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On the Ramanujan-Göllnitz-Gordon Continued Fraction

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Dedicated to Professor B. Gordon

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Abstract. We derive many new identities involving the Ramanujan-Göllnitz-Gordon continued fraction $H(q)$. These include relations between $H(q)$ and $H(q^n)$, which are established using modular equations of degree n. We also evaluate explicitly $H(q)$ at $q = e^{-\pi\sqrt{n}/2}$ for various positive integers n. Using results of M. Deuring, we show that $H(\pm e^{-\pi\sqrt{n}/2})$ are units for all positive integers n.

Key words: continued fraction, units, singular moduli, modular equations

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1. Introduction

Let

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1 - aq^n)$$

and let the Ramanujan-Göllnitz-Gordon continued fraction be defined as

$$H(q) := \frac{q^{\frac{1}{2}}}{1+q} \, \frac{}{+} \, \frac{q^2}{1+q^3} \, \frac{}{+} \, \frac{q^4}{1+q^5} \, \frac{}{+} \, \frac{q^6}{1+q^7} \, \frac{}{+} \, \cdots, \qquad |q| < 1.$$

On page 229 of his second notebook [13], Ramanujan recorded a product representation of $H(q)$, namely,

$$H(q) = q^{\frac{1}{2}} \frac{(q;q^8)_\infty (q^7;q^8)_\infty}{(q^3;q^8)_\infty (q^5;q^8)_\infty}. \tag{1.1}$$

Without any knowledge of Ramanujan's work, Göllnitz [10] and Gordon [11] rediscovered and proved (1.1) independently. Shortly thereafter, Andrews [1] proved (1.1) as a corollary of a more general result.

In addition to (1.1), Ramanujan offered two other identities [13, p. 229] for $H(q)$:

$$\frac{1}{H(q)} - H(q) = \frac{\varphi(q^2)}{q^{1/2}\psi(q^4)} \tag{1.2}$$

and

$$\frac{1}{H(q)} + H(q) = \frac{\varphi(q)}{q^{1/2}\psi(q^4)}, \tag{1.3}$$

where

$$\varphi(q) := \sum_{k=-\infty}^{\infty} q^{k^2}$$

and

$$\psi(q) := \sum_{k=0}^{\infty} q^{k(k+1)/2}.$$

Proofs of identities (1.2) and (1.3) can be found in Berndt's book [3, p. 221].

In this paper, we will establish several identities which will give us a better understanding of $H(q)$. Some of these identities are motivated by identities involving the Rogers-Ramanujan continued fraction [4]

$$F(q) := \frac{q^{\frac{1}{5}}}{1} \; + \frac{q}{1} + \frac{q^2}{1} \; + \frac{q^3}{1} \; + \cdots$$

and Ramanujan's cubic continued fraction [7]

$$G(q) = \frac{q^{\frac{1}{3}}}{1} \; + \frac{q+q^2}{1} \; + \frac{q^2+q^4}{1} \; + \frac{q^3+q^6}{1} \; + \cdots.$$

In Section 2, we will develop some basic identities satisfied by $H(q)$. These identities are proved by using (1.1) and some elementary theta-functions identities. In particular, we will reprove (1.2) and (1.3) by an approach slightly different from that in [3]. Then, by using these identities, we are able to establish relations between $H(q)$ and $H(-q)$, and $H(q)$ and $H(q^2)$.

Let, as customary,

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

where $(a)_k = (a)(a+1)\cdots(a+k-1)$, and $|z| < 1$. We say that the *modulus* β has degree n over the *modulus* α when

$$\frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)} = n \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)}. \tag{1.4}$$

A *modular equation of degree n* is a relation between α and β which is induced by (1.4). In Section 3, we will prove an important theorem which will allow us to deduce relations between $H(q)$ and $H(q^n)$ from modular equations of degree n and vice versa. In particular, we will use this theorem to obtain relations between $H(q)$ and the two continued fractions $H(q^3)$ and $H(q^4)$.

In Section 4, we will establish some explicit formulas for evaluating $H(e^{-\pi\sqrt{n}/2})$ in terms of Ramanujan-Weber class invariants. Using these formulas, we derive many numerical continued fractions. One such example, which first appeared in [10, (2.32)], is

$$H(e^{-\pi}) = \sqrt{4+2\sqrt{2}} - \sqrt{3+2\sqrt{2}}.$$

Observe that $H(e^{-\pi})$ is a unit. In fact, in the final section, we will show that $H(\pm e^{-\pi\sqrt{n}/2})$ *is a unit when n is a positive integer*. This is an analogue of a result recently established by Berndt et al. [4], which states that $F(e^{-\pi\sqrt{n}})$ *is a unit when n is a positive rational number*.

We conclude this introduction with the remark that the results established for $H(q)$ are also valid for

$$H_1(q) := \frac{q^{\frac{1}{2}}}{1} \, {}_{+} \, \frac{q+q^2}{1} \, {}_{+} \, \frac{q^4}{1} \, {}_{+} \, \frac{q^3+q^6}{1} \, {}_{+} \, \frac{q^8}{1} \, {}_{+} \cdots .$$

This follows from the identities (1.1) and

$$H_1(q) = q^{\frac{1}{2}} \frac{(q;q^8)_\infty (q^7;q^8)_\infty}{(q^3;q^8)_\infty (q^5;q^8)_\infty}, \tag{1.5}$$

which was stated by Ramanujan [13, p. 290; 14, p. 44] and first proved by Selberg [15]. Other proofs of (1.5) have also been given by Andrews [2] and Ramanathan [12].

2. Some identities satisfied by *H*(*q*)

Let

$$f(a,b) := \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}, \qquad |ab| < 1.$$

By Jacobi's triple product identity, we have [3, p. 35, Entry 19]

$$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty. \tag{2.1}$$

Using (2.1), we rewrite (1.1) as

$$H(q) = q^{\frac{1}{2}} \frac{f(-q,-q^7)}{f(-q^3,-q^5)}. \tag{2.2}$$

Theorem 2.1.

(i)
$$H^2(q) = \frac{\varphi(q) - \varphi(q^2)}{\varphi(q) + \varphi(q^2)},$$

(ii)
$$\frac{H^{-1}(q) - H(q)}{H^{-1}(q) + H(q)} = \frac{\varphi(q^2)}{\varphi(q)},$$

(iii)
$$H^{-2}(q) - H^2(q) = \frac{\varphi(q)\varphi(q^2)}{q\psi^2(q^4)},$$

(iv)
$$H^{-1}(q) - H(q) = \frac{\varphi(q^2)}{q^{1/2}\psi(q^4)},$$

and

(v)
$$H^2(q)H^2(-q) = -H^2(q^2).$$

Proof of (i): We recall from [3, p. 51] that

$$\varphi(-q) + \varphi(q^2) = 2\frac{f^2(q^3, q^5)}{\psi(q)} \tag{2.3}$$

and

$$\varphi(-q) - \varphi(q^2) = -2q\frac{f^2(q, q^7)}{\psi(q)}. \tag{2.4}$$

Dividing (2.4) by (2.3), and replacing q by $-q$, we find that

$$q\frac{f^2(-q, -q^7)}{f^2(-q^3, -q^5)} = \frac{\varphi(q) - \varphi(q^2)}{\varphi(q) + \varphi(q^2)}.$$

The result follows by (2.2). □

Proof of (ii): We first rewrite Theorem 2.1(i) as

$$H^2(q) = \frac{1 - \dfrac{\varphi(q^2)}{\varphi(q)}}{1 + \dfrac{\varphi(q^2)}{\varphi(q)}}. \tag{2.5}$$

Since $b = \frac{1-a}{1+a}$ is equivalent to $a = \frac{1-b}{1+b}$, we find that

$$\frac{1 - H^2(q)}{1 + H^2(q)} = \frac{\varphi(q^2)}{\varphi(q)},$$

by (2.5), and this clearly implies Theorem 2.1(ii). □

Proof of (iii): We recall from [3, p. 40, Entry 25(v), (vi)] that

$$\varphi^2(q)+\varphi^2(-q)=2\varphi^2(q^2) \tag{2.6}$$

and

$$\varphi^2(q)-\varphi^2(-q)=8q\psi^2(q^4). \tag{2.7}$$

By Theorem 2.1(i) and (2.6), we deduce that

$$\begin{aligned} H^{-2}(q)-H^2(q) &= \frac{\varphi(q)+\varphi(q^2)}{\varphi(q)-\varphi(q^2)}-\frac{\varphi(q)-\varphi(q^2)}{\varphi(q)+\varphi(q^2)}=\frac{4\varphi(q)\varphi(q^2)}{\varphi^2(q)-\varphi^2(q^2)} \\ &= \frac{4\varphi(q)\varphi(q^2)}{\varphi^2(q)-\dfrac{\varphi^2(q)+\varphi^2(-q)}{2}}=\frac{8\varphi(q)\varphi(q^2)}{\varphi^2(q)-\varphi^2(-q)}. \end{aligned} \tag{2.8}$$

Using (2.7), we simplify (2.8) to obtain

$$H^{-2}(q)-H^2(q)=\frac{\varphi(q)\varphi(q^2)}{q\psi^2(q^4)},$$

which is Theorem 2.1(iii). □

Proof of (iv): By Theorem 2.1(ii) and (iii), we have

$$\begin{aligned} (H^{-1}(q)-H(q))^2 &= (H^{-2}(q)-H^2(q))\frac{H^{-1}(q)-H(q)}{H^{-1}(q)+H(q)} \\ &= \frac{\varphi(q)\varphi(q^2)}{q\psi^2(q^4)}\frac{\varphi(q^2)}{\varphi(q)}=\frac{\varphi^2(q^2)}{q\psi^2(q^4)}. \end{aligned} \tag{2.9}$$

Taking square roots on both sides of (2.9), we deduce Theorem 2.1(iv). □

Proof of (v): From [3, p. 46, Entry 30(iv)], we have

$$f(a,b)f(-a,-b)=f(-a^2,-b^2)\varphi(-ab). \tag{2.10}$$

By (2.2) and (2.10), we deduce that

$$\begin{aligned} H^2(q)H^2(-q) &= -q^2\left(\frac{f(-q,-q^7)f(q,q^7)}{f(-q^3,-q^5)f(q^3,q^5)}\right)^2 \\ &= -q^2\left(\frac{f(-q^2,-q^{14})\varphi(-q^8)}{f(-q^6,-q^{10})\varphi(-q^8)}\right)^2=-H^2(q^2). \end{aligned}$$

This completes the proof of Theorem 2.1(v). □

By using Theorem 2.1, we can find relations between $H(q)$ and $H(-q)$, and $H(q)$ and $H(q^2)$.

Theorem 2.2. *Let* $u = H(q)$, $v = H(-q)$, *and* $w = H(q^2)$. *Then*

(i) $$\left(\frac{1}{u} - u\right)^2 + \left(\frac{1}{v} - v\right)^2 = 0$$

and

(ii) $$u^2 = w\frac{1-w}{1+w}.$$

Proof of (i): Squaring both sides of the identity (iv) in Theorem 2.1, we arrive at

$$\left(\frac{1}{u} - u\right)^2 = \frac{\varphi^2(q^2)}{q\psi^2(q^4)}. \tag{2.11}$$

Replacing q by $-q$ in (2.11), we find that

$$\left(\frac{1}{v} - v\right)^2 = -\frac{\varphi^2(q^2)}{q\psi^2(q^4)}. \tag{2.12}$$

Adding (2.11) and (2.12), we complete the proof. □

Proof of (ii): We first rewrite (i) as

$$\left(\frac{1}{u} - u\right)^2 = -\frac{1}{v^2} + 2 - v^2. \tag{2.13}$$

From (2.13) and Theorem 2.1(v), we have

$$\left(\frac{1}{u} - u\right)^2 = \left(\frac{u}{w}\right)^2 + 2 + \left(\frac{w}{u}\right)^2. \tag{2.14}$$

Simplifying (2.14), we obtain

$$\frac{1}{u} - u = \frac{u}{w} + \frac{w}{u},$$

which yields the desired result, after some simple manipulations. □

3. Modular equations of degree n and relations between $H(q)$ and $H(q^n)$

Theorem 3.1. *If*

$$q = \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)}\right), \tag{3.1}$$

then

$$\left(\frac{H^{-1}(\sqrt{q}) - H(\sqrt{q})}{2}\right)^4 = \frac{1}{\alpha}. \tag{3.2}$$

Proof: If we replace q by $\sqrt{q}$ in Theorem 2.1(iv), divide both sides by 2, and then raise both sides to the fourth power, we find that

$$\left(\frac{H^{-1}(\sqrt{q}) - H(\sqrt{q})}{2}\right)^4 = \frac{\varphi^4(q)}{16q\psi^4(q^2)}. \tag{3.3}$$

On the other hand, we recall from [3, p. 40, Entry 25(vii)] that

$$\varphi^4(q) - \varphi^4(-q) = 16q\psi^4(q^2). \tag{3.4}$$

Substituting (3.4) into (3.3), we find that

$$\left(\frac{H^{-1}(\sqrt{q}) - H(\sqrt{q})}{2}\right)^4 = \frac{\varphi^4(q)}{\varphi^4(q) - \varphi^4(-q)} = \frac{1}{1 - \frac{\varphi^4(-q)}{\varphi^4(q)}}. \tag{3.5}$$

From [3, p. 100, Entry 5], we know that the identity (3.1) implies that

$$\alpha = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}. \tag{3.6}$$

Combining (3.6) and (3.5), we complete the proof. □

Now, let α and q be related by (3.1). If β has degree n over α, then Theorem 3.1 gives us

$$\left(\frac{H^{-1}(q^{n/2}) - H(q^{n/2})}{2}\right)^4 = \frac{1}{\beta}. \tag{3.7}$$

Hence, by (3.2), (3.7), and any given modular equation of degree n, we can obtain a relation between $H(\sqrt{q})$ and $H(q^{n/2})$. Replacing q by q^2 will then yield a relation between $H(q)$ and $H(q^n)$. We illustrate these ideas with $n = 3$ and 4.

Corollary 3.2. *Let $u = H(q)$, $v = H(q^3)$, and $w = H(q^4)$. Then*

(i) $$3uv(1-uv)(u+v) + (u^3 - v)(1 + uv^3) = 0$$

and

(ii) $$u = \sqrt{\sqrt{\left(\frac{2w(1-w)}{1+w^2}\right)^2 + \frac{w(1-w)}{1+w}} - \frac{2w(1-w)}{1+w^2}}.$$

Proof of (i): Let

$$x := H(\sqrt{q}) \qquad \text{and} \qquad y := H(q^{3/2}),$$

where q is given by (3.1). When β has degree 3 over α, we have [3, p. 231, Entry 5(xiii)]

$$\left(\frac{\beta}{\alpha}\right)^{1/4} - \left(\frac{\alpha}{\beta}\right)^{1/4} = 2((\alpha\beta)^{1/8} - (\alpha\beta)^{-1/8}). \tag{3.8}$$

By Theorem 3.1, (3.8) is equivalent to

$$\frac{x^{-1}-x}{y^{-1}-y} - \frac{y^{-1}-y}{x^{-1}-x} = 2\left(\frac{2}{\sqrt{(x^{-1}-x)(y^{-1}-y)}} - \frac{\sqrt{(x^{-1}-x)(y^{-1}-y)}}{2}\right). \tag{3.9}$$

Using *Mathematica*, we simplify (3.9) to arrive at

$$\begin{aligned}&(x^4y^3 - 3x^3y^2 + x^3 - 3x^2y^3 + 3x^2y - xy^4 + 3xy^2 - y)\\ &\quad\times (x^3y^4 - 3x^2y^3 + y^3 - 3x^3y^2 + 3xy^2 - x^4y + 3x^2y - x) = 0.\end{aligned}$$

The second factor in the product is a non-zero function when $|q| \to 0$, since there is only one term x with leading term $q^{1/4}$. Hence, we conclude that

$$x^4y^3 - 3x^3y^2 + x^3 - 3x^2y^3 + 3x^2y - xy^4 + 3xy^2 - y = 0,$$

i.e.,

$$3xy(1-xy)(x+y) + (x^3-y)(1+xy^3) = 0.$$

Replacing q by q^2, we complete the proof. □

Proof of (ii): When β has degree 4 over α, we have [3, p. 215, Eq. (24.22)]

$$\sqrt{\beta} = \left(\frac{1-(1-\alpha)^{1/4}}{1+(1-\alpha)^{1/4}}\right)^2.$$

Replacing α by $1-\beta$ and β by $1-\alpha$, we arrive at another modular equation of degree 4, which is

$$\sqrt{1-\alpha} = \left(\frac{1-\beta^{1/4}}{1+\beta^{1/4}}\right)^2, \tag{3.10}$$

by [3, p. 216, Entry 24(v)].

If

$$q^2 = \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)}\right),$$

then, by Theorem 3.1, we deduce that

$$\left(\frac{u^{-1}-u}{2}\right)^4 = \frac{1}{\alpha} \tag{3.11}$$

and

$$\left(\frac{w^{-1}-w}{2}\right)^4 = \frac{1}{\beta}. \tag{3.12}$$

Squaring both sides of (3.10) and combining it with (3.11) and (3.12), we find that

$$1 - \left(\frac{2}{u^{-1}-u}\right)^4 = \left(\frac{w^{-1}-w-2}{w^{-1}-w+2}\right)^4. \tag{3.13}$$

Upon simplifying (3.13) using *Mathematica*, we arrive at

$$\begin{aligned}&(u^4w^3 + u^4w^2 + u^4w + u^4 - 4u^2w^3 + 4u^2w + w^4 - w^3 + w^2 - w)\\ &\quad\times(1 + u^4w^4 - u^4w^3 + u^4w^2 - u^4w - 4u^2w^3 + 4u^2w + w^3 + w^2 + w) = 0.\end{aligned}$$

Clearly, the second factor in the product does not vanish when $|q| \to 0$. Therefore,

$$u^4w^3 + u^4w^2 + u^4w + u^4 - 4u^2w^3 + 4u^2w + w^4 - w^3 + w^2 - w = 0. \tag{3.14}$$

Rearranging (3.14), we find that

$$(w+1)(w^2+1)u^4 - 4w(w-1)(w+1)u^2 + w(w-1)(w^2+1) = 0. \tag{3.15}$$

By regarding (3.15) as a quadratic equation of u^2, we conclude that

$$u^2 = \frac{-2w(1-w)}{1+w^2} + \sqrt{\left(\frac{2w(1-w)}{1+w^2}\right)^2 + \frac{w(1-w)}{1+w}}, \tag{3.16}$$

where the $+$ sign in front of the radical is verified by letting $0 < q < 1$. Finally, taking the square roots on both sides of (3.16) yields the result. □

4. Explicit formulas for the evaluations of *H*(*q*)

Let $q_n := e^{-\pi\sqrt{n}}$, and let the corresponding value of α in (3.1) be denoted by α_n; $\sqrt{\alpha_n}$ is called a singular modulus. Then, by applying Theorem 3.1 and solving (3.2) for $H(q_n)$, we have

$$H(e^{-\pi\sqrt{n}/2}) = \sqrt{\sqrt{\frac{1}{\alpha_n}}+1} - \sqrt{\sqrt{\frac{1}{\alpha_n}}}. \tag{4.1}$$

It is known from [3, p. 97] that $\alpha_1 = 1/2$, $\alpha_2 = (\sqrt{2}-1)^2$, and $\alpha_4 = (\sqrt{2}-1)^4$. Hence, by using (4.1), we deduce that

$$H(e^{-\pi/2}) = \sqrt{\sqrt{2}+1} - \sqrt{\sqrt{2}}, \tag{4.2}$$

$$H(e^{-\pi/\sqrt{2}}) = \sqrt{\sqrt{2}+2} - \sqrt{\sqrt{2}+1}, \tag{4.3}$$

and

$$H(e^{-\pi}) = \sqrt{4+2\sqrt{2}} - \sqrt{3+2\sqrt{2}}. \tag{4.4}$$

In fact, more values of $H(q)$ can be obtained simply by using (4.1) and known values of α_n [5]. But this process does not always give us elegant radicals for the values of $H(q)$. Hence, we require another expression for the right hand side of (4.1).

Theorem 4.1. *Let the Ramanujan-Weber class invariants be defined by*

$$G_n := 2^{-1/4} {q_n}^{-1/24} \left(-q_n; q_n^2\right)_\infty \tag{4.5}$$

and

$$g_n := 2^{-1/4} {q_n}^{-1/24} \left(q_n; q_n^2\right)_\infty, \tag{4.6}$$

where $q_n = e^{-\pi\sqrt{n}}$, *and set* $p = G_n^{12}$ *and* $p_1 = g_n^{12}$. *Then*

(i) $$H(e^{-\pi\sqrt{n}/2}) = \sqrt{\sqrt{p(p+1)}+\sqrt{p(p-1)}+1} - \sqrt{\sqrt{p(p+1)}+\sqrt{p(p-1)}}$$
$$= \sqrt{p_1+\sqrt{p_1^2+1}+1} - \sqrt{p_1+\sqrt{p_1^2+1}}$$

and

(ii) $$H(e^{-\pi\sqrt{n}}) = \sqrt{\left(\sqrt{p+1}+\sqrt{p}\right)^2\left(\sqrt{p}+\sqrt{p-1}\right)^2+1}$$
$$- \left(\sqrt{p+1}+\sqrt{p}\right)\left(\sqrt{p}+\sqrt{p-1}\right)$$
$$= \left(\sqrt{\sqrt{p(p+1)}+\sqrt{p(p-1)}+1} - \sqrt{\sqrt{p(p+1)}+\sqrt{p(p-1)}}\right)$$
$$\times \left(\sqrt{\sqrt{p(p+1)}+\sqrt{p(p-1)}} - \sqrt{\sqrt{p(p+1)}+\sqrt{p(p-1)}-1}\right)$$

$$= \left(\sqrt{p_1+\sqrt{p_1^2+1}+1}-\sqrt{p_1+\sqrt{p_1^2+1}}\right)$$
$$\times\left(\sqrt{p_1+\sqrt{p_1^2+1}}-\sqrt{p_1+\sqrt{p_1^2+1}-1}\right).$$

Proof of (i): Since [5]

$$G_n=(4\alpha_n(1-\alpha_n))^{-1/24}, \tag{4.7}$$

we deduce that

$$\frac{1}{\alpha_n}=\left(\sqrt{p(p+1)}+\sqrt{p(p-1)}\right)^2. \tag{4.8}$$

Using (4.8) in (4.1), we obtain the first equality of (i).

Next, from [5],

$$\frac{1}{\sqrt{\alpha_n}}-\sqrt{\alpha_n}=2g_n^{12}. \tag{4.9}$$

Hence,

$$\frac{1}{\sqrt{\alpha_n}}=p_1+\sqrt{p_1^2+1}. \tag{4.10}$$

From (4.10) and (4.1), we deduce the second equality. □

Proof of (ii): By (4.7) and (4.8), we deduce that

$$\frac{1}{1-\alpha_n}=\left(\sqrt{p(p+1)}-\sqrt{p(p-1)}\right)^2. \tag{4.11}$$

From [3, p. 213, Eq. (24.12)], (4.8) and (4.11),

$$\begin{aligned}\alpha_{4n}&=\left(\sqrt{\sqrt{\frac{1}{\alpha_n}}}-\sqrt{\sqrt{\frac{1}{\alpha_n}}-1}\right)^4\\&=\left(\sqrt{p+1}+\sqrt{p-1}\right)^4\left(\sqrt{p}-\frac{\sqrt{p+1}+\sqrt{p-1}}{2}\right)^4\\&=\left(\sqrt{p+1}-\sqrt{p}\right)^4\left(\sqrt{p}-\sqrt{p-1}\right)^4.\end{aligned} \tag{4.12}$$

Using (4.12) in (4.1), we obtain the first equality of (ii). Next, from Theorem 2.2(i) and (3.2), we observe that

$$H(-e^{-\pi\sqrt{n}/2})=-i\left(\sqrt{\sqrt{\frac{1}{\alpha_n}}}-\sqrt{\sqrt{\frac{1}{\alpha_n}}-1}\right). \tag{4.13}$$

Using (4.1), (4.13) and Theorem 2.1(v), we obtain

$$H(e^{-\pi\sqrt{n}}) = \left(\sqrt{\sqrt{\frac{1}{\alpha_n}}+1} - \sqrt{\sqrt{\frac{1}{\alpha_n}}}\right)\left(\sqrt{\sqrt{\frac{1}{\alpha_n}}} - \sqrt{\sqrt{\frac{1}{\alpha_n}}-1}\right). \tag{4.14}$$

Substituting (4.8) and (4.10) into (4.14), respectively, we arrive at the second and third equalities of (ii). □

Examples. Let $n = 1$. Since $G_1 = 1$, Theorem 4.1 yields (4.2) and (4.4). Let $n = 2$. Since $g_2 = 1$ [16, p. 721], Theorem 4.1 gives (4.3) and

$$H(e^{-\pi\sqrt{2}}) = \left(\sqrt{\sqrt{2}+2} - \sqrt{\sqrt{2}+1}\right)\left(\sqrt{\sqrt{2}+1} - \sqrt{\sqrt{2}}\right).$$

When $n = 3$, $G_3^{12} = 2$ [16, p. 721] and hence,

$$H(e^{-\pi\sqrt{3}/2}) = \sqrt{\sqrt{6}+\sqrt{2}+1} - \sqrt{\sqrt{6}+\sqrt{2}}$$

and

$$\begin{aligned} H(e^{-\pi\sqrt{3}}) &= \sqrt{(\sqrt{3}+\sqrt{2})^2(\sqrt{2}+1)^2+1} - (\sqrt{3}+\sqrt{2})(\sqrt{2}+1) \\ &= \left(\sqrt{\sqrt{6}+\sqrt{2}+1} - \sqrt{\sqrt{6}+\sqrt{2}}\right)\left(\sqrt{\sqrt{6}+\sqrt{2}} - \sqrt{\sqrt{6}+\sqrt{2}-1}\right). \end{aligned}$$

It is clear that the continued fractions we discovered are all units. It is therefore natural to conjecture that $H(e^{-\pi\sqrt{n}/2})$ is a unit when n is a positive integer. To prove this, by (4.1), it suffices to show that $1/\alpha_n$ is an algebraic integer for each positive integer n.

5. $H(\pm e^{-\pi\sqrt{n}/2})$ is a unit when n is a positive integer

We first recall some basic definitions from algebraic number theory. An *order* $\mathfrak{O}_f$ with *conductor* f in a quadratic field K is a subset $\mathfrak{O}_f \subset K$ such that

(i) $\mathfrak{O}_f$ is a subring of K containing 1,
(ii) $\mathfrak{O}_f$ is a finitely generated $\mathbb{Z}$-module,
(iii) $\mathfrak{O}_f$ contains a $\mathbb{Q}$-basis of K, and
(iv) $[\mathfrak{O}_K : \mathfrak{O}_f] = f$,

where $\mathfrak{O}_K = \mathfrak{O}_1$ is the ring of integers of K.

If α_1 and α_2 generate an $\mathfrak{O}_f$-ideal $\mathfrak{a}_{\mathfrak{O}_f}$ over $\mathbb{Z}$, we say that $[\alpha_1, \alpha_2]$ is a basis of $\mathfrak{a}_{\mathfrak{O}_f}$. An ideal $\mathfrak{a}_{\mathfrak{O}_f}$ is said to be *proper* if $\mathfrak{a}_{\mathfrak{O}_f}$ is coprime to f. For more details on orders of an imaginary quadratic field, we refer the reader to [8].

If η is an algebraic integer and $\mathfrak{a}$ is an $\mathfrak{O}_K$ ideal, we write $\eta \approx \mathfrak{a}$ to mean that $\eta\mathfrak{O}_\Omega = \mathfrak{a}\mathfrak{O}_\Omega$ in some large number field Ω. Similarly, if η_1 and η_2 are algebraic integers, we write $\eta_1 \approx \eta_2$ to mean that η_1/η_2 is a unit.

Theorem 5.1. *Suppose p is a prime such that $p^t \| f$, where f is a positive integer. Let a, b, c and d be integers, and $P := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a matrix with determinant p and $\Delta(\tau) = q(q;q)_\infty^{24}$, where $q = e^{2\pi i\tau}$, with $\mathrm{Im}(\tau) > 0$. Define*

$$\varphi_P(\tau) = p^{12}\frac{\Delta\left(P\begin{pmatrix}\tau_1\\ \tau_2\end{pmatrix}\right)}{\Delta\begin{pmatrix}\tau_1\\ \tau_2\end{pmatrix}}, \tag{5.1}$$

where $\tau = \frac{\tau_1}{\tau_2}$ and $\Delta\left(\begin{smallmatrix}\tau_1\\ \tau_2\end{smallmatrix}\right) = \tau_2^{-12}\Delta(\tau)$. Let $[\alpha_1, \alpha_2]$ be a basis of a proper $\mathfrak{O}_f$-ideal $\mathfrak{a}_{\mathfrak{O}_f}$ and set $\alpha = \frac{\alpha_1}{\alpha_2}$. The action of P on the basis $[\alpha_1, \alpha_2]$ is defined as

$$P[\alpha_1, \alpha_2] := [a\alpha_1 + b\alpha_2, c\alpha_1 + d\alpha_2].$$

1. *When p splits completely in K, namely $p = \mathfrak{p}\bar{\mathfrak{p}}$, then*
   (1.1) *$\varphi_P(\alpha)$ is a unit if $P[\alpha_1, \alpha_2]$ is a basis of a proper $\mathfrak{O}_{fp}$-ideal,*
   (1.2) *$\varphi_P(\alpha) \approx p^{12}$ if $P[\alpha_1, \alpha_2]$ is a basis of a proper $\mathfrak{O}_{fp^{-1}}$-ideal, and*
   (1.3) *if $p \nmid f$, then $\varphi_P(\alpha) \approx \bar{\mathfrak{p}}^{12}$ and $\varphi_P(\alpha) \approx \mathfrak{p}^{12}$ when $P[\alpha_1, \alpha_2]$ is a basis of $\mathfrak{a}_{\mathfrak{O}_f}\mathfrak{p}_{\mathfrak{O}_f}$ and $P[\alpha_1, \alpha_2]$ is a basis of $\mathfrak{a}_{\mathfrak{O}_f}\bar{\mathfrak{p}}_{\mathfrak{O}_f}$, respectively.*
2. *When p ramifies in K, namely $p = \mathfrak{p}^2$, then*
   (2.1) *$\varphi_P(\alpha) \approx p^{6/p^{t+1}}$ if $P[\alpha_1, \alpha_2]$ is a basis of a proper $\mathfrak{O}_{fp}$-ideal,*
   (2.2) *$\varphi_P(\alpha) \approx p^{12-6/p^t}$ if $P[\alpha_1, \alpha_2]$ is a basis of a proper $\mathfrak{O}_{fp^{-1}}$-ideal, and*
   (2.3) *$\varphi_P(\alpha) \approx p^6$ if $P[\alpha_1, \alpha_2]$ is a basis of $\mathfrak{a}_{\mathfrak{O}_f}\mathfrak{p}_{\mathfrak{O}_f}$.*
3. *When p is inert in K, then*
   (3.1) *$\varphi_P(\alpha) \approx p^{12/p^t(p+1)}$ if $P[\alpha_1, \alpha_2]$ is a basis of a proper $\mathfrak{O}_{fp}$-ideal, and*
   (3.2) *$\varphi_P(\alpha) \approx p^{12(1-1/p^{t-1}(p+1))}$ if $P[\alpha_1, \alpha_2]$ is a basis of a proper $\mathfrak{O}_{fp^{-1}}$-ideal.*

Proof: See [9, p. 43]. □

Corollary 5.2. *Let the Ramanujan-Weber class invariants be defined by* (4.5) *and* (4.6). *Then*
(i) *for $n \equiv 1 \pmod 4$, G_n is a unit,*
(ii) *for $n \equiv 3 \pmod 8$, $2^{-1/12}G_n$ is a unit,*
(iii) *for $n \equiv 7 \pmod 8$, $2^{-1/4}G_n$ is a unit, and*
(iv) *for $n \equiv 2 \pmod 4$, g_n is a unit.*

Proof: Throughout the proof, we will assume that $n = f^2 d$, where d is squarefree. We will also let $K = \mathbb{Q}(\sqrt{-d})$.

(i) When $n \equiv 1 \pmod 4$, $d \equiv 1 \pmod 4$, $\mathfrak{O}_K = \mathbb{Z}[\sqrt{-d}]$ and $(2) = \mathfrak{p}^2$ ramifies in $\mathfrak{O}_K$. Let $[\sqrt{-n}, 1]$ be a basis of $\mathfrak{O}_f$ and $P = \left(\begin{smallmatrix} 1 & f \\ 0 & 2 \end{smallmatrix}\right)$. Since $P[\sqrt{-n}, 1] = [\sqrt{-n} + f, 2]$ is a basis of $\mathfrak{p}\mathfrak{O}_f$, we deduce from Theorem 5.1(2.3) that

$$\varphi_P(\sqrt{-n}) \approx 2^6. \tag{5.2}$$

By (5.2), (5.1) and observing that f is odd, we conclude that

$$2^{-6}\frac{\Delta\left(\frac{\sqrt{-n}+1}{2}\right)}{\Delta\left(\sqrt{-n}\right)} \tag{5.3}$$

is a unit, which implies the desired result, since (5.3) reduces to $-G_n^{24}$ by using the definition of Δ appeared in Theorem 5.1 and (4.5).

(ii) When $n \equiv 3 \pmod 8$, $d \equiv 3 \pmod 8$, $\mathfrak{O}_K = \mathbb{Z}[\frac{1+\sqrt{-d}}{2}]$ and (2) is inert in $\mathfrak{O}_K$. Let $[\sqrt{-n} + f, 1]$ be a basis of a $\mathfrak{O}_{2f}$-ideal and $P = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$. Since $P[\sqrt{-n} + f, 1]$ is a basis of a $\mathfrak{O}_f$-ideal, we conclude that

$$\varphi_P(\sqrt{-n} + f) \approx 2^8,$$

by Theorem 5.1(3.2). Now, by a similar argument as in (i), we obtain

$$(2^{1/4}G_n)^{24} \approx 2^8,$$

which implies that $2^{-1/12}G_n$ is a unit.

(iii) When $n \equiv 7 \pmod 8$, $d \equiv 7 \pmod 8$, $\mathfrak{O}_K = \mathbb{Z}\left[\frac{1+\sqrt{-d}}{2}\right]$ and (2) splits completely in $\mathfrak{O}_K$. Let $[\sqrt{-n} + f, 1]$ be a basis of a $\mathfrak{O}_{2f}$-ideal and $P = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$. Since $P[\sqrt{-n} + f, 1]$ is a basis of a $\mathfrak{O}_f$-ideal, we find that

$$\varphi_P(\sqrt{-n} + f) \approx 2^{12},$$

by Theorem 5.1(1.2). Therefore, $2^{-1/4}G_n$ is a unit by (4.5) and (5.1).

(iv) When $n \equiv 2 \pmod 4$, $d \equiv 2 \pmod 4$, $\mathfrak{O}_K = \mathbb{Z}[\sqrt{-d}]$ and $(2) = \mathfrak{p}^2$ ramifies in $\mathfrak{O}_K$. Let $[\sqrt{-n}, 1]$ be a basis of $\mathfrak{O}_f$ and $P = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$. Then $P[\sqrt{-n}, 1]$ is a basis of $\mathfrak{p}\mathfrak{O}_f$. By Theorem 5.1(2.3), we deduce that

$$\varphi_P(\sqrt{-n}) \approx 2^6. \tag{5.4}$$

Hence, by (5.4) and (5.1), we find that

$$\frac{\Delta\left(\frac{\sqrt{-n}}{2}\right)}{\Delta\left(\sqrt{-n}\right)} \approx 2^6, \tag{5.5}$$

which implies that g_n is a unit, since the left hand side of (5.5) reduces to $2^6 g_n^{24}$ by the definition of Δ and (4.6). This completes the proof of the corollary. □

Remarks. The results given in Corollary 5.2 were briefly mentioned in [6] without any proofs. There is a misprint on page 291, line 14 of [6]. The statement "...$\sigma^3(\omega)$ is a real unit..." should be replaced by "...$2^{-1}\sigma^3(\omega)$ is a real unit...."

From Corollary 5.2, we deduce the following theorem.

Theorem 5.3. *Let*

$$\sqrt{n} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha_n\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha_n\right)}.$$

Then ${\alpha_n}^{-1}$ *is an algebraic integer for every positive integer* n*.*

Proof: From (4.7) we deduce that

$$\left(\frac{1}{\alpha_n}\right)^2 - 4G_n^{24}\left(\frac{1}{\alpha_n}\right) + 4G_n^{24} = 0. \tag{5.6}$$

By Corollary 5.2(i), (ii) and (iii), we find that G_n^{24} is an algebraic integer when n is odd. Hence, by (5.6), α_n^{-1} is an algebraic integer when n is odd.

By Corollary 5.2(iv) and (4.9), we conclude that α_n^{-1} is a unit when $n \equiv 2 \pmod 4$. In order to complete the proof of the theorem, it remains to show that α_n^{-1} is a unit when $n \equiv 0 \pmod 4$.

Let $m \equiv \pm 1,\ 2 \pmod 4$ and k be a positive integer. We will show by induction that α_n^{-1} is a unit when $n = 4^k m$. Since α_m^{-1} is an algebraic integer when $4 \nmid m$, we conclude from (4.12) that α_{4m} is a unit. This proves the case $k = 1$.

Now, suppose $\alpha_{4^{k-1}m}^{-1}$ is an algebraic integer. Then applying (4.12) again, we conclude that $\alpha_{4^k m}^{-1}$ is an algebraic integer. This completes the proof of the theorem. □

Combining (4.1) and Theorems 5.3 and 2.1(v), we deduce the following corollary.

Corollary 5.3. *For every positive integer* n, $H(\pm e^{-\pi\sqrt{n}/2})$ *is a unit.*

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