REPRESENTATIONS OF INTEGERS AS SUMS OF SQUARES

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Let $\mathbb{Z}$ be the set of integers. One of the central problems in Number Theory is to study the representations of integers as sums of elements chosen from a certain subset $\mathcal{A}$ of $\mathbb{Z}$. When $\mathcal{A}$ is the set of prime numbers, we are led to the famous Goldbach Conjecture (which states that every even integer is the sum of two odd primes) and Vinogradov's Theorem (which states that every odd integer is a sum of three primes). When $\mathcal{A}$ is the set of integers of the form k^n for some $n > 1$, we are dealing with what is known as the Waring Problem. In this article, we set $\mathcal{A}$ to be the set of squares, namely, $\mathcal{A} = \{1, 4, 9, 16, 25, \cdots\}$. This may be viewed as Waring's Problem with $n = 2$. To facilitate our discussion, let $r_k(n)$ be the total number of ways of representing n as a sum of k squares.

The problem of representing integers as sums of squares has a long and interesting history. It was Diophantus (325 - 409 A.D.) who showed that every integer of the form $4m + 3$ cannot be represented as a sum of two squares. In other words, Diophantus showed that $r_2(4m + 3) = 0$ (The readers are encouraged to try to establish this). In 1632, Girard conjectured that n is a sum of two squares if the prime divisors of n of the form $4m + 3$ occur in n in an even power. For example, $n = 3^2 \cdot 5 = 3^2 + 6^2$ while $n = 3^3 \cdot 5$ cannot be represented as a sum of two squares. This conjecture was first proved by Euler in 1749. Euler's proof, however, did not give an explicit formula for $r_2(n)$ and the formula was later discovered by Legendre in 1798 and Gauss in 1801. Legendre and Gauss showed that

$$r_2(n) = 4(d_1(n) - d_3(n)),$$

where $d_j(n)$ is the number of divisors of n of the form $4m + j$.

In 1770, Lagrange showed that every positive integer is a sum of four squares. Lagrange's proof of his Four Squares Theorem involved an identity closely related to the *quaternions* but he did not give any formula for $r_4(n)$.

In 1829, Jacobi observed that $r_k(n)$ can be obtained from the series expansion of the function $\varphi^k(q)$ where

$$\varphi(q) = \sum_{k=-\infty}^{\infty} q^{k^2}.$$

Using the theory of elliptic functions to express $\varphi^k(q)$ in terms of Lambert series, Jacobi was able to establish explicit formulas for $r_k(n)$ for $k = 2, 4, 6$ and 8. His formula for $k = 4$ yields Lagrange's Theorem immediately. Jacobi's work marks the beginning of an important Chapter in the development of mathematics, namely, the use of complex analysis in the study of number theoretic problems. Since then, mathematicians such as Glaisher, Ramanujan, R. Rankin, and others succeeded in deriving formulas for $r_{2k}(n)$ with $k > 4$. The most general statement in this topic states that

$$r_{2k}(n) = \delta_{2k}(n) + e_{2k}(n),$$

where $\delta_{2k}(n)$ is the coefficient of q^n in the expansion of certain Lambert series and $e_{2k}(n)$ is the coefficient of q^n of the Fourier expansion of certain cusp form. This general result was obtained by Rankin using the theory of modular forms in 1965.

The subject was believed to be "dead" until recently. In 1994, V. Kac and M. Wakimoto announced their new conjectural formulas for $\varphi^k(n)$ when $k = 4l^2$ and $k = 4l(l+1)$. Their formulas were discovered as a consequence of their study of certain affine superalgebras. These conjectures were later proved by S. Milne using Jacobi's elliptic functions, Hankel's determinants and continued fractions. The simplest formula arising from Milne's Theorem is

$$\begin{aligned}\varphi^{24}(q) &= \frac{1}{3^2}\det\begin{vmatrix}16\mathcal{G}_4(q)+1 & 16\mathcal{G}_6(q)-2\\ 32\mathcal{G}_6(q)-4 & 32\mathcal{G}_8(q)+17\end{vmatrix}\\ &= 1+\frac{16}{9}\left(17\mathcal{G}_4(q)+8\mathcal{G}_6(q)+2\mathcal{G}_8(q)\right)\\ &\qquad +\frac{512}{9}\left(\mathcal{G}_4(q)\mathcal{G}_8(q)-\mathcal{G}_6^2(q)\right),\end{aligned}$$

where

$$\mathcal{G}_{2s}(q) = \sum_{k=1}^{\infty}\frac{k^{2s-1}q^k}{1-(-q)^k}.$$

In general, Milne's formula expresses $\varphi^k(q)$ with $k = 4l(l+1)$ as a sum of at most l products of $\mathcal{G}_{2s}(q), s \in \mathbb{N}$. For example, when $l = 3$, there will be terms involving products of three $\mathcal{G}_{2s}(q), s \in \mathbb{N}$. Recently, motivated by Milne's formula for 24 squares, the author observes that for every even k, $\varphi^k(q)$ can be expressed as a sum of products of 2 series (instead of l) analogous to $\mathcal{G}_{2s}(q)$. In the case when $k = 8l$, the author discovers the following:

Conjecture. Let

$$\sec^2 u = \sum_{k=0}^{\infty} a_{2k}\frac{u^{2k}}{2k!},$$

and suppose that $\dfrac{A_k}{B_k} = \dfrac{a_{2k}}{2^{2k+3}}$, where $k > 1$ and $gcd(A_k, B_k) = 1$. If $s \equiv 0 \pmod 4$, then

$$\varphi^{2s}(q) = \sum_{m+n=s} a_{m,n}\mathcal{E}_{2m}(q)\mathcal{E}_{2n}(q),$$

where

$$\mathcal{E}_{2k+2}(q) = A_k - (-1)^k B_k \mathcal{G}_{2k+2}(q), \quad \text{and} \quad a_{m,n} \in \mathbb{Q}.$$

When $k = 32$, the above conjecture produces the identity

$$\varphi^{32}(q) = \frac{1}{4725}\left\{-400\mathcal{E}_6(q)\mathcal{E}_{10}(q) + 16\mathcal{E}_{12}(q)\mathcal{E}_4(q) + 21\mathcal{E}_8^2(q)\right\}.$$

Since 32 is neither a square nor a product of two consecutive integers, this identity is not contained in Milne's Theorems and it is indeed new. The author is currently working on proving the above conjecture and its analogues for case $k = 8l+4$ and $k = 4l+2$. These new results will hopefully provide new insights in the study of sums of squares and more generally, in the development of the theory of modular forms via Eisenstein series.