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Weight characterization of an averaging operator

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Abstract

Let $0 < \alpha < 1$ and $T_\alpha : f \mapsto (1/[(1-\alpha)x])(\int_{\alpha x}^{x} f)$, $x \geqslant 0$. A factorization theorem is given, which provides a weight characterization of the space of all positive functions f such that $T_\alpha f$ belongs to L_w^p, $1 < p < \infty$, w a weight function. This theorem yields a two-sided estimate for the norm of $T_\alpha f$. An analogous result holds for $\alpha = 0$. In the latter case, it is also shown that the averaging Hardy operator T_0 and its dual T_0^* are comparable in L_w^p, $1 < p < \infty$, if w belongs to the Muckenhoupt weight class A_p.

Keywords: Weight functions; A_p-weight class; Factorization theorem

1. Introduction

Let $0 < \alpha < 1$ and T_α denote the operator defined by

$$T_\alpha : f \mapsto T_\alpha f(x) = \frac{1}{(1-\alpha)x}\left(\int_{\alpha x}^{x} f\right), \quad x \geqslant 0, \tag{1}$$

for f measurable on $[0, \infty)$.

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In the limit case where $\alpha = 0$, T_α coincides formally with the Hardy averaging operator $T_0 : f \mapsto (1/x)(\int_0^x f)$, the mapping properties of which are well known in weighted L^p-spaces (cf. [1,9]; see also [6,10]). For $0 < \alpha < 1$, the operator T_α differs from T_0 in the sense that the averages $(T_\alpha f)(x)$ are calculated on a moving interval $[\alpha x, x]$, the length of which increases as $x \to +\infty$, as a constant multiple of its distance from the origin. Actually, T_α, $0 < \alpha < 1$, belongs to a class of operators that has been considered in [8]. A characterization of the pairs of weights (v, u) for which T_α is bounded from L_v^p into L_u^p can be deduced from Theorem 2.2 of [8] (see Theorem A below). Also, the class of weights which is determined in this way is strictly larger than that obtained in [9] for the boundedness of $T_0 : L_v^p \to L_u^p$. Note here that extensive mapping properties of operators, in weighted Lebesgue and Banach function spaces were recently studied in [5] and [7], and are concerned with operators more general than T_α.

In this paper, we consider other aspects of the study of T_α, $0 < \alpha < 1$, and T_0. The main result is a factorization theorem characterizing the space of all non-negative functions f such that $T_\alpha f \in L_w^p$, $1 < p < \infty$, w a weight function (Theorem 1). This result yields a two-sided estimate for the norm of $T_\alpha f$ (Corollary 1). A similar result holds for $\alpha = 0$ (Corollary 2). This part of our study was motivated by the factorization theorems previously established by Bennett in the unweighted case, for the discrete Cesaro and Copson operators [2]. Our paper also includes a comparability result for the Hardy operator T_0 and its adjoint T_0^* in L_w^p, w in the Muckenhoupt class A_p (Theorem 2). A discrete analogue is also mentioned in the context of weighted sequence spaces (Theorem 3). These results extend an earlier result obtained by Boas in the L_w^p case, where $w = |x|^\beta$, $-1 < \beta < p - 1$ [3], and a result by Bennett in the discrete unweighted case [2].

Throughout this paper, we adopt the usual notation and conventions. The conjugate index p' of p, $p \neq 1$, is $p' = p/(p-1)$. Positive constants are denoted C, C', c, c', sometimes with subscripts. Functions are assumed to be measurable. Weight functions denoted u, v, w are positive a.e. and locally integrable. The operator T_α is defined by (1). We use a similar notation when $\alpha = 0$, in which case T_0 denotes the Hardy averaging operator. For short, the latter will also be denoted T, when no confusion may arise.

To conclude this introduction, we state the following known result, which will be needed in our study:

Theorem A [8, Theorem 2.2]. *Let $1 < p < \infty$ and $0 < \alpha < 1$. Let v, u be weight functions. Then,*

$$\|T_\alpha f\|_{p,u} \leqslant C\|f\|_{p,v} \quad \forall f \geqslant 0 \text{ a.e. on } (0,\infty) \tag{2}$$

if and only if

$$C_{p,\alpha} \equiv \sup\left(\int_t^x \frac{u(s)}{s^p}\,ds\right)^{1/p}\left(\int_{\alpha x}^t v(s)^{1-p'}\,ds\right)^{1/p'} < \infty,$$

where the supremum is taken over $\{(x,t):\ 0 < t \leqslant x \leqslant t/\alpha\}$.

Moreover, if $\|T_\alpha\|$ denotes the best constant in (2), then

$$C_{p,\alpha} \leqslant \|T_\alpha\| \leqslant 2p^{1/p}(p')^{1/p'}C_{p,\alpha}.$$

2. A factorization theorem for T_α

We prove the following result:

Theorem 1. *Let* $0 < \alpha < 1$, $1 < p < \infty$, $w > 0$, $v > 0$ *a.e. Assume* f *is non-negative on* $[0, \infty)$. *Then,* $T_\alpha f \in L^p_w$ *if and only if* f *admits a factorization*

$$f = gh, \quad g \geqslant 0,\ h > 0 \text{ on } [0, \infty), \tag{3}$$

such that $g \in L^p_v$ *and*

$$\|h\|_{w,v} \equiv \sup\left(\int_t^x \frac{w(s)}{s^p}\,ds\right)^{1/p}\left(\int_{\alpha x}^t v^{1-p'}h^{p'}\,dt\right)^{1/p'} < \infty,$$

where the supremum is taken over the set $\{(x,t)\colon 0 < t \leqslant x < t/\alpha\}$.

Furthermore,

$$\inf\left(\|g\|_{p,v}\|h\|_{w,v}\right) \leqslant \|T_\alpha f\|_{p,w} \leqslant c_p \inf\left(\|g\|_{p,v}\|h\|_{w,v}\right), \tag{4}$$

where the infimum is taken over all factorizations (3) *satisfying the above conditions, and* $c_p = 2p^{1/p}(p')^{1/p'}$.

Proof. (i) Suppose $f = gh$ with $g \in L^p_v$, $\|h\|_{w,v} < \infty$. Then, considering $h^{-p}v$ as a weight function, we see that $\|f\|_{p,h^{-p}v} = \|g\|_{p,v}$ and, on the other hand, $\|h\|_{w,v}$ coincides with the constant $C_{p,\alpha}$ in Theorem A. From the latter, it follows that

$$\|T_\alpha f\|_{p,w} \leqslant 2p^{1/p}\left(p'\right)^{1/p'}\|h\|_{w,v}\|g\|_{p,v}.$$

The inequality in the right side of (4) follows.

(ii) Suppose $\|T_\alpha f\|_{p,w} < \infty$. Hereafter, we show that there exists $g \in L^p_v$ such that $\|T_\alpha f\|_{p,w} = \|g\|_{p,v}$ and such that the function $h = f/g$ satisfies $\|h\|_{w,v} \leqslant 1$. The existence of the required factorization will then follow, as well as the estimate in the left side of (4).

To do this, note first that

$$\begin{aligned}\int_0^\infty (T_\alpha f)^p w\,dx &= \int_0^\infty (T_\alpha f)\left[(T_\alpha f)^{p-1}w\right]dx\\ &= \int_0^\infty f\left(T_\alpha^\star\left[(T_\alpha f)^{p-1}w\right]\right)dx = \int_0^\infty fb\,dx,\end{aligned}$$

where $b \equiv b_f = T_\alpha^\star[(T_\alpha f)^{p-1}w]$ and $T_\alpha^\star$ is defined by

$$T_\alpha^\star\varphi(s) = \frac{1}{(1-\alpha)}\int_s^{s/\alpha}\frac{\varphi(x)}{x}\,dx,$$

$\varphi \geqslant 0$ on $[0, \infty)$.

If we define g, h respectively by $g = (fb)^{1/p} v^{-1/p}$ and $h = f/g = f^{1/p'} v^{1/p} b^{-1/p}$, we obtain

$$\|g\|_{p,v} = \left(\int_0^\infty f b\, dx \right)^{1/p} = \|T_\alpha f\|_{p,w}.$$

In order to estimate $\|h\|_{w,v}$, we now observe that

$$\int_{\alpha x}^{t} v^{1-p'} h^{p'}\, d\xi = \int_{\alpha x}^{t} f(\xi) b(\xi)^{1-p'}\, d\xi.$$

For $\alpha x < \xi < t$, $b(\xi)$ can be minorized as follows:

$$b(\xi) = \frac{1}{(1-\alpha)} \int_{\xi}^{\xi/\alpha} \frac{w(s)}{s} \big[T_\alpha f(s)\big]^{p-1}\, ds$$
$$\geqslant \frac{1}{(1-\alpha)} \int_{t}^{x} \frac{w(s)}{s} \big[T_\alpha f(s)\big]^{p-1}\, ds \equiv \tilde{b}(x,t),$$

which yields

$$\left(\int_{\alpha x}^{t} v^{1-p'} h^{p'}\, d\xi \right)^{1/p'} \leqslant \left(\int_{\alpha x}^{t} f(\xi)\, d\xi \right)^{1/p'} \big(\tilde{b}(x,t)\big)^{-1/p}.$$

Hence,

$$\tilde{h}(x,t) \equiv \left(\int_{t}^{x} \frac{w(s)}{s^p}\, ds \right)^{1/p} \left(\int_{\alpha x}^{t} v^{1-p'} h^{p'}\, d\xi \right)^{1/p'}$$
$$\leqslant \left\{ \int_{t}^{x} \frac{w(s)}{s^p} \left(\int_{\alpha x}^{t} f\, d\xi \right)^{p-1} ds \right\}^{1/p} \big(\tilde{b}(x,t)\big)^{-1/p},$$

where the integral on the right side can be majorized by

$$(1-\alpha)^{p-1} \int_{t}^{x} \frac{w(s)}{s} \left[\frac{1}{(1-\alpha)s} \int_{\alpha s}^{s} f\, d\xi \right]^{p-1} ds$$
$$= (1-\alpha)^{p-1} \int_{t}^{x} \frac{w(s)}{s} \big[T_\alpha f(s)\big]^{p-1}\, ds = (1-\alpha)^p \tilde{b}(x,t).$$

This yields,

$$\tilde{h}(x,t) \leqslant (1-\alpha) \tilde{b}(x,t)^{1/p} \tilde{b}(x,t)^{-1/p} = 1-\alpha$$

and, as a consequence, $\|h\|_{w,v} \leqslant 1$, as needed.

The proof of Theorem 1 is thus completed. □

Noting that to each given $v > 0$, there corresponds a family $\mathcal{F}(v)$ of factorizations satisfying (3), (4), we can state

Corollary 1. *Let $0 < \alpha < 1$, $1 < p < \infty$, $w > 0$ a.e. Assume f is non-negative on $[0,\infty)$. Let $\mathcal{F}(v)$ denote the family of all factorizations $f = gh$ with $g \in L_v^p$ and $\|h\|_{w,v} < \infty$. Then,*

$$\sup_{v>0} \inf_{\mathcal{F}(v)} \big(\|g\|_{p,v}\|h\|_{w,v}\big) \leqslant \|T_\alpha f\|_{p,w} \leqslant c_p \inf_{v>0} \inf_{\mathcal{F}(v)} \big(\|g\|_{p,v}\|h\|_{w,v}\big).$$

It is not difficult to see that Theorem 1 is still valid for $\alpha = 0$. We thus obtain the following factorization result for the Hardy averaging operator $T_0 : f \mapsto (1/x)\int_0^x f\,dt$. This is the weighted integral analogue of a result obtained by Bennett for the discrete Hardy operator in the unweighted case [2].

Corollary 2. *Let $1 < p < \infty$, $w > 0$, $v > 0$ a.e. Assume f is non-negative on $[0,\infty)$. Then, $T_0 f \in L_w^p$ if and only if f admits a factorization $f = gh$, $g \geqslant 0$, $h > 0$ on $[0,\infty)$, with $g \in L_v^p$ and*

$$\|h\|_{w,v} = \sup_{t>0} \left(\int_t^\infty \frac{w(s)}{s^p}\,ds \right)^{1/p} \left(\int_0^t v^{1-p'} h^{p'}\,ds \right)^{1/p'} < \infty.$$

Furthermore,

$$\inf_{\mathcal{F}(v)} \big(\|g\|_{p,v}\|h\|_{w,v}\big) \leqslant \|T_0 f\|_{p,w} \leqslant c_p \inf_{\mathcal{F}(v)} \big(\|g\|_{p,v}\|h\|_{w,v}\big),$$

where $\mathcal{F}(v)$ denotes the family of all factorizations $f = gh$ satisfying (3), (4) with $\alpha = 0$. Also,

$$\sup_{v>0} \inf_{\mathcal{F}(v)} \big(\|g\|_{p,v}\|h\|_{w,v}\big) \leqslant \|T_0 f\|_{p,w} \leqslant c_p \inf_{v>0} \inf_{\mathcal{F}(v)} \big(\|g\|_{p,v}\|h\|_{w,v}\big).$$

3. Comparability of T_0 and T_0^*

In relation to Corollary 2, let us note that the particular factorization $f = gh$, with $g = f \in L_w^p$, $1 < p < \infty$, $h = 1$, corresponds to the condition

$$\beta(p,w) = \|1\|_{w,w} = \sup_{t>0} \left(\int_t^\infty \frac{w(s)}{s^p}\,ds \right)^{1/p} \left(\int_0^t w^{1-p'}\,ds \right)^{1/p'} < \infty. \tag{5}$$

The latter is necessary and sufficient for the boundedness of T_0 from L_w^p into itself, for $1 < p < \infty$. Also, this condition is weaker than the condition

$$A_{p,w} = \sup_{I \subset \mathbb{R}^+} \left(\frac{1}{|I|} \int_I w \right)^{1/p} \left(\frac{1}{|I|} \int_I w^{1-p'} \right)^{1/p'} < \infty,$$

where I denotes any interval of $\mathbb{R}^+$ and $|I|$ its measure (cf. [1,6,9,10]). This remark leads us to prove the following comparability result regarding T_0 and $T_0^\star$ in L_w^p. The latter extends a result obtained by Boas, with a different method, in the case $w(x) = |x|^\beta$, $-1 < \beta < p - 1$ [3].

Hereafter, in Theorem 2, we use the abbreviation T to denote the Hardy averaging operator T_0. Also, we write $w \in A_p$ instead of $A_{p,w} < \infty$. We denote

$$\mathcal{C}es(p,w) = \{f \text{ real valued: } \|T(|f|)\|_{p,w} < \infty\},$$
$$\mathcal{C}op(p,w) = \{f \text{ real valued: } \|T^\star(|f|)\|_{p,w} < \infty\}.$$

This notation is a variant of that introduced by Bennett in his study of unweighted sequence spaces [2]. The analogy between the above spaces and those considered by Bennett will be explained in the context of Theorem 3, at the end of this section.

We now state

Theorem 2. *Let $w \in A_p$, $1 < p < \infty$. Then,*

$$C' \|T^\star f\|_{p,w} \leqslant \|Tf\|_{p,w} \leqslant C \|T^\star f\|_{p,w} \tag{6}$$

holds for every $f \geqslant 0$. As a consequence,

$$\mathcal{C}es(p,w) = \mathcal{C}op(p,w).$$

In (6)*, the inequalities are to be understood in the sense that, if the right-hand side is finite, so is the left-hand side, and the inequality holds.*

Proof. We shall actually prove the theorem, under the weaker assumption $\max(\beta(p,w), \beta(p', w^{1-p'})) < \infty$, where β is given by (5). The announced result will follow, since the conditions $A_{p,w} < \infty$ and $A_{p', w^{1-p'}} < \infty$ are equivalent and imply the above "β-type" conditions.

(i) First, suppose f non-negative. Then,

$$T(T^\star f) = Tf + T^\star f = T^\star(Tf).$$

Since $\beta(p,w) < \infty$, this yields

$$\big|\|Tf\|_{p,w} - \|T^\star f\|_{p,w}\big| \leqslant \|Tf + T^\star f\|_{p,w} = \|T(T^\star f)\|_{p,w} \leqslant c_p \beta(p,w) \|T^\star f\|_{p,w}$$

and, as a consequence,

$$\|Tf\|_{p,w} \leqslant \big(1 + c_p \beta(p,w)\big) \|T^\star f\|_{p,w} \quad \forall f \geqslant 0.$$

A similar argument can be used to estimate $T^\star(Tf)$. As a result, one then obtains

$$\|T^\star f\| \leqslant \big(1 + c_{p'} \beta(p', w^{1-p'})\big) \|Tf\|_{p,w}.$$

It follows that (6) holds with the constants $C = (1 + c_p \beta(p,w))$, $C' = (1 + c_{p'} \beta(p', w^{1-p'}))^{-1}$.

(ii) Suppose now that f is real valued. Then, we may replace f by $|f|$ in (6). The resulting inequalities then show that the spaces $\mathcal{C}es(p,w)$ and $\mathcal{C}op(p,w)$ coincide. □

To end this section, we mention a discrete analogue of Theorem 2. The operators under consideration here are

$$T^d : [a] \to T^d[a] = \left(\frac{1}{n} \sum_{k=1}^{n} a_k \right)_{n \geqslant 1},$$

$$(T^d)^\star : [a] \to (T^d)^\star[a] = \left(\sum_{k=n}^{\infty} \frac{a_k}{k} \right)_{n \geqslant 1},$$

where $[a] = (a_k)_{k \geqslant 1}$. The discrete weight class, denoted A_p^d, $1 < p < \infty$, is the set of all sequences $[w] = (w_n)_{n \geqslant 1}$ such that

$$A_p^d[w] = \sup \left(\frac{1}{|I|} \sum_{n \in I} w_n \right)^{1/p} \left(\frac{1}{|I|} \sum_{n \in I} w_n^{1-p'} \right)^{1/p'} < \infty,$$

where the supremum is taken over all $I = \{n \in \mathbb{N}^+ \colon N \leqslant n \leqslant M\}$ with $1 \leqslant N < M$, and $|I| = M - N + 1$. Also, ℓ_w^p, $1 < p < \infty$, denotes the space

$$\ell_w^p = \left\{ [a] \colon \sum_{k=1}^{\infty} w_k |a_k|^p < \infty \right\}.$$

The analogues of the function spaces $\mathcal{C}es(p, w)$, $\mathcal{C}op(p, w)$ are, in this case,

$$ces(p, w) = \left\{ [a] \text{ real valued: } \left\| T^d\big([|a|]\big) \right\|_{\ell_w^p} < \infty \right\},$$

$$cos(p, w) = \left\{ [a] \text{ real valued: } \left\| (T^d)^\star\big([|a|]\big) \right\|_{\ell_w^p} < \infty \right\}.$$

In this context, the following result provides a weighted extension of a result established by Bennett in the case $w(x) = 1$, $x \in \mathbb{R}$ [2].

Theorem 3. *Suppose* $[w] \in A_p^d$, $1 < p < \infty$. *Then, for all* $[a] \geqslant 0$,

$$C' \left\| (T^d)^\star[a] \right\|_{\ell_w^p} \leqslant \left\| T^d[a] \right\|_{\ell_w^p} \leqslant C \left\| (T^d)^\star[a] \right\|_{\ell_w^p}.$$

As a consequence,

$$ces(p, w) = cop(p, w).$$

The proof relies on arguments quite similar to those used in the proof of Theorem 2. In fact, since the weight characterization of the discrete Hardy operator and its dual are quite similar to the continuous case, the result follows in the same way. We refer, for instance, to [4] and references given therein, for related definitions and questions on weighted discrete inequalities. As previously, we refer to [2] for a thorough discussion of the unweighted discrete case.

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