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On p-adic stochastic dynamics, supersymmetry and the Riemann conjecture

Carlos Castro

Center for Theoretical Studies of Physical Systems, Clark Atlanta University, Atlanta, GA 30314, USA

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Dedicated to the memory of Michael Conrad

Abstract

We construct (assuming the quantum inverse scattering problem has a solution) the operator that yields the zeroes of the Riemann zeta function by defining explicitly the supersymmetric quantum mechanical model (SUSY QM) associated with the p-adic stochastic dynamics of a particle undergoing a Brownian random walk. The zig-zagging occurs after collisions with an infinite array of scattering centers that *fluctuate randomly*. Arguments are given to show that this physical system can be modelled as the scattering of the particle about the infinitely many locations of the prime numbers positions. We are able then to reformulate such a p-adic stochastic process, that has an underlying hidden Parisi–Sourlas supersymmetry, as the *effective* motion of a particle in a potential which can be expanded in terms of an infinite collection of p-adic harmonic oscillators with fundamental (Wick-rotated imaginary) frequencies $\omega_p = \mathrm{i}\,\log p$ (p is a prime) and whose harmonics are $\omega_{p,n} = \mathrm{i}\,\log p^n$. The p-adic harmonic oscillator potential allows us to determine a one-to-one correspondence between the amplitudes of oscillations a_n (and phases) with the imaginary parts of the zeroes of the Riemann zeta function, λ_n, after solving the inverse scattering problem. We review our recent proof of the Riemann hypothesis that the non-trivial zeroes of zeta are of the form $s = 1/2 + \mathrm{i}\lambda_n$ and the solution to the quantum inverse scattering problem.

1. Introduction

The Riemann conjecture, that the non-trivial zeroes of the zeta function lie on the vertical line $z = 1/2 + \mathrm{i}y$ of the complex plane, remains one of the most important unsolved problems in pure Mathematics. Hilbert and Polya suggested long ago that the zeroes of zeta might have an spectral interpretation in terms of the eigenvalues of a suitable self-adjoint trace class linear differential operator. Finding such an operator, if it exists, would be tantamount to proving the Riemann conjecture. There is a related analogy with the Laplace–Beltrami operator in the hyperbolic plane, a surface of constant negative curvature. The motion of a billiard ball in such surfaces is a typical example of classical chaotic motion. The Selberg zeta function associated with such a Laplace–Beltrami operator admits zeroes which can be related to the energy eigenvalues of the operator. Since the zeroes of the Riemann zeta function are deeply connected with the distribution of primes by the explicit Hadamard formula below it has been suggested by many authors that the spectral properties of the zeroes of zeta may be associated with the statistical fluctuations of the energy levels of a quantum system whose underlying classical dynamics are chaotic. Random matrix theory [1] has numerous physical applications in all branches of science. Connections between the asymptotic distribution of primes, $x/\log x$ and atmospheric turbulence in the cloud formation and the distribution of vortices and eddies in the form of the logarithmic spiral with

E-mail address: castro@ctsps.cau.edu (C. Castro).

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the golden mean winding number have been analyzed by Selvam [2]. Montgomery has shown [3] that the two-level correlation function of the imaginary parts of the zeroes of zeta is exactly the same expression found by Wigner and Dyson using random matrices techniques: the two-level spectral density correlation function of the Brownian-like discrete level statistical dynamics associated with the random matrix model of a Gaussian Unitary Ensemble (GUE) turned out to be

$$1 - \left[\frac{\sin(\pi x)}{\pi x}\right]^2. \tag{1}$$

The function $(\sin(\pi x))/(\pi x)$ appears very naturally as well in the self-similarity of the iterated symbolic dynamics of the binary Fibonacci "rabbit" sequence of numbers. It is the Fourier amplitude spectrum of the iterated binary sequence and the golden mean plays a fundamental role in generating a long range but aperiodic order, a one-dimensional quasi-crystal, see the book by Schroeder [20]. The golden mean is the limit as $n = \infty$ of the ratio of two successive Fibonacci numbers. Physically the relation (1) means that there is a "repulsion" of the energy levels; i.e., the probability of two energy levels being very close becomes very small. For signals of Chaos in M(atrix) theory (Yang–Mills) and its deep relation to the holographic properties in string and M-theory, we refer to the important work of Volovich and his co-workers [4]. Since string theory has a deep connection to the statistical properties of random surfaces, index theory for fractal p-branes in Cantorian fractal space–time was considered by the author and Mahccha [16] in connection to the Riemann zeta function. The spectrum of drums (membranes) with fractal boundaries bears deep relations to the zeta function for those fractal strings that are not Minkowski measurable [18]. A random walk based on p-adic numbers has been studied by Albeverio and Karkowski [25]. In this article we will combine all those ideas with the fundamental inclusion of supersymmetry. Parisi and Sourlas [5] discovered in the late 1970s that there is a hidden supersymmetry in classical stochastic differential equations. The existence of stationary solutions to the Fokker–Planck equation associated with the stochastic Langevin equation can be reformulated in terms of an unbroken supersymmetric (SUSY) quantum mechanical (QM) model; more precisely, with an imaginary time Schröedinger equation: a diffusion equation involving a dual diffusion process forward and backward in time. Chapline and his co-workers [6,7] have used these ideas to reformulate QM as information fusion. Ord and Nottale [8,9] following the path pioneered by Feynman's path integral formulation of QM, have shown that the QM equations can be understood from an underlying fractal dynamics of a particle zig-zagging back and forth in space–time spanning a fractal trajectory; i.e., a particle undergoing a random walk or Brownian Motion. Armitage [20] proposed long ago a random walk (diffusion) approximation to the Riemann Hypothesis: a random walk approach to the Ornstein–Uhlenbeck process (or Fokker–Planck equation) to exhibit a polynomial whose zeroes, under a suitable limiting process, ought to be zeroes of the Riemann zeta. El Naschie [10] has suggested also that this dual diffusion process could be the clue to prove the Riemann conjecture. We will propose here thephysical dynamical model that furnishes, in principle, once a solution of the quantum inverse scattering problem exists, the sought-after Hilbert–Polya operator which yields the zeroes of zeta. If a solution to this quantum inverse scattering problem exists this could be instrumental in proving the Riemann conjecture. We will make use of all of these ideas of random matrix theory, Brownian motion, random walk, fractals, quantum chaos, stochastic dynamics, etc. within the framework of the supersymmetric QM model associated with the Langevin dynamics and the Fokker–Planck equation of a particle moving in a randomly fluctuating medium; i.e., *noise* due to the random fluctuations of the infinite array of particles (atoms) located along the one-dimensional quasi periodic crystal. By random fluctuations we mean those fluctuations with respect to their equilibrium configurations which, for example, could be assumed to be the locations of the integers. Because the distribution of the prime numbers is connected to the distribution of the zeroes of zeta, by the explicit Hadamard formula, the main idea of this work is to recast this physical problem in terms of the scattering of the particle by scattering centers situated at the prime numbers, and in this fashion we have effectively a random process with an underlying Parisi–Sourlas hidden supersymmetry, and hence, a well defined SUSY QM problem. Watkins [11] has also suggested that an infinite array of (charged) particles located at the positions of the prime numbers could be relevant in describing the physical system which provides the evolution dynamics linked to the zeroes of zeta. Pitkanen [12] has refined Riemann's conjecture within the language of p-adic numbers by constraining the imaginary parts of the zeroes of zeta to be members of complex rational Pythagorean phases and Berry and Keating [13] have proposed that the SUSY QM Hamiltonian:

$$H = xp - \mathrm{i} = Q^2 \tag{2}$$

is relevant to generate the imaginary parts of the zeroes. The imaginary time Schröedinger equation (diffusion equation) that we propose is

$$-D\frac{\partial}{\partial t}K_{\pm}(x,t) = H_{\pm}K(x,t), \tag{3}$$

where D is the diffusion constant which can be set to unity, in the same way that one can set $\hbar = m = 1$, where m is the particle's mass subject to the random walk. The *isospectral* partner Hamiltonians, H_+, H_- are, respectively,

$$H\pm = -\frac{D^2}{2}\frac{\partial}{\partial x^2} + \frac{1}{2}\Phi^2 \pm \frac{D}{2}\frac{\partial}{\partial x}\Phi \tag{4}$$

the transition-probability density solution of the Fokker–Plank equation, $m^{\pm}(x,x_0,t)$, for the particle arriving at x, in a given time t, after having started at x_0 is

$$m^{\pm}(x,x_0,t) = \exp\left[-\frac{1}{D}(U_{\pm}(x) - U_{\pm}(x_0))\right]K_{\pm}(x,t). \tag{5}$$

The Fokker–Planck equation obeyed by the transition-probability density is

$$\frac{\partial}{\partial t}m(x,x_0,t) = \frac{D}{2}\frac{\partial^2}{\partial x^2}m(x,x_0,t) + \frac{\partial}{\partial x}\Phi(x)m(x,x_0,t) \tag{6a}$$

and the associated Langevin dynamical equation:

$$\frac{\partial x}{\mathrm{d}t} = F(x) + \xi(t). \tag{6b}$$

$F = \Phi(x)$ is the drift momentum experienced by the particle. The quantity $\xi(t)$ is the *noise* term due to the random fluctuations of the medium in which the particle is immersed. The drift potential $U(x)$ associated with the stochastic Langevin equation is defined to be

$$U_{\pm}(x) = -(\pm)\int_0^x \mathrm{d}z\,\Phi(z), \tag{7}$$

$\Phi(x)$ is precisely the SUSY QM potential as we shall see below. The two partners *isospectral* (same eigenvalues) Hamiltonians can be *factorized*:

$$H_+ = \frac{1}{2}\left(D\frac{\partial}{\partial x} + \Phi(x)\right)\left(D\frac{\partial}{\partial x} - \Phi(x)\right) = \mathscr{L}^-\mathscr{L}^+, \tag{8}$$

$$H_- = \frac{1}{2}\left(D\frac{\partial}{\partial x} - \Phi(x)\right)\left(D\frac{\partial}{\partial x} + \Phi(x)\right) = \mathscr{L}^+\mathscr{L}^-. \tag{9}$$

If SUSY is unbroken, there is a *zero* eigenvalue $\lambda_0 = 0$ whose eigenfunction corresponding to the H_- Hamiltonian is the ground state

$$\Psi_0^-(x) = C\mathrm{e}^{-(1/D)\int_0^x \mathrm{d}z\,\Phi(z)}, \tag{10}$$

where C is a normalization constant. Notice that the random "momentum" term $\xi(t)$ appearing in Langevin's equation can be simply recast in terms of the other quantities as

$$\frac{\partial x}{\mathrm{d}t} = F(x) + \xi(t) \;\Rightarrow\; \frac{\partial x}{\mathrm{d}t} - F(x) = \xi(t), \tag{11}$$

which in essence means that the random potential term (in units of $\hbar = m = 1$) is

$$\xi(t) = p - F(x) = p + \Phi(x), \tag{12}$$

which is just the $\mathscr{L}^-$ operator used to factorize the Hamiltonian H_-. One has two random potential terms: $\xi^{\pm}(t)$ corresponding to the two operators $\mathscr{L}^-$, $\mathscr{L}^+$ associated with the two isospectral Hamiltonian partners $H_{\pm}$. An immediate question soon arises. Since the imaginary parts of the zeroes of zeta do not start at zero, the first zero is at $y = 14.1347\ldots$, how can we reconcile the fact that the ground state eigenfunction has zero for eigenvalue (by virtue of SUSY)? The answer to this question has been discussed by Pitkanen [12] using theoretical arguments involving p-adic numbers. This was based on earlier work by Julia [22] who constructed the fermionic version of the zeta function and had shown that the bosonic and fermionic zeta functions can be recast as partition functions of systems of p-adic bosonic/fermionic oscillators in a thermal bath of temperature T. The frequencies of those oscillators are $\log p$, for

$p = 2, 3, 5, \ldots$ a prime number and the inverse temperature $1/T$ corresponds to the z coordinate present in the $\zeta(z)$, where *real* $z > 1$. The pole at $z = 1$ naturally corresponds to the limiting Hagedorn temperature. By virtue of SUSY the zeroes of the fermionic zeta function coincided precisely with the zeroes of the bosonic zeta with the fundamental difference that the fermionic zeta had an additional zero precisely at $z = 1/2 = 1/2 + 0\mathrm{i}$; i.e., the imaginary part of the first zero of the fermionic partition function is precisely *zero*! Therefore, in this SUSY QM model we naturally should expect to have a zero eigenvalue associated with the supersymmetric ground state. Such a ground state does *not* break SUSY and ensures that the associated Fokker–Planck equation has a *stationary* solution in the $t = \infty$ limit (equilibrium configuration is attained at $t = \infty$). Such a stationary solution is given precisely by the square of the modulus of the ground state solution to the SUSY QM model [5]:

$$\lim m^-_{t\to\infty}(x, x_0 = 0, t) = P(x) = |\Psi^-_0(x)|^2 = C^2 \exp\left[-\frac{2}{D}U_-(x)\right]. \tag{13}$$

Notice that the ground state solution is explicitly given in terms of the potential function $U_-(x)$ given by the integral of the SUSY potential $\Phi(x)$; i.e., $\partial/(\partial x)(-U_\pm(x)) = \pm\Phi(x)$. One should notice that the *ordinary* harmonic oscillator corresponds roughly speaking to the case: $\Phi \sim x$ so the operators $\mathscr{L}^+$, $\mathscr{L}^-$ match the raising and lowering operators in this restricted case. However this is a very special case and *not* the SUSY QM model studied here. The physical model we are studying is *not* an ordinary *real* harmonic oscillator but instead it is a p-adic one related to Julia's original p-adic bosonic/fermionic harmonic oscillator formulation of the zeta function in terms of p-adic partition functions. In principle, if one has the list of the imaginary parts of *all* the zeroes of zeta one can *equate* them to the infinite number of eigenvalues:

$$\lambda_0 = 0, \quad \lambda_n = \lambda^-_n = \lambda^+_n, \quad n = 1, 2, 3, \ldots \tag{14}$$

The main problem then is to find the SUSY potential $\Phi(x)$ associated with the zeroes of the bosonic/fermionic zeta. This would require solving the inverse quantum scattering method (that gave rise to quantum groups). To do this is a formidable task since it one requires to have the list of the *infinite* number of zeroes to begin with and then to solve the inverse scattering method problem. Wavelet analysis is very suitable for solving inverse scattering methods. Not surprisingly, Kozyrev has given convincing arguments that wavelet analysis is nothing but p-adic harmonic analysis [17]. What type of SUSY potential do we expect to get? Is it related to the scattering of the particle by an infinite array of atoms located at the prime numbers? Is it related to the Coulomb potential felt by the particle due to the infinite array of charges located at the prime numbers? Is it related to a chaotic one-dimensional billiard ball with the bouncing (scattering) back and forth from an infinite array of obstacles located at the prime numbers? Is it just the p-adic stochastic Brownian motion modelled by Pitkanen's p-adic bosonic/fermionic oscillators? Since the properties of the zeta function are associated with the distribution of prime numbers it is sensible to pose this list of questions. We will try to answer those questions shortly. The ordinary QM potential associated with the SUSY $\Phi(x)$ potential is defined as

$$V_\pm(x) = \frac{1}{2}\Phi^2(x) \pm \frac{D}{2}\frac{\partial}{\partial x}\Phi(x). \tag{15}$$

The potential $V(x)$ should be symmetric (to preserve supersymmetry) under the exchange $x \to -x$ and this entails that the SUSY potential $\Phi(x)$ has to be an odd function: $\Phi(-x) = -\Phi(x)$. Clearly since we do not have at hand the algorithm to generate all the zeroes of zeta (nor the prime numbers) one cannot write explicitly the ansatz for the potential nor solve the inverse scattering problem (which would yield the form of the potential). Nevertheless, Euler was confronted with a similar problem of finding out all the prime numbers when he wrote the (so called adelic) product formula of the Riemann zeta which relates an infinite summation over the integers to an infinite product over the primes:

$$\text{For real } z > 1, \quad \zeta(z) \equiv \sum^\infty \frac{1}{n^z} = \prod_p (1 - p^{-z})^{-1}. \tag{16}$$

Euler's formula can be derived simply by writing any integer as a product of powers of primes and using the summation formula for a geometric series with growth parameter p^{-z}. Euler's formula is a simple proof of why the number of primes is infinite because the product diverges at $z = 1$. The product is an infinite product over *all* primes. Despite the fact that we do not have the list of *all* the prime numbers nor an algorithm to generate them, this does nor prevent us from evaluating such an infinite product; i.e., in computing the value of the zeta function by performing the sum of the Dirichlet series over all integers! This will provide us with the fundamental clue for writing an ansatz for the SUSY potential $\Phi(x)$ and its associated potentials $V_\pm(x)$ giving us the SUSY QM model which yields all the imaginary parts of the zeroes of the zeta function as the eigenvalues of such a SUSY QM model. The reader could ask why should we go to all this trouble and take a tortuous route of writing down the SUSY QM model instead of solving directly the ordinary

QM inverse scattering problem; i.e., finding the ordinary potential $V_{\pm}(x)$ of an ordinary QM problem? If one followed such a procedure one would lose the deep underlying stochastic dynamics of the problem. One would not have discovered the underlying hidden supersymmetry associated with the stochastic Langevin dynamics; nor its associated Fokker–Planck equation; nor be able to notice that the ground state is supersymmetric and its eigenvalue is precise given by *zero*; nor to construct the ground state solution explicitly in terms of the SUSY potential $\Phi(x)$ as shown in Eq. (13). Also one would fail to notice the crucial *factorization* properties of the Hamiltonian, and that the potential $V \pm (x)$ must be symmetric with respect to the origin while the SUSY potential $\Phi(x)$ is antisymmetric, etc. The SUSY QM is very restricted and that narrows down the inverse scattering problem. The simplest analogy one can give is that of a person who fails to recognize the sine function because instead there is the infinite Taylor expansion of the sine function. The crux of this work is to write down the form of the potential associated with the SUSY QM model; the only assumption is based on writing the sought-after SUSY potential in terms of an infinite product, similar to the Euler adelic product form of the zeta and to well-known relation between the *sums* of gamma functions in terms of *products* of zeta functions present in the scattering formulae of p-adic open strings [4,14]. We proceed as follows:

$$\Phi(x) \equiv \sum_n V(|x - x_n|) = \prod_p W(x_p), \tag{17}$$

where $x_p \equiv p^{-x}$ and $W = W(x_p)$ is a function to be determined. This is our ansatz for the form of the SUSY potential. We shall call this ansatz for the potential the adelic potential condition since the product is taken over all the primes. All we are assuming is that a potential of this form can be found. The zeta function has a similar form (although it is not symmetric with respect to the origin): one has an infinite summation over all the integers of the series n^{-z} (playing the role of the potential) expressed as an infinite product over all the primes of functions of p^{-z}. We are just recasting the potential felt by the particle due to the infinite interactions with the objects situated at x_n, an infinite sum of terms, in terms of the infinite product over all the primes of functions of $x_p \equiv p^{-z}$. We emphasize that what is equal to the imaginary parts of the zeroes of zeta are the *eigenvalues* λ_n of the SUSY QM model. We have just recast the infinite numerical input parameters x_n of the potential in terms of the location of the infinite number of primes. For example, based on the Euler adelic formula for the zeta function, one could have chosen the potential to have precisely the zeta function form:

$$\text{For } x > 0, \quad \Phi(x) = \prod_p W(x_p) = \prod_p (1 - x_p)^{-1}, \quad x_p \equiv p^{-x}, \tag{18}$$

where x is the location of the particle executing the p-adic stochastic Brownian motion. For applications of p-adic numbers in physics we refer to [4,14]. Due to the antisymmetry requirement of $\Phi(x)$, the SUSY potential for $x < 0$ must be taken to be an exact mirror copy of the $x > 0$ region to ensure that supersymmetry is unbroken. As a result, the partner potentials $V_{\pm}$ are

$$V_{\pm}(x) = \frac{1}{2}\Phi^2 \pm \frac{D}{2}\frac{\partial}{\partial x}\Phi(x). \tag{19}$$

But how can we be so sure that the eigenvalues will be precisely equal to the imaginary parts of the zeroes of zeta? This would have been an *amazing* coincidence! For this reason we must have an unknown function $W = W(x_p)$ to be *determined* by solving the quantum inverse scattering method. Assuming that the eigenvalues are precisely the imaginary parts of the zeroes of zeta, in principle, we have defined the quantum inverse scattering problem. Using wavelet analysis or p-adic harmonic analysis one could find the SUSY potential $\Phi(x) = \prod_p W(x_p)$, where $x_p \equiv p^{-x}$, where x is the location of the particle. How does one achieve such a numerical feat? One could expand the function $W(x_p)$ in a Taylor series assuming that the potential is analytic, except at some points representing the location of the infinite array of particles, obstacles of the chaotic one-dimensional billiard ball, or the atoms (scattering centers) of the one-dimensional quasi periodic crystal. The adelic condition for the SUSY potential becomes then, for $x > 0$:

$$\Phi(x) = \prod_p \sum_j (a_{jp}p^{jx} + b_{jp}p^{-jx}) - x \leftrightarrow -x, \tag{20a}$$

where the antisymmetrization condition must be imposed on the SUSY potential $\Phi(x) = -\Phi(-x)$ to ensure that $\Phi(0) = 0$. Recasting the Taylor series as a Dirichlet series by simply rewriting

$$p^{-jx} = e^{-xj\log p} = e^{i^2 xj\log p} \rightarrow \Phi(x) = \prod_p \sum_j a_{jp}\cos[ix(\log p^j)] + ia_{jp}\sin[ix(\log p^j)] - x \leftrightarrow -x, \tag{20b}$$

where once again $\Phi(x) = -\Phi(-x)$, allows us to expand the adelic potential in terms of an *infinite* collection of p-adic harmonic oscillators with fundamental imaginary frequencies $\omega_p = i \log p$ and whose harmonics are suitable *powers* of the fundamental frequencies: $\omega_{p,j} = \mathrm{i}\ \log p^j = \mathrm{i}j \log p$. One can recast the imaginary argument trigonometric functions in terms of hyperbolic functions if one wishes. Hyperbolic functions (potentials) are very natural in SUSY QM models. A whole class of solvable potentials, like the shape–invariant partner potentials, have been discussed amply in the book by Junker [5]. We have then recast the quantum inverse scattering problem (solving for the SUSY potential) as the problem of solving for the amplitudes a_{pj} (and phases) of the (imaginary frequencies) p-adic harmonic oscillators by simply writing the adelic condition on the SUSY potential in terms of a p-adic Fourier expansion (p-adic harmonic analysis). This is attained by means of performing the usual Wick rotation in Euclidean QFT: $\omega \to \mathrm{i}\omega$. One could Wick rotate the imaginary time Schröedinger equation (a diffusion equation) to an ordinary Schröedinger equation by the usual Wick rotation trick $t \to \mathrm{i}t$.

Recently we [26] have proposed a solution to the quantum-inverse scattering problem using the fermionic phase approximation of Comtet, Bandrauk and Campbell (CBC), see the book by Junker [5]. This stems from the following SUSY Schröedinger equation associated with the H^+ Hamiltonian:

$$\left(\frac{\partial}{\partial x} + \Phi\right)\left(-\frac{\partial}{\partial x} + \Phi\right) = \lambda_n^{(+)}\psi_n^{(+)}$$

in units of $\hbar = 2m = 1$. SUSY imposes that Φ is an odd function of x so it vanishes at the origin. Hence Φ^2 is an even function of x and the left/right turning points must be symmetrically located: $x_n = x_n^{\mathrm{r}} = -x_n^{\mathrm{l}}$ for all orbitals $n = 1, 2, 3, 4, \ldots$ The CBC formula for the fermionic phase approximation (which is not the same as the WKB) is given by the action-angle type of integrals, see the book by Junker [5]:

$$I_n(x_n; \lambda_n; a_{pj}; b_{pj}) = 4\int_0^{x_n} \mathrm{d}x\, \sqrt{\lambda_n - \Phi^2(x; a_{pj}, b_{pj})} = \pi n,$$

where $n = 1, 2, 3, \ldots$ and λ_n are the imaginary parts of the non-trivial zeroes of the Riemann zeta identified as the energy levels of the orbitals. The required crucial additional set of equations is given by the definition of the turning points:

$$\Phi^2(x_n; a_{pj}; b_{pj}) = \lambda_n, \quad n = 1, 2, 3, \ldots$$

In principle, these two equations will give us an estimate of the amplitudes of the p-adic harmonic oscillators: $a_{pj}; b_{pj}$ and the turning points x_n as functions of all the λ_n. We must emphasize that the CBC formula is only an approximation which becomes exact only in the very special case of *shape–invariant* potentials. A clear simplification is obtained by setting $a_{pj} = b_{pj}$. There are two impending questions: (1) Rather than solving the inverse scattering problem, via the CBC formula, it is more important to prove that it has a solution. And (2) is if these amplitudes a_{pj} are in fact related to the *entries* of a suitable random $N \times N$ matrix model in the large N limit where Universality sets in. We leave these questions for future investigation.

Our p-adic Fourier expansion condition on the SUSY potential coincides with Julia's ideas about the zeta function being the partition function of the adelic ensemble of an infinite system of p-adic oscillators with fundamental frequencies $\omega_p = \log p$, with the only difference that in our case we are performing the Wick rotation of those frequencies.

The infinite unknown amplitude coefficients a_{pj} will be determined numerically by solving the inverse quantum scattering problem in terms of the eigenvalues of the SUSY QM model = imaginary parts of the zeroes of zeta. Odlyzko [15] has computed the first 10^{20} (or more) zeroes of zeta. Having a list of 10^{20} zeroes should be enough data points to find the first 10^{20} numerical coefficients a_{pj} appearing in the Taylor expansion of the potential function $\Phi(x) = \prod_p W(x_p)$ that is being determined via inverse quantum scattering methods associated with this SUSY QM model that links stochastic dynamics, supersymmetry, chaos, etc. to the zeroes of the Riemann zeta function. To summarize, if one can find a solution of the inverse scattering problem that determines the (symmetric) potential $V_\pm(x)$ then one can: (1) propose the candidate for the Hilbert–Polya operator in the following form:

$$\mathscr{H} = 1/2(H_-H_+ + H_+H_-) + 1/4; \tag{21}$$

and (2) postulate that the eigenfunctions ψ_n of the Hamiltonian containing quartic derivative $\mathscr{H}$ is the *fusion* (or convolution) of two eigenfunctions ψ_{n-1}^+ and ψ_n^-. The *ordinary* product will not be suitable, as can be verified by simple inspection. The fusion rules of this type have been widely used in conformal field theories and in string theory. The fused quartic derivative Hamiltonian operator is automatically self-adjoint as a result of the *fusion* of the *self-adjoint* isospectral Hamiltonians H_+, H_- which characterize the two "dual" Nagasawa's diffusion equations [10] and its *real* eigenvalues are the product of the non-trivial zeroes of the Riemann zeta function and their complex conjugates:

$$\mathscr{H}\psi_n = (1/4 + \lambda_n^2)\psi_n = (1/2 + \mathrm{i}\lambda_n)(1/2 + \mathrm{i}\lambda_n)\Psi_n. \tag{22}$$

Notice that the value $n = 0$ is not included since ψ_{-1}^+ is *not* defined and the operator $\mathscr{H}$ is *quartic* in derivatives. Could this candidate for the Hilbert–Polya operator be instrumental in proving the Riemann conjecture? The Eguchi–Schild action for the string is the square of the Poisson bracket of the string embedding coordinates with respect to the worldsheet variables. The momentum conjugate to the Eguchi–Schild holographic area variables is called the area-momentum. The square of the area-momentum is in this sense quartic in derivatives. The fusion or convolution product of the two eigenfunctions of $H_\pm$ can be found by referring to the Fourier transform: the Fourier transform of an ordinary product equals the convolution product of their Fourier transforms. Hence the eigenfunctions of $\mathscr{H}$ can be written, by denoting F, F^{-1} the Fourier transform and its inverse:

$$\psi_n(x) = F^{-1}[F(\psi^+_{n-1} \ * F(\psi^-_n)], \quad n = 1, 2, 3, \ldots \tag{23}$$

As expected, the eigenfunction of the fused Hamiltonian is *not* the naive product of the eigenfunctions of their constituents. The 1/4 coefficient present in the eigenvalues of the fused-Hamiltonian operator is intrinsically related to the real part of the zeroes of zeta $(1/2 + \mathrm{i}\lambda_n)(1/2 + \mathrm{i}\lambda_n)$. The interpretation of the 1/4 coefficient appearing in the fused-Hamiltonian is as an *additive* constant due to normal ordering ambiguities in QFT, like a zero point energy of the ordinary Harmonic oscillator. From the conformal field theory and string theory point of view one constructs unitary irreducible highest-weight representations of the Virasoro algebra for suitable values of the central charges and weights associated with the ground states, c and h, respectively. It is very plausible that supersymmetry, superconformal invariance and representation theory may select and fix uniquely the value of 1/4 which then would be an elegant proof of the Riemman conjecture. Pitkanen [12] has used this proposal based on conformal field theory arguments to refine the Riemman conjecture and offered a plausible proof.

He has proposed a non-Hermitian Hilbert–Polya operator of the type $L_0 + V$ whose eigenvalues are $1/2 + \mathrm{i}y$ using superconformal field arguments. The inner product of two eigenfunctions was re-expressed in terms of the zeta function and the orthogonality of states amounted to the vanishing of zeta. L_0 is the zero mode of the Virasoro scaling generator in the complex plane and V is a suitable potential.

A coherent states interpretation may allow one to express the inner product of states in terms of a non-trivial metric with varying scalar curvature. A constant negative scalar curvature belongs to a hyperbolic geometry, like the Poincarè disc in the complex plane. The Selberg zeta function allows one to count the primitive periodic orbits (geodesies) of a chaotic billiard ball on a hyperbolic plane. The project will be to find out whether or not the Riemann zeta function can be extracted in this physical model using coherent states methods. The location of the zeroes of zeta will amount to a destructive interference between the holomorphic modes (right-moving) and the anti-holomorphic modes (left moving) associated with the coherent states. Fractal p-branes in Cantorian fractal space–time and its relation to the zeta function were considered by the author and Mahecha [16]. The role of fractal strings and the zeroes of zeta has appeared in the book by Lapidus and van Frankenhuysen [18]. Combining p-adic numbers and fractals we arrive at the notion of p-adic fractal strings. The fundamental question to ask would be how to establish a one-to-one correspondence between the zeroes of zeta and the spectrum of p-adic fractal strings. This would mean an establishment of a relation between the exact location of the *poles* of the scattering amplitudes of p-adic fractal strings (a generalization of the Veneziano formula in terms of Euler gamma functions) to the exact location of the zeroes of zeta, i.e., a one-to-one correspondence between the Regge trajectories in the complex angular momentum plane and the spectrum of the p-adic fractal strings with the zeroes of zeta. Since p-adic topology is the topology of Cantorian fractal space–time it is not surprising that the Golden Mean will play a fundamental role [2,10,16]. p-Adic fractals have been discussed in full detail by Pitkannen [12]. p-Adic fractals are roughly speaking just *fuzzy* fractals [19], as they should be, since Cantorian fractal space–time involves a von Neumann's non-commutative *pointless* geometry. Wavelet analysis, that is p-adic Harmonic analysis must play a fundamental role [17] in the classification of such spectrum. After all, the scattering of a particle off a p-adic fractal string should be another way to look at the p-adic stochastic motion discussed in this work.

2. Conclusion

We have been able to construct (assuming the quantum inverse scattering problem has a solution) the operator that yields the zeroes of the Riemann zeta function by defining explicitly the SUSY QM model associated with the p-adic stochastic dynamics of a particle undergoing a Brownian random walk. The zig-zagging occurs after collisions with an infinite array of scattering centers that fluctuate *randomly*. We argued why this physical system can be *reformulated* as the scatterings of the particle about the infinite locations of the prime numbers positions. Assuming that the SUSY potential admits a p-adic Fourier decomposition, we have reformulated such p-adic stochastic process which has an underlying hidden Parisi–Sourlas supersymmetry as the effective Brownian motion of a test particle moving in a background of a thermal gas of photons; i.e., the quanta excitations of an infinite collection of p-adic harmonic oscillators with fundamental imaginary frequencies given by $\omega_p = \mathrm{i}\ \log p$ and whose harmonics are $\omega_{p,n} = \mathrm{i}\ \log p^n$. That is what we called the adelic ansatz condition for the SUSY potential and that allowed us to determine a one-to-one correspondence between the amplitudes of oscillations a_n (and phases) with the imaginary parts of the zeroes of zeta λ_n, after solving the inverse scattering problem. The p-adic fractal strings and their spectrum may establish a one-to-one correspondence between the poles of their scattering amplitudes and the zeroes of zeta.

It was pointed out by the referee of this work that the "explicit formula" of prime number theory, which connects the zeroes with primes, derives from Euler's adelic product formula:

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}$$

and the Hadamard product over all the zeroes ρ of zeta:

$$-s(1-s)\pi^{-s/2}\Gamma(s/2)\zeta(s) = -s(1-s)Z(s) = \prod_{\rho=\text{zeroes}} \left[\left(1 - \frac{s}{\rho}\right)\left(1 - \frac{s}{1-\rho}\right)\right],$$

where $Z(s) = Z(1-s)$ the fundamental functional relation which establishes that the zeroes must be situated symmetrically with respect to the critical line Real $s = 1/2$. Due to the analytical properties of the $Z(s)$ and the relation $Z(s) = Z(1-s)$ it is not difficult to see that on the critical line given by the conditions: $1 - s^* = s$ and $1 - s = s^*$ allows us to see that on the critical line the function $Z(s)$ is real-valued:

$$Z(s)^* = Z(s^*) = Z(1-s) = Z(s) \quad \Rightarrow \quad Z(s) = Z(s)^* \quad \Rightarrow \quad Z(s = 1/2 + \mathrm{i}y) = \text{real}.$$

The distribution of zeroes is intrinsically connected to the distribution of primes via the Hadamard–Euler formulae. And the latter are precisely the points associated with the scattering process. We will show next how one can find a continuous family of operators with eigenvalues parametrized by the points s in the complex plane. We will see that the zeroes of zeta have a one-to-one correspondence with those orthogonal states to the vacuum represented by the SUSY ground state $s = 1/2 + \mathrm{i}0$. The Riemann hypothesis then follows naturally if, and only if, those orthogonal states have the same reflection symmetry properties like the zeroes of zeta have; i.e., if s is a zero, then $1 - s$ is also.

2.1. Noted added in proof: A recent proof of the Riemann hypothesis

We will be very brief and just outline the basic steps for our recent proof [26] of the Riemann hypothesis (RH) based in one, and only one assumption. The essence of our proposal is based in finding a continuous family of operators parametrized by k, l:

$$D_{k,l} = -\frac{\mathrm{d}}{\mathrm{d}lnt} + \frac{\mathrm{d}V}{\mathrm{d}lnt} + k,$$

where the potential term $V(t)$ is related to the Jacobi-theta series:

$$\mathrm{e}^{2V(t)} = \sum_{n=1}^{\infty} \mathrm{e}^{-\pi n^2 t^l},$$

whose eigenvalues s are complex-valued and whose eigenfunctions are

$$\psi_s(t) = t^{-s+k}\mathrm{e}^{V(t)}.$$

A Hermitian "inner product" (not necessarily positive definite) among any states is defined by the pairing:

$$\langle\psi_{s_1}(t)|\psi_{s_2}(t)\rangle = \int_0^{\infty} \frac{\mathrm{d}t}{t}\psi^*_{s_1}(t)\psi_{s_2}(t) = \frac{1}{l}Z\left[\frac{2}{l}(2k - s_2 - s_1^*)\right].$$

Such "inner product" among any pair of states can always be rewritten in terms of the products of a state ψ_s with respect to the SUSY ground state ψ_0, associated with the complex number, $s = 1/2 + \mathrm{i}0$ as follows:

$$\langle\psi_{s_1}|\psi_{s_2}\rangle = \langle\psi_{1/2+\mathrm{i}0}|\psi_s\rangle = \frac{1}{l}Z(as + b),$$

where the state ψ_s is associated with the complex number given in terms of s_1, s_2:

$$s = s_1^* + s_2 - \frac{1}{2}, \quad a(k,l) = -\frac{2}{l}, \quad b = \frac{4k-1}{l}.$$

If, and only if, the *orthogonal* states to the SUSY ground state have the same *symmetry* properties as the zeroes of zeta this implies, firstly, that $Z(as+b) = Z(s') = 0$, for the orthogonality to occur, where s' is a zero of zeta, and from this condition we must have the corresponding four equations

$$as+b = s', \quad a(1-s)+b = 1-s', \quad as^*+b = (s')^*, \quad a(1-s^*)+b = 1-(s')^*,$$

which are satisfied, if, and only if, the real valued coefficients a, b obey the crucial constraint: $a+2b=1$, that also implies that the composition of two affine transformations:

$$s_1 \to as_1+b = s_2, \quad s_2 \to as_2+b = s_3 = a(as_1+b)+b = a^2s_1+ab+b$$

belongs to the same symmetry class because if $a+2b=1$ then the composition also obeys such condition:

$$s_3 = As_1+B, \quad A=a^2, \quad B=ab+b, \quad A+2B = a(a+2b)+2b = a+2b = 1.$$

This is all one needs to show that the RH follows from assuming that the orthogonal states to the SUSY ground state $\psi_{1/2+i0}$, have identical symmetry properties as the zeroes of the Riemann zeta: symmetrically distributed with respect to the critical line Real $s=1/2$.

Therefore, given $a+2b=1$ and any zero $s' = as+b$, we can then infer that: $s' = 1/2 + a(s-1/2)$ and read-off the real parts: $x' = 1/2 + a(x-1/2)$, and establish a one-to-one mapping of the orthogonal states s to the zeroes s'. Since the coefficients $a = -2/l$ belong to a *continuous* family parametrized by l one would have had assigned, to any given x associated with an orthogonal state, a continuous family of zeroes by simply varying the values of l at will. Since the zeroes are assumed to be discrete then the only consistent solution for x' must be $x' = 1/2$ and the RH is proven [26].

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References

[1] Metha ML. Random matrices. New York: Academic Press; 1991;
Bandey A. Brownian motion of discrete spectra. Chaos, Solitons & Fractals 1995;5(7):1275.
[2] Selvam AM. Wave-particle duality in cloud formation. Universal Quantification for Self Organized Criticality in Atmospherical Flows. Available from: http://www.geocities.com/amselvam/pns97.html.
[3] Montgomery H. Distribution of the zeroes of the Riemann zeta function. In: Proceedings of the International Congress on Mathematics, vol. 1. Cannada: Vancouver; 1974. p. 379–81.
[4] Aref'eva I, Medvedev P, Rythchkov O, Volovich I. Chaos in matrix theory. Chaos, Solitons & Fractals 1999;10(2–3);
Vladimirov V, Volovich I, Zelenov E. *p*-Adic analysis in mathematical physics. Singapore: World Scientific; 1994.
[5] Parisi J, Sourlas N. Supersymmetric field theories and stochastic differential equations. Nucl Phys B 1982;206:321;
Junker G. Super-symmetric methods in quantum and statistical mechanics. Berlin: Springer; 1996;
Efetov K. Supersymmetry in disorder and chaos. Cambridge: Cambridge University Press; 1999;
Nakamura K. Quantum chaos. Cambridge: Cambridge University Press; 1995.
[6] Chapline G. Quantum mechanics as information fusion. Available from: quant-ph/9912019.
[7] Nagasawa M. Chaos, information and diffusion in quantum physics. Chaos, Solitons & Fractals 1996;7(5), special issue.
[8] Nottale L. The scale relativity program. Chaos, Solitons & Fractals 1999;10(2–3):459;
Nottale L. La relativite dans tous ses etats. Paris: Hachette Literature; 1998.
[9] Ord G. The spiral gravity model. Chaos, Solitons & Fractals 1999;10(2–3):499.
[10] El Naschie MS. Remarks on superstrings, fractal gravity, Nagasawa's diffusion and cantorian fractal spacetime. Chaos, Solitons & Fractals 1997;8(11):1873.
[11] Watkins M. Prime evolution notes. Available from: http://www.maths.ex.ac.uk/~mwatkins/mwatkins/zeta/evolutionnotes.htm.
[12] Pitkanen M. Topological geometrodynamics. Book on line: http://www.physics.helsinki.fi/matpitka/tgd.htmf.
[13] Berry M, Keating J. The Riemann zeroes and eigenvalue asymptotics. SIAM Rev 1999;41(2):236–66.
[14] Brekke L, Freund PGO. *p*-Adic numbers in physics. Phys Rep 1993;233(1).
[15] Odlyzko A. Supercomputers and the Riemann zeta function. In: Kartashev LP, Kartashev SL, editors. Proceedings of the Fourth International Conference on Supercomputing. 1989. p. 348–52.

[16] Castro C, Mahccha J. Comments on the Riemann Conjecture and index theory on cantorian fractal spacetime. Chaos, Solitons & Fractals 2002;13:1407–19.
[17] Kozyrcv S. Wavelets analysis as *p*-adic harmonic analysis. Available from: math.ph/0012019.
[18] Lapidus M, van Frankenhuysen M. Fractal geometry and number theory: fractal strings and zeroes of zeta functions. Basel, Boston: Birkhauser; 2000.
[19] Granik A. Private communication.
[20] Schroeder. Chaos, Fractals and Power laws.
[22] Julia B. See: http://www.maths.ex.ac.uk/~mwatkins/zeta/physics2.htm.
[25] Albeverio S, Karkowski W. A random walk on *p*-adics. Stochastic Processes and Applications 1994;53:1–22.
[26] Castro C, Granik A, Jorge Mahecha. On SUSY QM, fractal strings and steps towards a proof of the Riemann conjecture. Available from: hep-th/0107266. Chaos, Solitons & Fractals, submitted.