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Partition identities and congruences associated with the Fourier coefficients of the Euler products

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Dedicated to Professor Srinivasa Rao on the occasion of his 60th birthday

Abstract

In this article, we discuss two applications of the operator $U(m)$ (see (1.1)) defined on the product of two power series.

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1. Introduction

Let m be a positive integer and define the operator $U(m)$ on a formal power series $\sum_{n=0}^{\infty} a_n q^n$ by

$$\sum_{n=0}^{\infty} a_n q^n \bigg|_{U(m)} = \sum_{n=0}^{\infty} a_{mn} q^n.$$

The operator $U(m)$ acts on the product of two power series as follows:

$$\left(\sum_{n=0}^{\infty} b_n q^{mn} \sum_{n=0}^{\infty} a_n q^n \right) \Bigg|_{U(m)} = \sum_{n=0}^{\infty} b_n q^n \sum_{n=0}^{\infty} a_{mn} q^n. \tag{1.1}$$

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Relation (1.1) shows that under $U(m)$, we may "shift" the "m" from the power of q in the first series to the subscript of the coefficients of the second series. This fact was known to Atkin and O'Brien [1, (28)].

In Section 2, we prove, with the aid of (1.1), Ramanujan's famous congruences [7]

$$p(5n+4) \equiv 0 \pmod 5, \tag{1.2}$$

$$p(7n+5) \equiv 0 \pmod 7 \tag{1.3}$$

and

$$p(11n+6) \equiv 0 \pmod{11}, \tag{1.4}$$

where $p(n)$ denotes the number of unrestricted partitions of the nonnegative integer n.

It is obvious that (1.2) and (1.3) follows from Ramanujan's identities

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\prod_{n=1}^{\infty}\frac{(1-q^{5n})^5}{(1-q^n)^6}, \tag{1.5}$$

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\prod_{n=1}^{\infty}\frac{(1-q^{7n})^3}{(1-q^n)^4} + 49q\prod_{n=1}^{\infty}\frac{(1-q^{7n})^7}{(1-q^n)^8}. \tag{1.6}$$

Identities such as (1.5) and (1.6) are more difficult to establish than congruences (1.2) and (1.3). In [10, (1.15)], Zuckerman obtained the following analogue of (1.5) and (1.6):

$$\begin{aligned}\sum_{n=0}^{\infty} p(13n+6)q^n
&= 11\prod_{n=1}^{\infty}\frac{(1-q^{13n})}{(1-q^n)^2} + 468q\prod_{n=1}^{\infty}\frac{(1-q^{13n})^3}{(1-q^n)^4} + 6422q^2\prod_{n=1}^{\infty}\frac{(1-q^{13n})^5}{(1-q^n)^6}\\
&\quad + 43940q^3\prod_{n=1}^{\infty}\frac{(1-q^{13n})^7}{(1-q^n)^8} + 171366q^4\prod_{n=1}^{\infty}\frac{(1-q^{13n})^9}{(1-q^n)^{10}}\\
&\quad + 371293q^5\prod_{n=1}^{\infty}\frac{(1-q^{13n})^{11}}{(1-q^n)^{12}} + 371293q^6\prod_{n=1}^{\infty}\frac{(1-q^{13n})^{13}}{(1-q^n)^{14}}.\end{aligned} \tag{1.7}$$

In Section 3, we use (1.1) and results in [4] to establish identities associated with

$$\sum_{n=0}^{\infty} p_{-r}(l^k n + \delta_{l,k,r})q^n, \quad l = 5, 7 \text{ and } 13,$$

where

$$\delta_{l,k,r} = \begin{cases} \dfrac{r(1-l^k)}{24} & \text{if } k \text{ is even,} \\[2mm] \dfrac{r(1-l^{k+1})}{24} & \text{if } k \text{ is odd} \end{cases} \tag{1.8}$$

and

$$\prod_{n=1}^{\infty}(1-q^n)^r = \sum_{n=0}^{\infty} p_r(n)q^n. \tag{1.9}$$

When $(l,k,r) = (5,1,-1), (7,1,-1)$, and $(13,1,-1)$ we obtain (1.5)–(1.7) and when $(l,k,r) = (5,1,-2)$ and $(l,k,r) = (5,1,-3)$, we find that

$$\sum_{n=0}^{\infty} p_{-2}(5n-2)q^n = 10q\prod_{n=1}^{\infty}\frac{(1-q^{5n})^4}{(1-q^n)^6} + 125q^2\prod_{n=1}^{\infty}\frac{(1-q^{5n})^{10}}{(1-q^n)^{12}} \tag{1.10}$$

and

$$\sum_{n=0}^{\infty} p_{-3}(5n-3)q^n = 9q\prod_{n=1}^{\infty}\frac{(1-q^{5n})^3}{(1-q^n)^6} + 375q^2\prod_{n=1}^{\infty}\frac{(1-q^{5n})^9}{(1-q^n)^{12}}$$

$$+ 3125q^3\prod_{n=1}^{\infty}\frac{(1-q^{5n})^{15}}{(1-q^n)^{18}}. \tag{1.11}$$

Identities (1.10) and (1.11) appear to be new.

2. Ramanujan's congruences

Congruence properties of $p_r(n)$ (see (1.9)) were studied by Ramanujan, who deduced (1.2) and (1.3) from

$$p_4(5n+4) \equiv 0 \pmod 5 \quad \text{and} \quad p_6(7n+5) \equiv 0 \pmod 7,$$

respectively. In [9], Winquist showed (1.4) by proving that

$$p_{10}(11n+6) \equiv 0 \pmod{11}.$$

Since then, many congruences have been discovered for $p_r(n)$ (see for example [2,5]). In this section, we show that in order to obtain congruences for $p_r(n)$ of the type

$$p_r(ln-N) \equiv 0 \pmod l, \quad n \geqslant 1,$$

it suffices to check if l divides $\tau_N(lj)$, $1 \leqslant j \leqslant N$, where

$$\Delta^N(z) := q^N\prod_{n=1}^{\infty}(1-q^n)^{24N} = \sum_{n=0}^{\infty}\tau_N(n)q^n, \quad q = e^{2\pi i z}.$$

Note that $\tau_1(n)$ is the famous Ramanujan's τ-function.

Proof of (1.2): It is known that $\Delta(z)$ is an eigenform in $\mathscr{S}_{12}(SL_2(\mathbb{Z}))$, where $\mathscr{S}_k(SL_2(\mathbb{Z}))$ denotes the space of weight k cusp forms invariant under $SL_2(\mathbb{Z})$. Hence,

$$\Delta(z)|_{T_p} = \tau(p)\Delta(z),$$

where T_p is the Hecke operator defined by

$$\left.\sum_{n=0}^{\infty} a_n q^n\right|_{T_p} = \sum_{n=0}^{\infty}(a(pn) + p^{k-1}a(n/p))q^n$$

with k being the weight of the modular form $\sum_{n=0}^{\infty} a_n q^n$ invariant under $SL_2(\mathbb{Z})$. Note that since the coefficient of q^5 in $\Delta(z)$ is $\tau(5) = 4830$, we conclude that

$$\Delta(z)|_{T_5} = \tau(5)\Delta(z) \equiv 0 \pmod 5. \tag{2.1}$$

We now write

$$\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{5s}\prod_{n=1}^{\infty}(1-q^n)^r \equiv \prod_{n=1}^{\infty}(1-q^{5n})^s\sum_{n=0}^{\infty} p_r(n-1)q^n \pmod 5, \tag{2.2}$$

where r and s are integers. Since

$$\left.\sum_{n=0}^{\infty} a_n q^n\right|_{U(p)} \equiv \left.\sum_{n=0}^{\infty} a_n q^n\right|_{T_p} \pmod 5,$$

we find by (1.1), (2.2) and (2.1) that

$$\left.\prod_{n=1}^{\infty}(1-q^{5n})^s\sum_{n=0}^{\infty} p_r(n-1)q^n\right|_{U(5)} \equiv \prod_{n=1}^{\infty}(1-q^n)^s\sum_{n=0}^{\infty} p_r(5n-1)q^n$$

$$\equiv \Delta(z)|_{T_5} \equiv 0 \pmod 5. \tag{2.3}$$

This implies that $p_r(5n-1) \equiv 0 \pmod 5$ for all r satisfying the equation

$$24 = 5s + r$$

or

$$p_{24-5s}(5n-1) \equiv 0 \pmod 5, \quad s \in \mathbb{Z},$$

which immediately yields Ramanujan's congruences for $p(5n+4)$ and $p_4(5n+4)$.

Our computation shows that one only needs to know $\tau(5)$ in $\Delta(z)$ in order to deduce the above congruences. In general, we always obtain a collection of congruences of the form

$$p_{24-ls}(ln-1) \equiv 0 \pmod l$$

for each l satisfying

$$\tau(l) \equiv 0 \pmod l. \tag{2.4}$$

Questions involving primes satisfying (2.4) can be found in [8, 5.2(b)].

Proof of (1.3): To prove Ramanujan's congruences for $p(7n+5)$, we express $\Delta^2(z)|_{T_7}$ in terms of $\Delta^2(z)$ and $\Delta(z)Q^3(q)$, where

$$Q(q) = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}.$$

This turns out to be

$$\Delta^2(z)|_{T_7} = -985\,824\Delta(z)Q^3(q) - 525\,803\,656\Delta^2(z). \tag{2.5}$$

Note that the coefficients of $\Delta(z)Q^3(q)$ and $\Delta^2(z)$ in the above identities are both divisible by 7. Hence we conclude that

$$p_{48-7s}(7n-2) \equiv 0 \pmod 7, \quad s \in \mathbb{Z}.$$

In particular, we obtain (1.3), as well as the congruence for $p_6(7n+5)$.

It is clear from the above calculations that to obtain congruences such as

$$p_{24N-ls}(ln-N) \equiv 0 \pmod{l},$$

it suffices to compute the image of $\Delta^N(z)$ under T_l. If

$$\Delta^N(z)|_{T_l} = a_1B_1 + a_2B_2 + \cdots + a_NB_N,$$

where $N=$ dimension of $\mathscr{S}_k(SL_2(\mathbb{Z}))$, then each a_i is a $\mathbb{Z}$-linear combination of $\tau_N(lj)$ for N values of j, $1 \leqslant j \leqslant N$. For example, in order to verify that

$$\tau_N(lj) \equiv 0 \pmod{l}$$

holds, it suffices to verify it for $1 \leqslant j \leqslant N$. Therefore, to prove (1.4), it suffices to check that 11 divides $\tau_5(11j)$, $1 \leqslant j \leqslant 5$.

3. Partition identities

In this section, we give proofs of (1.5)–(1.7) and their generalizations.

We begin this section with the proof of (1.5). It is known that $\eta(25z)/\eta(z)$ is a modular function on $\Gamma_0(25)$ [6], where

$$\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n).$$

Since

$$\frac{\eta(25z)}{\eta(z)} = \prod_{n=1}^{\infty}(1-q^{25n})\sum_{n=0}^{\infty}p(n-1)q^n,$$

we conclude by (1.1) that

$$\prod_{n=1}^{\infty}(1-q^{5n})\sum_{n=0}^{\infty}p(5n-1)q^n = \left.\frac{\eta(25z)}{\eta(z)}\right|_{U(5)}.$$

Following the method illustrated in [4, Theorem 4], we find that $\frac{\eta(25z)}{\eta(z)}|_{U(5)}$ is an entire modular function on $\Gamma_0(5)$. It is known that these functions are polynomials in $h_5(z) := \eta^6(5z)/\eta^6(z)$ [3]. Hence, we conclude immediately that

$$\prod_{n=1}^{\infty}(1-q^{5n})\sum_{n=0}^{\infty}p(5n-1)q^n = 5\,\frac{\eta^6(5\tau)}{\eta^6(z)} \tag{3.1}$$

which is (1.5).

The proof of (1.6) and (1.7) is similar since $\eta(l^2z)/\eta(z)$ is an entire modular function on $\Gamma_0(l^2)$ and entire modular functions on $\Gamma_0(7)$ and $\Gamma_0(13)$ are polynomials in $\eta^4(7z)/\eta^4(z)$ and $\eta^2(13z)/\eta^2(z)$ [3], respectively.

The method of proof illustrated above yields the following:

Theorem 3.1 (Lehner [4, Theorem 4]). *Let $l > 3$ be an odd prime. Then*

$$\prod_{n=1}^{\infty}(1-q^{ln})\sum_{n=0}^{\infty} p(ln+(1-l^2)/24)q^n$$

is an entire modular function on $\Gamma_0(l)$.

It is also known that $U(l)$ sends an entire modular function $f(z)$ on $\Gamma_0(l)$ to an entire modular function on $\Gamma_0(l)$ if f satisfies the transformation formula [4, (2.2)]

$$f(-1/lz) = cf(z) \quad \text{or} \quad f(-1/lz) = c/f(z). \tag{3.2}$$

This is clearly satisfied by the functions $\eta^6(5z)/\eta^6(z)$, $\eta^4(7z)/\eta^4(z)$ and $\eta^2(13z)/\eta^2(z)$, for $l = 5, 7$ and 13, respectively.

In the case of $l = 5$, we apply $U(5)$ to the left-hand side of (3.1) to conclude that [10, (1.13)]

$$\begin{aligned}\prod_{n=1}^{\infty}(1-q^n)\sum_{n=0}^{\infty} p(25n-1)q^n \\ &= 63\cdot 5^2\left(\frac{\eta(5z)}{\eta(z)}\right)^6 + 52\cdot 5^5\left(\frac{\eta(5z)}{\eta(z)}\right)^{12} \\ &\quad + 63\cdot 5^7\left(\frac{\eta(5z)}{\eta(z)}\right)^{18} + 6\cdot 5^{10}\left(\frac{\eta(5z)}{\eta(z)}\right)^{24} + 5^{12}\left(\frac{\eta(5z)}{\eta(z)}\right)^{30}.\end{aligned} \tag{3.3}$$

To obtain identities associated with higher power of 5, we first multiply

$$\prod_{n=1}^{\infty}(1-q^n)\sum_{n=0}^{\infty} p(5^2n-1)q^n$$

by $\eta(25z)/\eta(z)$ and note that each function on the right-hand side satisfies (3.2). Therefore, by applying $U(5)$, we conclude that

$$\prod_{n=1}^{\infty}(1-q^{5n})\sum_{n=0}^{\infty} p(5^3n-26)q^n$$

is an entire modular function on $\Gamma_0(5)$ and is expressible in terms of $h_5(z)$. It is clear that when we pass from an identity involving k, where k is an odd integer, to the corresponding identity for $k+1$, we only need to apply $U(5)$ to

$$\prod_{n=1}^{\infty}(1-q^{5n})\sum_{n=0}^{\infty} p(5^kn+\delta_{5,k})q^n,$$

where $\delta_{5,k} := \delta_{5,k,1}$, with $\delta_{l,k,r}$ defined as in (1.8). To obtain an identity corresponding to $k+1$ from an identity involving k, where k is even, we have to first multiply the identity involving

$$\prod_{n=1}^{\infty}(1-q^n)\sum_{n=0}^{\infty} p(5^kn+\delta_{5,k})q^n$$

by $\eta(25z)/\eta(z)$ before applying $U(5)$. In this way, we obtain an expression for

$$\prod_{n=1}^{\infty}(1-q^{\varepsilon_5 n})\sum_{n=0}^{\infty}p(5^k n+\delta_{5,k})q^n$$

in terms of $h_5(z)$ for all $k \in \mathbb{N}$, with

$$\varepsilon_l = \begin{cases} l & \text{if } k \text{ is odd,} \\ 1 & \text{if } k \text{ is even.} \end{cases} \tag{3.4}$$

This method can be found in [4, Theorem 7], where the case $l = 11$ is discussed.

The advantage of using (1.1) to obtain partition identities is that one does not need to know the modular behavior of the expressions such as $\sum_{n=0}^{\infty} p(5^k n+\delta_{5,k})q^n$. The method can be modified to obtain identities for $\sum_{n=0}^{\infty} p_{-r}(5^k n+\delta_{5,k,r})q^n$, where $p_r(n)$ and $\delta_{l,k,r}$ are defined in (1.9) and (1.8), respectively. All we have to do is to use

$$(\eta(25\tau)/\eta(\tau))^r$$

and follow the arguments illustrated as above to conclude that $\prod_{n=1}^{\infty}(1-q^{\varepsilon_5 n})^r \sum_{n=0}^{\infty} p_{-r}(5^k n+\delta_{5,k,r})q^n$ is a polynomial in $h_5(z)$, where ε_5 is defined in (3.4). For $(k,r)=(5,-2)$ and $(5,-3)$, we obtain (1.10) and (1.11), respectively.

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