PARTITION IDENTITIES AND CONGRUENCES ASSOCIATED WITH THE FOURIER COEFFICIENTS OF THE EULER PRODUCTS

HENG HUAT CHAN AND RICHARD P. LEWIS

Abstract. In this article, we discuss two applications of the operator $U(m)$ (see (1.1)) defined on the product of two power series .

1. Introduction

Let m be a positive integer and define the operator $U(m)$ on a formal power series $\sum_{n=0}^{\infty} a_n q^n$ by

$$\sum_{n=0}^{\infty} a_n q^n \Big|_{U(m)} = \sum_{n=0}^{\infty} a_{mn} q^n.$$

The operator $U(m)$ acts on the product of two power series as follow:

$$\left(\sum_{n=0}^{\infty} b_n q^{mn} \sum_{n=0}^{\infty} a_n q^n\right)\Big|_{U(m)} = \sum_{n=0}^{\infty} b_n q^n \sum_{n=0}^{\infty} a_{mn} q^n. \tag{1.1}$$

The relation (1.1) shows that under $U(m)$, we may "shift" the "m" from the power of q in the first series to the subscript of the coefficients of the second series. This observation is known to A.O.L. Atkin and J.N. O'Brien [1, (28)].

In Section 2, we prove, with the aid of (1.1), Ramanujan's famous congruences

$$p(5n+4) \equiv 0 \pmod 5, \tag{1.2}$$

$$p(7n+5) \equiv 0 \pmod 7 \tag{1.3}$$

and

$$p(11n+6) \equiv 0 \pmod{11}, \tag{1.4}$$

where $p(n)$ denote the number of unrestricted partitions of the non-negative integer n.

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It is clear that (1.2) and (1.3) follows from Ramanujan's identities

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\prod_{n=1}^{\infty}\frac{(1-q^{5n})^5}{(1-q^n)^6}, \tag{1.5}$$

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\prod_{n=1}^{\infty}\frac{(1-q^{7n})^3}{(1-q^n)^4} + 49q\prod_{n=1}^{\infty}\frac{(1-q^{7n})^7}{(1-q^n)^8}. \tag{1.6}$$

Identities such as (1.5) and (1.6) are more difficult to establish than congruences (1.2) and (1.3). In [10, (1.15)], H.S. Zuckerman obtained the following analogue of (1.5) and (1.6):

(1.7)

$$\begin{aligned}\sum_{n=1}^{\infty} p(13n+6)q^n = 11\prod_{n=1}^{\infty}\frac{(1-q^{13n})}{(1-q^n)^2} + 468q\prod_{n=1}^{\infty}\frac{(1-q^{13n})^3}{(1-q^n)^4} + 6422q^2\prod_{n=1}^{\infty}\frac{(1-q^{13n})^5}{(1-q^n)^6}\\ + 43940q^3\prod_{n=1}^{\infty}\frac{(1-q^{13n})^7}{(1-q^n)^8} + 171366q^4\prod_{n=1}^{\infty}\frac{(1-q^{13n})^9}{(1-q^n)^{10}}\\ + 371293q^5\prod_{n=1}^{\infty}\frac{(1-q^{13n})^{11}}{(1-q^n)^{12}} + 371293q^6\prod_{n=1}^{\infty}\frac{(1-q^{13n})^{13}}{(1-q^n)^{14}}.\end{aligned}$$

In Section 3, we use (1.1) and results in [4] to establish identities associated with

$$\sum_{n=0}^{\infty} p_{-r}(l^k n + \delta_{l,k,r})q^n, l = 5, 7 \text{ and } 13,$$

where

$$\delta_{l,k,r} = \begin{cases} \dfrac{r(1-l^k)}{24} & \text{if } k \text{ is even} \\ \dfrac{r(1-l^{k+1})}{24} & \text{if } k \text{ is odd,}\end{cases} \tag{1.8}$$

and

$$\prod_{n=1}^{\infty}(1-q^n)^r = \sum_{n=0}^{\infty} p_r(n)q^n. \tag{1.9}$$

When $(l,k,r) = (5,1,-1), (7,1,-1)$, and $(13,1,-1)$ we obtain (1.5)–(1.7) and when $(l,k,r) = (5,1,-2)$ and $(l,k,r) = (5,1,-3)$, we find that

$$\sum_{n=0}^{\infty} p_{-2}(5n-2)q^n = 10q\prod_{n=1}^{\infty}\frac{(1-q^{5n})^4}{(1-q^n)^6} + 125q^2\prod_{n=1}^{\infty}\frac{(1-q^{5n})^{10}}{(1-q^n)^{12}}, \tag{1.10}$$

and

$$\sum_{n=0}^{\infty} p_{-3}(5n-3)q^n = 9q\prod_{n=1}^{\infty}\frac{(1-q^{5n})^3}{(1-q^n)^6} + 375q^2\prod_{n=1}^{\infty}\frac{(1-q^{5n})^9}{(1-q^n)^{12}} + 3125q^3\prod_{n=1}^{\infty}\frac{(1-q^{5n})^{15}}{(1-q^n)^{18}}. \tag{1.11}$$

Identities (1.10)–(1.11) appear to be new.

2. Ramanujan's Congruences

The congruence properties of $p_r(n)$ (see (1.9)) were studied by Ramanujan, who deduced (1.2) and (1.3) from

$$p_4(5n+4) \equiv 0 \pmod 5 \text{ and } p_6(7n+5) \equiv 0 \pmod 7,$$

respectively. In [9], L. Winquist showed (1.4) by proving that

$$p_{10}(11n+6) \equiv 0 \pmod{11}.$$

Since then, many congruences have been discovered for $p_r(n)$ (see for example [2] and [5]). In this section, we show that in order to obtain congruences for $p_r(n)$ of the type

$$p_r(ln-N) \equiv 0 \pmod l, n \geq 1,$$

it suffices to check if l divides $\tau_N(lj)$, $1 \leq j \leq N$, where

$$\Delta^N(z) := q^N \prod_{n=1}^{\infty}(1-q^n)^{24N} = \sum_{n=0}^{\infty}\tau_N(n)q^n, q = e^{2\pi i z}.$$

Note that $\tau_1(n)$ is the famous Ramanujan's τ-function.

Proof of (1.2). It is known that $\Delta(z)$ is an eigenform in $\mathcal{S}_{12}(SL_2(\mathbf{Z}))$, where $\mathcal{S}_k(SL_2(\mathbf{Z}))$ denote the space of weight k cusp forms invariant under $SL_2(\mathbf{Z})$. Hence,

$$\Delta(z)\Big|_{T_p} = \tau(p)\Delta(z),$$

where T_p is the Hecke operator defined by

$$\sum_{n=0}^{\infty} a_n q^n\Big|_{T_p} = \sum_{n=0}^{\infty}(a(pn) + p^{k-1}a(n/p))q^n,$$

with k being the weight of the modular form $\sum_{n=0}^{\infty} a_n q^n$ invariant under $SL_2(\mathbf{Z})$. Note that since the coefficient of q^5 in $\Delta(z)$ is $\tau(5) = 4830$, we conclude that

$$\Delta(z)\Big|_{T_5} = \tau(5)\Delta(z) \equiv 0 \pmod 5. \tag{2.1}$$

We now write
(2.2)
$$\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{5s}\prod_{n=1}^{\infty}(1-q^n)^r \equiv \prod_{n=1}^{\infty}(1-q^{5n})^s\sum_{n=0}^{\infty}p_r(n-1)q^n \pmod 5,$$
where r and s are integers. Since
$$\sum_{n=0}^{\infty}a_nq^n\Big|_{U(p)} \equiv \sum_{n=0}^{\infty}a_nq^n\Big|_{T_p} \pmod 5,$$
we find by (1.1), (2.2) and (2.1) that
$$\text{(2.3)}\quad \prod_{n=1}^{\infty}(1-q^{5n})^s\sum_{n=0}^{\infty}p_r(n-1)q^n\Big|_{U(5)} \equiv \prod_{n=1}^{\infty}(1-q^n)^s\sum_{n=0}^{\infty}p_r(5n-1)q^n$$
$$\equiv \Delta(z)\Big|_{T_5} \equiv 0 \pmod 5.$$
This implies that $p_r(5n-1) \equiv 0 \pmod 5$ for all r satisfying the equation
$$24 = 5s + r,$$
or
$$p_{24-5s}(5n-1) \equiv 0 \pmod 5, s \in \mathbf{Z},$$
which immediately yields Ramanujan's congruences for $p(5n+4)$ and $p_4(5n+4)$.

Our computation shows that one only needs to know $\tau(5)$ in $\Delta(z)$ in order to deduce the above congruences. In general, we always obtain a collection of congruences of the form
$$p_{24-ls}(ln-1) \equiv 0 \pmod l$$
for each l satisfying
$$\text{(2.4)}\qquad \tau(l) \equiv 0 \pmod l.$$
Questions involving primes satisfying (2.4) can be found in [8, 5.2 (b)].

Proof of (1.3). To prove Ramanujan's congruences for $p(7n+5)$, we express $\Delta^2(z)\Big|_{T_7}$ in terms of $\Delta^2(z)$ and $\Delta(z)Q^3(q)$, where
$$Q(q) = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}.$$
This turns out to be
$$\text{(2.5)}\qquad \Delta^2(z)\Big|_{T_7} = -985824\Delta(z)Q^3(q) - 525803656\Delta^2(z).$$
Note that the coefficients of $\Delta(z)Q^3(q)$ and $\Delta^2(z)$ in the above identities are both divisible by 7. Hence we conclude that
$$p_{48-7s}(7n-2) \equiv 0 \pmod 7, s \in \mathbf{Z}.$$

In particular, we obtain (1.3), as well as the congruence for $p_6(7n+5)$.

It is clear from the above calculations that to obtain congruences such as

$$p_{24N-ls}(ln-N) \equiv 0 \pmod{l},$$

it suffices to compute the image of $\Delta^N(z)$ under T_l. If

$$\Delta^N(z)\Big|_{T_l} = a_1B_1 + a_2B_2 + \cdots + a_NB_N,$$

where $N =$ dimension of $\mathcal{S}_k(SL_2(\mathbf{Z}))$, then each a_i is a $\mathbf{Z}$-linear combination of $\tau_N(lj), 1 \le j \le N$, where

$$\Delta^N(z) = \sum_{n=1}^{\infty} \tau_N(n)q^n.$$

Hence in order to verify that

$$\tau_N(lj) \equiv 0 \pmod{l}$$

holds, it suffices to verify it for $1 \le j \le N$. Therefore, to prove (1.4), it suffices to check that 11 divides $\tau_5(11j), 1 \le j \le 5$.

3. Partition identities

In this section, we give proofs of (1.5)–(1.7) and their generalizations.

We begin this section with the proof of (1.5). It is known that $\eta(25z)/\eta(z)$ is a modular function on $\Gamma_0(25)$ [6], where

$$\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n).$$

Since

$$\frac{\eta(25z)}{\eta(z)} = \prod_{n=1}^{\infty}(1-q^{25n})\sum_{n=0}^{\infty}p(n-1)q^n,$$

we conclude by (1.1) that

$$\prod_{n=1}^{\infty}(1-q^{5n})\sum_{n=0}^{\infty}p(5n-1)q^n = \frac{\eta(25z)}{\eta(z)}\Big|_{U(5)}.$$

Following the method illustrated in [4, Theorem 4], we find that $\frac{\eta(25z)}{\eta(z)}\Big|_{U(5)}$ is an entire modular function on $\Gamma_0(5)$. It is know that these functions are polynomials in $h_5(z) := \eta^6(5z)/\eta^6(z)$ [3]. Hence, we conclude immediately that

$$\prod_{n=1}^{\infty}(1-q^{5n})\sum_{n=0}^{\infty}p(5n-1)q^n = 5\frac{\eta^6(5\tau)}{\eta^6(z)}, \tag{3.1}$$

which is (1.5).

The proof of (1.6) and (1.7) is similar since $\eta(l^2z)/\eta(z)$ is an entire modular function on $\Gamma_0(l^2)$ and entire modular functions on $\Gamma_0(7)$ and $\Gamma_0(13)$ are polynomials in $\eta^4(7z)/\eta^4(z)$ and $\eta^2(13z)/\eta^2(z)$ [3], respectively.

The method of proof illustrated above yields the following

Theorem 3.1. [4, Theorem 4] *Let $l > 3$ be an odd prime. Then*

$$\prod_{n=1}^{\infty}(1-q^{ln})\sum_{n=0}^{\infty}p(ln+(1-l^2)/24)q^n$$

is an entire modular function on $\Gamma_0(l)$.

It is also known that $U(l)$ sends an entire modular function $f(z)$ on $\Gamma_0(l)$ to an entire modular function on $\Gamma_0(l)$ if f satisfies the transformation formula [4, (2.2)]

$$f(-1/lz) = cf(z) \text{ or } f(-1/lz) = c/f(z). \tag{3.2}$$

This is clearly satisfied by the functions $\eta^6(5z)/\eta^6(z)$, $\eta^4(7z)/\eta^4(z)$ and $\eta^2(13z)/\eta^2(z)$, for $l = 5, 7$ and 13, respectively.

In the case of $l = 5$, we apply $U(5)$ to the left hand side of (3.1) to conclude that [10, (1.13)]

(3.3)
$$\prod_{n=1}^{\infty}(1-q^n)\sum_{n=0}^{\infty}p(25n-1)q^n = 63\cdot 5^2\left(\frac{\eta(5z)}{\eta(z)}\right)^6 + 52\cdot 5^5\left(\frac{\eta(5z)}{\eta(z)}\right)^{12} + 63\cdot 5^7\left(\frac{\eta(5z)}{\eta(z)}\right)^{18} + 6\cdot 5^{10}\left(\frac{\eta(5z)}{\eta(z)}\right)^{24} + 5^{12}\left(\frac{\eta(5z)}{\eta(z)}\right)^{30}.$$

To obtain identities associated with higher power of 5, we first multiply

$$\prod_{n=1}^{\infty}(1-q^n)\sum_{n=0}^{\infty}p(5^2n-1)q^n$$

by $\eta(25z)/\eta(z)$ and note that each functions on the right hand side satisfies (3.2). Therefore, by applying $U(5)$, we conclude that

$$\prod_{n=1}^{\infty}(1-q^{5n})\sum_{n=0}^{\infty}p(5^3n-26)q^n.$$

is an entire modular function on $\Gamma_0(5)$ and is expressible in terms of $h_5(z)$. It is clear that when we pass from an identity involving odd k to $k+1$, we only need to apply $U(5)$ to

$$\prod_{n=1}^{\infty}(1-q^{5n})\sum_{n=0}^{\infty}p(5^kn-\delta_{5,k})q^n,$$

where $\delta_{5,k} := \delta_{5,k,1}$, with $\delta_{l,k,r}$ defined as in (1.8). When we need to obtain an identity involving odd $k+1$ from k, we have to first multiply the identity involving

$$\prod_{n=1}^{\infty}(1-q^n)\sum_{n=0}^{\infty}p(5^kn-\delta_{5,k})q^n$$

by $\eta(25z)/\eta(z)$ before applying $U(5)$. In this way, we obtain expressions of

$$\prod_{n=1}^{\infty}(1-q^{\epsilon_5 n})\sum_{n=0}^{\infty}p(5^kn-\delta_{5,k})q^n$$

in terms of $h_5(z)$ for all $k\in\mathbf{N}$, with

$$\epsilon_l=\begin{cases} l & \text{if } k \text{ is odd}\\ 1 & \text{if } k \text{ is even.}\end{cases} \tag{3.4}$$

This method can be found in [4, Theorem 7] where the case for $l=11$ is discussed.

The advantage of using (1.1) to obtain partition identities is that one does not need to know the modular behavior of the expressions such as $\sum_{n=0}^{\infty}p(5^kn-\delta_{5,k})q^n$. The method can be modified to obtain identities for $\sum_{n=0}^{\infty}p_{-r}(5^kn-\delta_{5,k,r})q^n$, where $p_r(n)$ and $\delta_{l,k,r}$ are defined in (1.9) and (1.8), respectively. All we have to do is to use

$$\left(\frac{\eta(25\tau)}{\eta(\tau)}\right)^r$$

and follow the arguments illustrated as above to conclude that $\prod_{n=1}^{\infty}(1-q^{\epsilon_5 n})^r\sum_{n=0}^{\infty}p_{-r}(5^kn-\delta_{5,k,r})q^n$ is a polynomial in $h_5(z)$, where ϵ_5 is defined in (3.4). For $(k,r)=(5,-2)$ and $(5,-3)$, we obtain (1.10) and (1.11) respectively.

We end this paper with the following conjectures regarding the congruences associated with $p_{-r}(n)$ for higher power of primes $l=5,7$ and 11, namely,

$$\begin{aligned}
p_{-2}(5^\alpha n+\delta_{5,\alpha,2}) &\equiv 0 \pmod{5^{[\alpha/2]+1}},\\
p_{-3}(11^\alpha n+\delta_{11,\alpha,3}) &\equiv 0 \pmod{11^\alpha},\\
p_{-4}(7^\alpha n+\delta_{7,\alpha,4}) &\equiv 0 \pmod{7^\alpha},
\end{aligned}$$

and

$$p_{-6}(5^\alpha n+\delta_{5,\alpha,6}) \equiv 0 \pmod{5^\alpha}.$$

There are many congruences of this type for other values of r and they are conjectured using the ideas we developed in Sections 2 and 3. These congruences are clearly the analogues of Ramanujan's congruences

$$\begin{aligned}
p(5^\alpha n+\delta_{5,\alpha,1}) &\equiv 0 \pmod{5^\alpha},\\
p(7^\alpha n+\delta_{7,\alpha,1}) &\equiv 0 \pmod{7^{[\alpha/2]+1}},
\end{aligned}$$

and

$$p(11^\alpha n + \delta_{11,\alpha,1}) \equiv 0 \pmod{11^\alpha}.$$

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Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543, Republic of Singapore

E-mail address: `chanhh@math.nus.edu.sg`

School of Mathematical Sciences, University of Sussex, BN1 9QH, Brighton, U.K.

E-mail address: `r.p.lewis@susx.ac.uk`