual product at the degrees $> n$, that is, by using the Taylor expansion at 0. As shall see, it is not the most satisfying as far as differential calculus is concerned, because it prevents the preservation of the Leibnitz rule. It can be used, however, as an universal product law, provided that a suitable reparameterization of the polynomials is performed.

Reparameterization of $\mathbb{R}_n[t]$ We will denote by Z the set of interpolation points which define the product law on $R_n[t]$ (see theorem 1), under the form $(z_i, o_i)_{1 \leq i \leq K}$, where z_i is the interpolation location and o_i the order of the

Hermite interpolation at that point, or equivalently, the multiplicity of the zero z_i of the polynomial $t^{n+1} - q(t)$ where q is defined in (2). The following theorem gives another representation of $\mathbb{R}_n[t]$.

Theorem 2 *Define the linear operator $\mathcal{L}$ by*

$$\begin{array}{rccl} \mathcal{L}: & \mathbb{R}_n[t] & \longrightarrow & \bigotimes_{i=1}^{i=K} \mathbb{R}_{o_i-1}[t] \\ & p & \longmapsto & (p_i)_{1\leq i\leq K} \text{ with } p_i(t) = p(t+z_i) \end{array} \tag{3}$$

On each space $\mathbb{R}_{o_i-1}[t]$ we use the product law defined by the Taylor expansion of the usual product at 0.
Then $\mathcal{L}$ is an algebraic isomorphism.

Proof. Because of the existence and uniqueness of the Hermite interpolation, $\mathcal{L}$ is a linear isomorphism. Since the product law over $\mathbb{R}_n[t]$ is defined by a Hermite interpolation of the product at the points z_i, the product $p * q$ of two elements of $\mathbb{R}_n[t]$ satisfies $(p*q)^{(j)}(z_i) = (pq)^{(j)}(z_i)$ for $0 \leq j < o_i$, or equivalently, $(p_i * q_i)^{(j)}(0) = (p_i q_i)^{(j)}(0)$, with $\mathcal{L}p = (p_i)$ and $\mathcal{L}q = (q_i)$. This is precisely the Taylor expansion of the product at 0 in $\mathbb{R}_{o_{i-1}}[t]$. $\Box$