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CONVERGENCE OF WEIGHTED AVERAGES OF MARTINGALES IN NONCOMMUTATIVE BANACH FUNCTION SPACES*

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Abstract Let $x = (x_n)_{n\geq 1}$ be a martingale on a noncommutative probability space $(\mathcal{M},\tau)$ and $(w_n)_{n\geq 1}$ a sequence of positive numbers such that $W_n = \sum\limits_{k=1}^{n} w_k \to \infty$ as $n\to\infty$. We prove that $x = (x_n)_{n\geq 1}$ converges in $E(\mathcal{M})$ if and only if $(\sigma_n(x))_{n\geq 1}$ converges in $E(\mathcal{M})$, where $E(\mathcal{M})$ is a noncommutative rearrangement invariant Banach function space with the Fatou property and $\sigma_n(x)$ is given by

$$\sigma_n(x) = \frac{1}{W_n}\sum_{k=1}^{n} w_k x_k, \quad n = 1, 2, \cdots.$$

If in addition, $E(\mathcal{M})$ has absolutely continuous norm, then, $(\sigma_n(x))_{n\geq 1}$ converges in $E(\mathcal{M})$ if and only if $x = (x_n)_{n\geq 1}$ is uniformly integrable and its limit in measure topology $x_\infty \in E(\mathcal{M})$.

Key words Weighted average; noncommutative martingales; noncommutative Banach function spaces; uniform integrability

2000 MR Subject Classification 46L52; 46L53; 60G42

1 Introduction

Let $(w_n)_{n\geq 1}$ be a sequence of positive numbers such that

$$W_n := \sum_{k=1}^{n} w_k \to \infty \quad \text{as } n \to \infty.$$

We call $(w_n)_{n\geq 1}$ a weight sequence. The weighted average $(\sigma_n(f))_{n\geq 1}$ of a martingale $f = (f_n)_{n\geq 1}$ with respect to $(w_n)_{n\geq 1}$ is given by

$$\sigma_n(f) = \frac{1}{W_n}\sum_{k=1}^{n} w_k f_k, \quad n = 1, 2, \cdots. \tag{1}$$

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Kazamaki and Tsuchikura [1] proved that $\sup_n \|\sigma_n(f)\|_p < \infty$ if and only if $\sup_n \|f_n\|_p < \infty$, where $1 \le p < \infty$. From this fact and the martingale convergence theorem, it follows that $f = (f_n)_{n\ge1}$ converges in L_p if and only if $(\sigma_n(f))_{n\ge1}$ converges in L_p for any $1 < p < \infty$. They also proved that a martingale $f = (f_n)_{n\ge1}$ converges almost surely if and only if $(\sigma_n(f))_{n\ge1}$ does. Kikuchi [2] extended these results to Banach function spaces. Zhang and Hou [3] extended them to noncommutative L_p-spaces.

In this article, we extend the above results to a martingale $x = (x_n)_{n\ge1}$ in a noncommutative rearrangement invariant (or simply, r.i.) Banach function spaces $E(\mathcal{M})$. First, we prove that a martingale $x = (x_n)_{n\ge1}$ in $E(\mathcal{M})$ is bounded if and only if its weighted average $(\sigma_n(x))_{n\ge1}$ is bounded. And then we obtain $x = (x_n)_{n\ge1}$ converges in $E(\mathcal{M})$ if and only if $(\sigma_n(f))_{n\ge1}$ converges in $E(\mathcal{M})$, where $E(\mathcal{M})$ is a noncommutative r.i. Banach function space with the Fatou property. In addition, if $E(\mathcal{M})$ has absolutely continuous norm, then, $(\sigma_n(f))_{n\ge1}$ converges in $E(\mathcal{M})$ if and only if $x = (x_n)_{n\ge1}$ is uniformly integrable and its limit in measure topology $x_\infty \in E(\mathcal{M})$.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mu)$ be a complete probability space and $L_0(\Omega)$ the set of all measurable functions on $(\Omega, \mathcal{F}, \mu)$. For $f \in L_0(\Omega)$, the decreasing rearrangement of f, denoted by $\mu(f)$, is defined as

$$\mu_t(f) = \inf\{s > 0 : \mu(|f| > s) \le t\}, \quad t \in [0,1].$$

Given $f, g \in L_0(\Omega)$, we say that f is submajorized by g, denoted by $f \prec\prec g$, if

$$\int_0^t \mu_s(f)\mathrm{d}s \le \int_0^t \mu_s(g)\mathrm{d}s \quad \text{for all } 0 \le t \le 1.$$

Definition 2.1 [4] (1) A Banach subspace $(E, \|\cdot\|_E)$ of $L_0(\Omega)$ is called a Banach function space on $(\Omega, \mathcal{F}, \mu)$ if $f \in L_0(\Omega), g \in E$ and $|f| \le |g|$ imply $f \in E$ and $\|f\|_E \le \|g\|_E$.

(2) A Banach function space E is called r.i. if $f \in L_0(\Omega), g \in E$ and $\mu(f) = \mu(g)$ imply that $f \in E$ and $\|f\|_E = \|g\|_E$.

(3) A Banach function space E is called fully symmetric if $f \in L_0(\Omega), g \in E$ and $f \prec\prec g$ imply that $f \in E$ and $\|f\|_E \le \|g\|_E$.

(4) We say that a Banach function space E has the Fatou property if $0 \le f_n \uparrow f, f_n \in E$ and $\sup_n \|f_n\|_E < \infty$ imply that $f \in E$ and $\|f\|_E = \sup_n \|f_n\|_E$.

It is known that any r.i. Banach function space on the interval $(0,1)$ (with respect to Lebesgue's measure) with the Fatou property is fully symmetric, as was shown by Luxemburg [5]. And if E is an r.i. Banach function space on $(0,1)$, then,

$$L_\infty(0,1) \subset E \subset L_1(0,1) \tag{2}$$

with continuous embeddings, as was shown in [6]. The classical L_p-spaces, Orlicz spaces, and Lorentz spaces on $(0,1)$ are r.i. Banach function spaces.

Let $\mathcal{M}$ be a finite von Neumann algebra acting on a Hilbert space H and τ a normal faithful trace with $\tau(1) = 1$. The couple $(\mathcal{M}, \tau)$ is called a noncommutative probability space. We refer to [7, 8] for noncommutative integration and more historical references. Let $L_0(\mathcal{M})$ denote the topological $*$-algebra of all measurable operators with respect to $(\mathcal{M},\tau)$. The topology of $L_0(\mathcal{M})$ is determined by the convergence in measure. For $x \in L_0(\mathcal{M})$, the generalized singular value function $\mu(x)$ of x is defined by

$$\mu_t(x) = \inf\{s \geq 0 : \ \tau(e_s^{\perp}(|x|)) \leq t\}, \quad t \geq 0,$$

where $e_s^{\perp}(|x|)$ is the spectral projection of $|x|$ associated with the interval (s,∞). The function $t \to \mu_t(x)$ from $(0,1)$ to $[0,\infty]$ is right continuous and non-increasing. Given $x, y \in L_0(\mathcal{M})$, we say that x is submajorized by y, denoted by $x \prec\prec y$, if $\int_0^t \mu_s(x)\mathrm{d}s \leq \int_0^t \mu_s(y)\mathrm{d}s$ for all $0 \leq t \leq 1$. For more complete study of $\mu(\cdot)$, we refer the reader to [9].

Let $E(0,1)$ be an r.i. Banach function space on $(0,1)$. Set

$$E(\mathcal{M},\tau) = \{x \in L_0(\mathcal{M}) : \mu(x) \in E(0,1)\}$$

and

$$\|x\|_{E(\mathcal{M},\tau)} = \|\mu(x)\|_{E(0,1)}, \quad x \in E(\mathcal{M},\tau).$$

Then, $(E(\mathcal{M},\tau), \|\cdot\|)$ is a Banach space. $E(\mathcal{M},\tau)$ (or simply, $E(\mathcal{M})$) is called a noncommutative r.i. Banach function space corresponding to $E(0,1)$. If $E(0,1)$ is fully symmetric, then, $E(\mathcal{M})$ is fully symmetric, that is, $x \in L_0(\mathcal{M}), y \in E(\mathcal{M})$ and $x \prec\prec y$ imply that $x \in E(\mathcal{M})$ and $\|x\|_{E(\mathcal{M})} \leq \|y\|_{E(\mathcal{M})}$. We say that $E(\mathcal{M})$ has the Fatou property if $E(0,1)$ has the Fatou property. If $E(0,1) = L_p(0,1)$ $(1 \leq p \leq \infty)$, then, $E(\mathcal{M})$ coincides with $L_p(\mathcal{M})$, the usual noncommutative L_p-space associated with $(\mathcal{M},\tau)$. It follows from (2) that any noncommutative r.i. Banach function space satisfies

$$L_\infty(\mathcal{M}) \subset E(\mathcal{M}) \subset L_1(\mathcal{M})$$

with continuous embeddings [4].

For more detailed discussions about noncommutative Banach function spaces, see [4, 10, 11].

Let us now recall the general setup for noncommutative martingales. Let $(\mathcal{M}_n)_{n\geq 1}$ be an increasing filtration of von Neumann subalgebras of $\mathcal{M}$, such that the union of $\mathcal{M}_n$'s is weak*-dense in $\mathcal{M}$. For each $n \geq 1$, let $\mathcal{E}_n$ be the conditional expectation operator from $L_p(\mathcal{M},\tau)$ onto $L_p(\mathcal{M}_n, \tau|_{\mathcal{M}_n})$. A noncommutative martingale with respect to the filtration $(\mathcal{M}_n)_{n\geq 1}$ is a sequence $x = (x_n)_{n\geq 1}$ in $L_1(\mathcal{M})$, such that

$$\mathcal{E}_n(x_{n+1}) = x_n \quad \text{for all } n \geq 1.$$

If in addition, $x_n \in E(\mathcal{M})$ for each $n \geq 1$, then, x is called an $E(\mathcal{M})$-martingale. In this case, we set

$$\|x\|_{E(\mathcal{M})} := \sup_n \|x_n\|_{E(\mathcal{M})}.$$

If $\|x\|_{E(\mathcal{M})} < \infty$, then, x is called a bounded $E(\mathcal{M})$-martingale. For more information of noncommutative martingales, see the seminal article of Pisier and Xu [12] and the sequels to it.

3 Main Results and Proofs

In this section, we keep all notations introduced in the preliminaries. Unless specified, all adapted sequences are understood to be with respect to a fixed filtration of von Neumann subalgebras $(\mathcal{M}_n)_{n\geq 1}$ of $\mathcal{M}$.

Lemma 3.1 Suppose that $E(\mathcal{M})$ is a fully symmetric r.i. Banach function space. Then, $\|\mathcal{E}_n(x)\|_{E(\mathcal{M})} \leq \|x\|_{E(\mathcal{M})}$ for any $x \in E(\mathcal{M})$ and $n \geq 1$.

Proof For any $x \in L_0(\mathcal{M})$, it is known that we have the following formula

$$\int_0^t \mu_s(x)\mathrm{d}s = \inf\{\|x_1\|_1 + t\|x_2\|_\infty : x = x_1 + x_2, x_1 \in L_1(\mathcal{M}), x_2 \in L_\infty(\mathcal{M})\},\ 0 \leq t \leq 1,$$

see [9]. For any $x = x_1 + x_2 \in E(\mathcal{M})$ with $x_1 \in L_1(\mathcal{M})$ and $x_2 \in L_\infty(\mathcal{M})$, we have

$$\int_0^t \mu_s(\mathcal{E}_n(x))\mathrm{d}s \leq \|\mathcal{E}_n(x_1)\|_1 + t\|\mathcal{E}_n(x_2)\|_\infty \leq \|x_1\|_1 + t\|x_2\|_\infty$$

because $\mathcal{E}_n$ is an operator with norm one from $L_1(\mathcal{M})$(or $L_\infty(\mathcal{M})$) into itself. Taking the infimum of the right-hand side, we obtain

$$\int_0^t \mu_s(\mathcal{E}_n(x))\mathrm{d}s \leq \int_0^t \mu_s(x)\mathrm{d}s,\ 0 \leq t \leq 1.$$

Therefore, $\mathcal{E}_n(x) \in E(\mathcal{M})$ and $\|\mathcal{E}_n(x)\|_{E(\mathcal{M})} \leq \|x\|_{E(\mathcal{M})}$ for any $x \in E(\mathcal{M})$ because $E(\mathcal{M})$ is fully symmetric. This completes the proof of Lemma 3.1.

Theorem 3.2 Let $E(\mathcal{M})$ be a fully symmetric r.i. Banach function space, $x = (x_n)_{n\geq 1}$ an $E(\mathcal{M})$-martingale and $(\sigma_n(x))_{n\geq 1}$ the weighted average of x given by (1). Then, we have

$$\sup_n \|x_n\|_{E(\mathcal{M})} = \sup_n \|\sigma_n(x)\|_{E(\mathcal{M})}.$$

Proof For any $n \geq 1$, using Minkowski's inequality, we have

$$\|\sigma_n(x)\|_{E(\mathcal{M})} = \left\|\frac{w_1x_1 + \cdots + w_nx_n}{W_n}\right\|_{E(\mathcal{M})} \leq \frac{1}{W_n}\sum_{k=1}^n w_k\|x_k\|_{E(\mathcal{M})} \leq \sup_n \|x_n\|_{E(\mathcal{M})}.$$

Then, we get $\sup\limits_n \|\sigma_n(x)\|_{E(\mathcal{M})} \leq \sup\limits_n \|x_n\|_{E(\mathcal{M})}$. Conversely, observe that, for all $m \geq n$, we have

$$\begin{aligned}\mathcal{E}_n(\sigma_m(x)) &= \frac{1}{W_m}\mathcal{E}_n(w_1x_1 + \cdots + w_nx_n) + \frac{1}{W_m}\mathcal{E}_n(w_{n+1}x_{n+1} + \cdots + w_mx_m)\\ &= \frac{W_n}{W_m}\sigma_n(x) + \frac{W_m - W_n}{W_m}x_n\end{aligned}$$

as $(x_n)_{n\geq 1}$ is a martingale. Because $E(\mathcal{M})$ is fully symmetric, $\mathcal{E}_n$ is contractive on $E(\mathcal{M})$ by Lemma 3.1. So, we obtain

$$\begin{aligned}\frac{W_m - W_n}{W_m}\|x_n\|_{E(\mathcal{M})} &\leq \frac{W_n}{W_m}\|\sigma_n(x)\|_{E(\mathcal{M})} + \|\mathcal{E}_n(\sigma_m(x))\|_{E(\mathcal{M})}\\ &\leq \frac{W_n}{W_m}\|\sigma_n(x)\|_{E(\mathcal{M})} + \|\sigma_m(x)\|_{E(\mathcal{M})}\\ &\leq \left(\frac{W_n}{W_m} + 1\right)\sup_n \|\sigma_n(x)\|_{E(\mathcal{M})}.\end{aligned}$$

Letting $m \to \infty$, we obtain $\|x_n\|_{E(\mathcal{M})} \le \sup_n \|\sigma_n(x)\|_{E(\mathcal{M})}$ because $\lim_{m\to\infty} W_m = +\infty$. We get the converse inequality. The proof is completed.

In what follows, for a martingale $x = (x_n)_{n\ge1}$, we investigate the relationship among its uniform integrability, the $E(\mathcal{M})$-convergence of $(x_n)_{n\ge1}$ and the $E(\mathcal{M})$-convergence of $(\sigma_n(x))_{n\ge1}$.

Recall that a subset K of $L_1(\mathcal{M})$ is said to be uniformly integrable if it is bounded and for every sequence of projections $(p_n)_{n\ge1}$ in $\mathcal{M}$ with $p_n \downarrow 0$(for the strong operator topology), it holds that

$$\lim_{n\to\infty} \sup\{\|p_n x p_n\|_1 : x \in K\} = 0.$$

It is clear that K is uniformly integrable if and only if K is bounded, and for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for any projection $p \in \mathcal{M}$ with $\tau(p) < \delta$, $\sup\{\|pxp\|_1 : x \in K\} < \varepsilon$. It is known that a martingale $x = (x_n)_{n\ge1}$ is uniformly integrable if and only if there exists $x_\infty \in L_1(\mathcal{M})$ such that $x_n = \mathcal{E}_n(x_\infty)$ for all $n \ge 1$ [13]. In this case, the sequence $(x_n)_{n\ge1}$ converges to x_∞ in $L_1(\mathcal{M})$.

Lemma 3.3 [14] Let $(x_n)_{n\ge1}$ be a sequence in $L_1(\mathcal{M})$ and $x \in L_1(\mathcal{M})$. Then, $(x_n)_{n\ge1}$ converges to x in $L_1(\mathcal{M})$ if and only if $(x_n)_{n\ge1}$ is uniformly integrable and $(x_n)_{n\ge1}$ converges to x in measure.

Let Φ be an Orlicz function on $[0,\infty)$, that is, an increasing and convex function satisfying $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$. Then, the Orlicz space $L_\Phi(0,1)$ is a fully symmetric r.i. Banach function space on $(0,1)$. Let $(L_\Phi(\mathcal{M}), \|\cdot\|_{L_\Phi(\mathcal{M})})$ be the fully symmetric noncommutative r.i. Banach function space corresponding to $L_\Phi(0,1)$. The norm $\|\cdot\|_{L_\Phi(\mathcal{M})}$ on $L_\Phi(\mathcal{M})$ is equivalent to

$$\|x\|_\Phi = \inf\left\{c > 0 : \tau(\Phi(\frac{|x|}{c})) < 1\right\}.$$

If $\Phi(t) = t^p$ with $1 \le p < \infty$, then, $L_\Phi(\mathcal{M}) = L_p(\mathcal{M})$.

We will use the noncommutative criterion for uniform integrability which we have proved in [3]. For the reader's convenience, we give its proof below.

Lemma 3.4 (Noncommutative criterion for uniform integrability) A nonempty family $K \subset L_1(\mathcal{M})$ is uniformly integrable if and only if there exists a nonnegative increasing convex function Φ such that $\Phi(0) = 0$, $\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$ and

$$\sup_{x\in K} \|\Phi(|x|)\|_1 < \infty. \tag{3}$$

Proof Suppose that $M := \sup_{x\in K} \|\Phi(|x|)\|_1 < \infty$ holds for some nonnegative convex increasing function Φ such that $\lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty$. For any $n \ge 1$, there exists $c_n > 0$, such that $\Phi(t) \ge nMt$ for $t \ge c_n$. Then, we have

$$e^\perp_{c_n}(|x|)\Phi(|x|)e^\perp_{c_n}(|x|) \ge nMe^\perp_{c_n}(|x|)|x|e^\perp_{c_n}(|x|)$$

and

$$M \ge \tau\left(\Phi(|x|)\right) \ge \tau\left(e^\perp_{c_n}(|x|)\Phi(|x|)e^\perp_{c_n}(|x|)\right) \ge nM\tau(e^\perp_{c_n}(|x|)|x|e^\perp_{c_n}(|x|))$$

for all $x \in K$. Hence, $\sup_{x\in K} \tau\left(e^\perp_{c_n}(|x|)|x|e^\perp_{c_n}(|x|)\right) \le \frac{1}{n}$ and $\sup_{x\in K} \|x\|_1 < \infty$. For any $\varepsilon > 0$, there exists an integer $n_0 \ge 1$, such that $\frac{1}{n_0} < \varepsilon$. Taking a projection p in $\mathcal{M}$ such that $\tau(p) < \frac{\varepsilon}{c_{n_0}}$, we have

$$\begin{aligned}\|pxp\|_1 &\le \left\|pxpe^{\perp}_{c_{n_0}}(|x|)\right\|_1 + \left\|pxpe_{(0,c_{n_0}]}(|x|)\right\|_1\\ &\le \left\|e^{\perp}_{c_{n_0}}(|x|)xe^{\perp}_{c_{n_0}}(|x|)\right\|_1 + \left\|e_{(0,c_{n_0}]}(|x|)xp\right\|_1\\ &< \varepsilon + \left\|e_{(0,c_{n_0}]}(|x|)x\right\|\tau(p)\\ &\le \varepsilon + c_{n_0}\cdot\tau(p)\\ &< 2\varepsilon\end{aligned}$$

for all $x \in K$. So, K is uniformly integrable.

Conversely, suppose that K is uniformly integrable. Because $e^{\perp}_c(|x|) \downarrow 0$ as $c \to \infty$, we can choose $(c_n)_{n\ge 0}$ with $c_0 = 0$, $c_n > c_{n-1}$ for $n \ge 1$ and $c_n \to \infty$ as $n \to \infty$, such that

$$\sup_{x\in K}\tau\left(e^{\perp}_{c_n}(|x|)|x|e^{\perp}_{c_n}(|x|)\right) \le \frac{1}{2^n}.$$

Denote $A_n = [c_{n-1}, c_n)$. Define $\phi(s) = \sum\limits_{n=1}^{\infty} n\chi_{A_n}(s)$, $\Phi(t) = \int_0^t \phi(s)\mathrm{d}s,\ t \ge 0$. Because $\phi(s)$ is increasing and $\phi(s) \to \infty$ as $s \to \infty$, $\Phi(t)$ is convex, $\Phi(0) = 0$, and $\lim\limits_{t\to\infty}\frac{\Phi(t)}{t} = \infty$. Noting that $\Phi(t)\chi_{A_n}(t) \le nt\chi_{A_n}(t)$ for all $n \ge 1$ and any $x \in K$, we have

$$\begin{aligned}\tau(\Phi(|x|)) &= \sum_{n=1}^{\infty}\tau\left(e_{A_n}(|x|)\Phi(|x|)e_{A_n}(|x|)\right)\\ &\le \sum_{n=1}^{\infty} n\tau\left(e_{A_n}(|x|)|x|e_{A_n}(|x|)\right)\\ &= \sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\tau(e_{A_n}(|x|)|x|e_{A_n}(|x|))\\ &= \tau(|x|) + \sum_{k=2}^{\infty}\tau\left(e^{\perp}_{c_{k-1}}(|x|)|x|e^{\perp}_{c_{k-1}}(|x|)\right)\\ &\le \tau(|x|) + \sum_{k=2}^{\infty}\frac{1}{2^{k-1}}\\ &= \|x\|_1 + 1 < \infty\end{aligned}$$

and $\sup\limits_{x\in K}\|\Phi(|x|)\|_1 < \infty$. The criterion is established.

Note that we may assume $M = \sup\limits_{x\in K}\|\Phi(|x|)\|_1 \le 1$ by taking $M^{-1}\Phi$ instead of Φ if necessary. Hence, (3) can be replaced by $\sup\limits_{x\in K}\|x\|_\Phi \le 1$.

Lemma 3.5 Let $E(\mathcal{M})$ be an r.i. Banach function space with the Fatou property, $x = (x_n)_{n\ge 1}$ an $E(\mathcal{M})$-martingale and $(\sigma_n(x))_{n\ge 1}$ the weighted average of x given by (1). If $(\sigma_n(x))_{n\ge 1}$ converges in $E(\mathcal{M})$, then, $(x_n)_{n\ge 1}$ is uniformly integrable and $(x_n)_{n\ge 1}$ converges in measure to $x_\infty \in E(\mathcal{M})$.

Proof Assume that $(\sigma_n(x))_{n\ge 1}$ converges in $E(\mathcal{M})$. Because $E(\mathcal{M})$ has the Fatou property, $E(\mathcal{M})$ is fully symmetric. As $\sup\limits_n\|\sigma_n(x)\|_{E(\mathcal{M})} < \infty$, we have $\sup\limits_n\|x_n\|_{E(\mathcal{M})} < \infty$ by Theorem 3.2. Because $E(\mathcal{M}) \subset L_1(\mathcal{M})$, $(\sigma_n(x))_{n\ge 1}$ converges in $L_1(\mathcal{M})$. By Lemma

3.3, $(\sigma_n(x))_{n\geq 1}$ is uniformly integrable. Then, there exists a nonnegative increasing convex function Φ, such that $\Phi(0)=0$, $\lim\limits_{t\to\infty}\frac{\Phi(t)}{t}=\infty$, and $\sup\limits_n \|\sigma_n(x)\|_\Phi \leq 1$ by Lemma 3.4 and $\sup\limits_n \|x_n\|_\Phi = \sup\limits_n \|\sigma_n(x)\|_\Phi \leq 1$ by Theorem 3.2. This implies the uniform integrability of $(x_n)_{n\geq 1}$ by Lemma 3.4.

The uniform integrability of $(x_n)_{n\geq 1}$ implies that $(x_n)_{n\geq 1}$ converges in measure. Let x_∞ be the limit in measure of $(x_n)_{n\geq 1}$. It is known that $\mu_s(x_\infty) \leq \liminf\limits_{n\to\infty} \mu_s(x_n) := \lambda(s)$. Observing that $\inf\limits_{k\geq n} \mu_s(x_k) \uparrow \lambda(s)$ as $n\to\infty$ and

$$\sup_n \Big\| \inf_{k\geq n} \mu_s(x_k) \Big\|_{E(0,1)} \leq \sup_n \|\mu_s(x_n)\|_{E(0,1)} = \sup_n \|x_n\|_{E(\mathcal{M})} < \infty,$$

we know that $\lambda(s)\in E(0,1)$, because $E(0,1)$ has the Fatou property. It follows that $\mu_s(x_\infty)\in E(0,1)$ and hence $x_\infty \in E(\mathcal{M})$. The proof is completed.

The following lemma shows that if the weight sequence $(w_n)_{n\geq 1}$ increases very rapidly, then $(x_n)_{n\geq 1}$ and $(\sigma_n(x))_{n\geq 1}$converge simultaneously in any r.i. Banach function space $E(\mathcal{M})$ with the Fatou property.

Lemma 3.6 Let $E(\mathcal{M})$ be an r.i. Banach function space with the Fatou property, $x=(x_n)_{n\geq 1}$ an $E(\mathcal{M})$-martingale and $(\sigma_n(x))_{n\geq 1}$ the weighted average of x given by (1).

(a) If $(x_n)_{n\geq 1}$ converges in $E(\mathcal{M})$, then, $(\sigma_n(x))_{n\geq 1}$ converges in $E(\mathcal{M})$.

(b) If $(\sigma_n(x))_{n\geq 1}$ converges in $E(\mathcal{M})$ and

$$\varliminf_{n\to\infty} \frac{w_n}{W_n} > 0, \tag{4}$$

then, $x=(x_n)_{n\geq 1}$ converges in $E(\mathcal{M})$.

Proof (a) Assume that $(x_n)_{n\geq 1}$ converges to x_∞ in $E(\mathcal{M})$. Because $\|x_n - x_\infty\|_{E(\mathcal{M})}\to 0$ and $W_n\to\infty$ as $n\to\infty$, the inequality

$$\|\sigma_n(x)-x_\infty\|_{E(\mathcal{M})} \leq \frac{1}{W_n}\sum_{k=1}^{n} w_k \|x_k - x_\infty\|_{E(\mathcal{M})}$$

yields that $(\sigma_n(x))_{n\geq 1}$ converges to x_∞ in $E(\mathcal{M})$.

(b) Assume that $(\sigma_n(x))_{n\geq 1}$ converges in $E(\mathcal{M})$. Then, $(x_n)_{n\geq 1}$ converges in measure to some $x_\infty\in E(\mathcal{M})$ by Lemma 3.5. Suppose that $(\sigma_n(x))_{n\geq 1}$ converges to y in $E(\mathcal{M})$. For every $n\geq 2$,

$$\sigma_n(x) - y = \frac{w_n}{W_n}(x_n - y) + \frac{W_{n-1}}{W_n}(\sigma_{n-1}(x)-y),$$

and therefore,

$$\begin{aligned}\frac{w_n}{W_n}\|x_n - y\|_{E(\mathcal{M})} &\leq \|\sigma_n(x)-y\|_{E(\mathcal{M})} + \frac{W_{n-1}}{W_n}\|\sigma_{n-1}(x)-y\|_{E(\mathcal{M})} \\ &\leq \|\sigma_n(x)-y\|_{E(\mathcal{M})} + \frac{W_{n-1}}{W_n}\|\sigma_{n-1}(x)-y\|_{E(\mathcal{M})}.\end{aligned} \tag{5}$$

Because $(\sigma_n(x))_{n\geq 1}$ converges to y in $E(\mathcal{M})$, combining (4) and (5), we obtain $\|x_n - y\|_{E(\mathcal{M})}\to 0$ as $n\to\infty$. So, $x_\infty = y$. The proof is completed.

Theorem 3.7 Let $E(\mathcal{M})$ be an r.i. Banach function space with the Fatou property, $x=(x_n)_{n\geq 1}$ an $E(\mathcal{M})$-martingale and $(\sigma_n(x))_{n\geq 1}$ the weighted average of x given by (1). Then, $(x_n)_{n\geq 1}$ converges in $E(\mathcal{M})$ if and only if $(\sigma_n(x))_{n\geq 1}$ converges in $E(\mathcal{M})$.

Proof By Lemma 3.6, $(x_n)_{n\geq 1}$ converging in $E(\mathcal{M})$ implies $(\sigma_n(x))_{n\geq 1}$ converging in $E(\mathcal{M})$. Now, assume that $(\sigma_n(x))_{n\geq 1}$ converges in $E(\mathcal{M})$. Suppose first that $\sup\limits_n \frac{W_{n+1}}{W_n} < \infty$. According to Lemma 3.5, $(x_n)_{n\geq 1}$ is uniformly integrable. Then, there exists $x_\infty \in E(\mathcal{M})$, such that $x_n = \mathcal{E}_n(x_\infty)$ for all $n \geq 1$ and $(x_n)_{n\geq 1}$ converges to x_∞ in measure. As in the proof of Lemma 3.6, we verify that $(\sigma_n(x))_{n\geq 1}$ converges to x_∞ in $E(\mathcal{M})$. We have

$$\mathcal{E}_n(\sigma_{n+1}(x) - x_\infty) = \frac{W_n}{W_{n+1}}(\sigma_n(x) - x_n)$$

and

$$\frac{W_n}{W_{n+1}}\|\sigma_n(x) - x_n\|_{E(\mathcal{M})} \leq \|\mathcal{E}_n(\sigma_{n+1}(x) - x_\infty)\|_{E(\mathcal{M})} \leq \|\sigma_{n+1}(x) - x_\infty\|_{E(\mathcal{M})}$$

for each $n \geq 1$ because $E(\mathcal{M})$ has the Fatou property. Then, it follows that

$$\begin{aligned}\|x_n - x_\infty\|_{E(\mathcal{M})} &\leq \frac{W_{n+1}}{W_n}\left(\frac{W_n}{W_{n+1}}\|\sigma_n(x) - x_n\|_{E(\mathcal{M})} + \frac{W_n}{W_{n+1}}\|\sigma_n(x) - x_\infty\|_{E(\mathcal{M})}\right)\\ &\leq \sup_n \frac{W_{n+1}}{W_n}\left(\|\sigma_{n+1}(x) - x_\infty\|_{E(\mathcal{M})} + \|\sigma_n(x) - x_\infty\|_{E(\mathcal{M})}\right).\end{aligned}$$

Because $(\sigma_n(x))_{n\geq 1}$ converges to x_∞ in $E(\mathcal{M})$, we verify $(x_n)_{n\geq 1}$ converges to x_∞ in $E(\mathcal{M})$.

Now, we assume that $\sup\limits_n \frac{W_{n+1}}{W_n} = \infty$, or equivalently, $\sup\limits_n \frac{w_{n+1}}{W_n} = \infty$. Choose a subsequence $(w_{n_i})_{i\geq 1}$, such that $\lim\limits_{i\to\infty} \frac{w_{n_i}}{W_{n_i-1}} = \infty$. For each integer $i \geq 1$, set $v_i = w_{n_i}$ and $V_k = \sum\limits_{i=1}^k v_i$. Because $\lim\limits_{i\to\infty} \frac{w_{n_i}}{W_{n_i-1}} = \infty$, we have $v_i \to \infty$ as $i \to \infty$ and hence that $V_k \to \infty$ as $k \to \infty$. Furthermore, because $V_k \leq W_{n_k-1} + v_k$, we have

$$1 + \frac{W_{n_k-1}}{w_{n_k}} = \frac{W_{n_k}}{v_k} \geq \frac{W_{n_k}}{V_k} \geq 1 \geq \frac{v_k}{V_k} \geq \frac{v_k}{W_{n_k-1} + v_k} \geq 1 - \frac{W_{n_k-1}}{w_{n_k}}.$$

So,

$$\lim_{k\to\infty} \frac{W_{n_k}}{V_k} = \lim_{k\to\infty} \frac{v_k}{V_k} = 1.$$

Now, for each $k \geq 1$, put $\rho_k(x) = \frac{1}{V_k}\sum\limits_{i=1}^k v_i x_{n_i}$. In other words, $(\rho_k(x))_{k\geq 1}$ is the weighted average of the martingale $\overline{x} = (x_{n_k}, \mathcal{M}_{n_k})_{k\geq 1}$ with respect to $(v_k)_{k\geq 1}$. We show that $(\rho_k(x))_{k\geq 1}$ converges to x_∞ in $E(\mathcal{M})$. Observe that

$$\sigma_{n_k}(x) - x_\infty = \frac{V_k}{W_{n_k}}(\rho_k(x) - x_\infty) + \frac{1}{W_{n_k}}\sum_{m=1}^{n_k}{}' w_m(x_m - x_\infty), \tag{6}$$

where $\sum'$ denotes the sum over $m \leq n_k$ such that $m \neq n_i$ for any $i \geq 1$. Note that $\|x_m\|_{E(\mathcal{M})} = \|\mathcal{E}_m(x_\infty)\|_{E(\mathcal{M})} \leq \|x_\infty\|_{E(\mathcal{M})}$ and by (6), we have

$$\begin{aligned}\frac{V_k}{W_{n_k}}\|\rho_k(x) - x_\infty\|_{E(\mathcal{M})} &\leq \|\sigma_{n_k}(x) - x_\infty\|_{E(\mathcal{M})} + \frac{1}{W_{n_k}}\sum_{m=1}^{n_k}{}' w_m\|x_m - x_\infty\|_{E(\mathcal{M})}\\ &\leq \|\sigma_{n_k}(x) - x_\infty\|_{E(\mathcal{M})} + \frac{2}{W_{n_k}}\|x_\infty\|_{E(\mathcal{M})}\sum_{m=1}^{n_k}{}' w_m\\ &= \|\sigma_{n_k}(x) - x_\infty\|_{E(\mathcal{M})} + 2\|x_\infty\|_{E(\mathcal{M})}(1 - \frac{V_k}{W_{n_k}}).\end{aligned}$$

From $\lim\limits_{k\to\infty}\frac{W_{n_k}}{V_k}=1$ and the above inequalities, it follows that $\|\rho_k(x)-x_\infty\|_{E(\mathcal{M})}\to 0$ as $k\to\infty$.

Now, we prove that $(x_n)_{n\geq 1}$ converges in $E(\mathcal{M})$. Because $(\rho_k(x))_{k\geq 1}$ converges to x_∞ in $E(\mathcal{M})$ and $\lim\limits_{k\to\infty}\frac{v_k}{V_k}=1$, Lemma 3.6 shows that $\overline{x}=(x_{n_k})_{k\geq 1}$ converges to x_∞ in $E(\mathcal{M})$. If $m,n\geq n_k$, then,

$$\begin{aligned}\|x_n-x_m\|_{E(\mathcal{M})}&\leq\|\mathcal{E}_n(x_\infty-x_{n_k})\|_{E(\mathcal{M})}+\|\mathcal{E}_m(x_\infty-x_{n_k})\|_{E(\mathcal{M})}\\&\leq 2\|x_\infty-x_{n_k}\|_{E(\mathcal{M})}\to 0\end{aligned}$$

as $k\to\infty$. This means that $(x_n)_{n\geq 1}$ is a Cauchy sequence in $E(\mathcal{M})$ and hence it converges in $E(\mathcal{M})$. The theorem is established.

A noncommutative Banach function space $E(\mathcal{M})$ is said to have absolutely continuous norm if, for any $x\in E(\mathcal{M})$, $\|xp_n\|_{E(\mathcal{M})}\to 0$ as $n\to\infty$ whenever $(p_n)_{n\geq 1}$ is a sequence of projections in $\mathcal{M}$ and $\tau(p_n)\to 0$ as $n\to\infty$.

Theorem 3.8 Let $E(\mathcal{M})$ be an r.i. Banach function space with the Fatou property, $x=(x_n)_{n\geq 1}$ an $E(\mathcal{M})$-martingale and $(\sigma_n(x))_{n\geq 1}$ the weighted average of x given by (1). If $E(\mathcal{M})$ has absolutely continuous norm, then, the following three conditions are equivalent:

(a) $x=(x_n)_{n\geq 1}$ is uniformly integrable and its limit in measure topology $x_\infty\in E(\mathcal{M})$;

(b) $x=(x_n)_{n\geq 1}$ converges in $E(\mathcal{M})$;

(c) $(\sigma_n(x))_{n\geq 1}$ converges in $E(\mathcal{M})$.

Proof Because (b) $\Rightarrow$ (c) by Theorem 3.7 and (c) $\Rightarrow$ (a) by Lemma 3.5, we only need prove that (a) $\Rightarrow$ (b).

First suppose that $x_\infty\in L_\infty(\mathcal{M})$. Because $x=(x_n)_{n\geq 1}$ is uniformly integrable, $(x_n)_{n\geq 1}$ converges to x_∞ in $L_1(\mathcal{M})$ and $x_n=\mathcal{E}_n(x_\infty)$ for each $n\geq 1$. For any $\varepsilon>0$, we have

$$\begin{aligned}\|x_n-x_\infty\|_{E(\mathcal{M})}&\leq\left\|(x_n-x_\infty)e_{(0,\varepsilon]}(|x_n-x_\infty|)\right\|_{E(\mathcal{M})}\\&\quad+\left\|(x_n-x_\infty)e_\varepsilon^\perp(|x_n-x_\infty|)\right\|_{E(\mathcal{M})}\\&\leq\varepsilon+2\|x_\infty\|_\infty\left\|e_\varepsilon^\perp(|x_n-x_\infty|)\right\|_{E(\mathcal{M})}.\end{aligned}$$

Because $(x_n)_{n\geq 1}$ converges to x_∞ in measure, $\tau(e_\varepsilon^\perp(|x_n-x_\infty|))\to 0$ as $n\to\infty$ for any $\varepsilon>0$. And because $E(\mathcal{M})$ has absolutely continuous norm, it follows that $\|e_\varepsilon^\perp(|x_n-x_\infty|)\|_{E(\mathcal{M})}\to 0$ as $n\to\infty$. Letting $\varepsilon\to 0$, we obtain

$$\lim_{n\to\infty}\|x_n-x_\infty\|_{E(\mathcal{M})}=0.$$

Now, assume that $x_\infty\in E(\mathcal{M})$. Because $L_\infty(\mathcal{M})$ is dense in $E(\mathcal{M})$, we can choose $y\in L_\infty(\mathcal{M})$, such that $\|y-x_\infty\|_{E(\mathcal{M})}<\varepsilon$ for any given $\varepsilon>0$. And we have

$$\begin{aligned}\|x_n-x_\infty\|_{E(\mathcal{M})}&\leq\|\mathcal{E}_n(x_\infty-y)\|_{E(\mathcal{M})}+\|\mathcal{E}_n(y)-y\|_{E(\mathcal{M})}+\|y-x_\infty\|_{E(\mathcal{M})}\\&\leq 2\|x_\infty-y\|_{E(\mathcal{M})}+\|\mathcal{E}_n(y)-y\|_{E(\mathcal{M})}\\&\leq 2\varepsilon+\|\mathcal{E}_n(y)-y\|_{E(\mathcal{M})}.\end{aligned}\tag{7}$$

Because $y\in L_\infty(\mathcal{M})\subset L_1(\mathcal{M})$, $(\mathcal{E}_n(y))_{n\geq 1}$ is uniformly integrable. By what we have proved in the first part of the proof, $\|\mathcal{E}_n(y)-y\|_{E(\mathcal{M})}\to 0$ as $n\to\infty$. Letting $\varepsilon\to 0$ in (7), we get $\lim\limits_{n\to\infty}\|x_n-x_\infty\|_{E(\mathcal{M})}=0$. The proof is completed.

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