# Alexander Numbering of Knotted Surface Diagrams

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ABSTRACT. A formula that relates triple points, branch points, and their distance from infinity is presented.

## 1. Introduction

An *Alexander numbering* of a classical knot diagram is depicted in Fig. 1. In this figure, the planar regions that are separated by arcs of the curve are labeled with integers. The unbounded region is labeled 0. The indices in two regions that are separated by an arc differ by 1; the region into which a normal arrow points has the larger index. Such numberings were used by J.W. Alexander [**1**] to give a combinatorial defintion of his now well-known polynomial invariant (See also [**18**]). An Alexander numbering for oriented knotted closed surfaces is defined similarly in Section 3 where indices are also assigned to the crossing points of a knotted surface diagram.

The title of this paper is an intentional pun. Alexander is also the given name of JSC's second son, who at age 8, looked at the diagram of Fig. 2 (which is SK's diagram of the 3-twist spun trefoil and at the time was on the blackboard of JSC's office), and observed that the sum of the encircled labels added to 0. Alexander Carter's observation on this diagram does not hold for every chart of knotted surfaces (see Fig. 3), but there is a relation among the indices of the triple points and branch points that we develop herein. These diagrams represent surface braids, and will be explained in Section 2.

1.1. HISTORICAL REMARKS. A formula between Whitney degree [**27**] and the Alexander numbering was given in [**22**], which was based on the integral calculus with respect to the Euler characteristic [**26**]. The Alexander numbering for surfaces in 3-space was used in [**11**] and [**23**]. Whitney's formula [**28**] relates the normal Euler class to the Euler characteristic (see also [**5**]). He conjectured that for non-orientable surfaces the number of achievable values is limited. This conjecture was proven by Massey in [**21**]; see [**13**] for a geometric proof. Banchoff relates the branch points to the normal Euler class in [**3**]. Banchoff's triple point formula [**2**] also relates singularities of surface maps to their intrinsic topology. Li shows [**20**] when a daisy graph is realizable as the multiple point set of a general position map. Generalizations of Banchoff's formula appear in [**12**] and [**19**]. Relation between
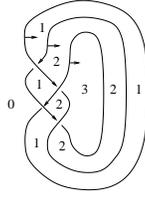
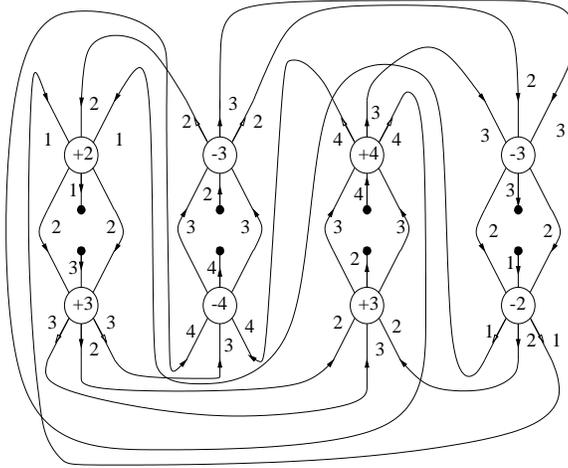FIGURE 1.  Alexander numbering of a classical knot



FIGURE 2. A braid chart of the 3-twist spun trefoil

branch points and triple points are found in [**4**] and [**8**]. A further generalization is presented in this paper.

Here is how the paper will be developed.  Section 2 summarizes the notation also found in [**9**]. Section 3 defines the method of numbering. Section 4 contains the main results: Theorem 4.2 and its consequences.

## 2.  Notation

In this section we review knotted surface diagrams and the chart description of surface braids that will be used in the paper. More details can be found in [**9**].

2.1. GENERIC PROJECTIONS AND DIAGRAMS.  For classical knots and links, under-arcs are broken in the projections to define knot diagrams. We generalize this notion to knotted surfaces as broken surface diagrams. First we develop some notation. Let $f : F \to \mathbf{R}^4$ denote a smooth embedding of a closed surface $F$ into 4-dimensional space.
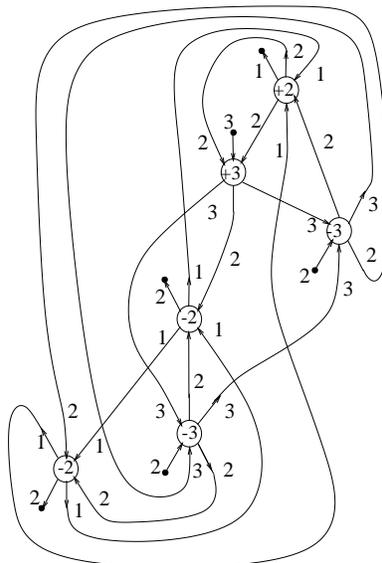
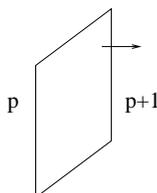FIGURE 3.   A braid chart of Fox's [10] Example 12



FIGURE 4.   Alexander numbering changes by 1 through a sheet

By deforming the map $f$ slightly by an ambient isotopy of $\mathbf{R}^4$, we may assume that $f$ is a general position map. So any point of the image surface has a neighborhood $U$ in 3-space such that $p_v(f(F)) \cap U$ looks like one of the pictures indicated in Fig. 4 or the left of Fig. 5. In other words, there is a diffeomorphism of $U$ into $\mathbf{R}^3$ such that the image of $F$ coincides with the intersection of 1, 2 (*a double point curve*), or 3 coordinate planes (*a triple point*), or is like the cone on a figure 8 (*a branch point*). The neighborhood of a double point curve, a triple point, a branch point are depicted in Figure 5 (A), (B), and (C), respectively. When the surface is oriented, we take normal vectors $\vec{n}$ to the projection of the surface such that the triple $(\vec{v}_1, \vec{v}_2, \vec{n})$ matches the orientaion of 3-space, where $(\vec{v}_1, \vec{v}_2)$ defines the orientation of the surface. Such normal vectors are defined on the projection at all points other than the isolated branch points.

There is an immersion in $\mathbf{R}^3$ of a compact 1-manifold with boundary whose image is the closure of the set

$$\{x \in p(f(F)) | p(f(x_1)) = x = p(f(x_2)) \text{ for some } x_1 \neq x_2, \quad \text{where } x_1, x_2 \in F\}.$$
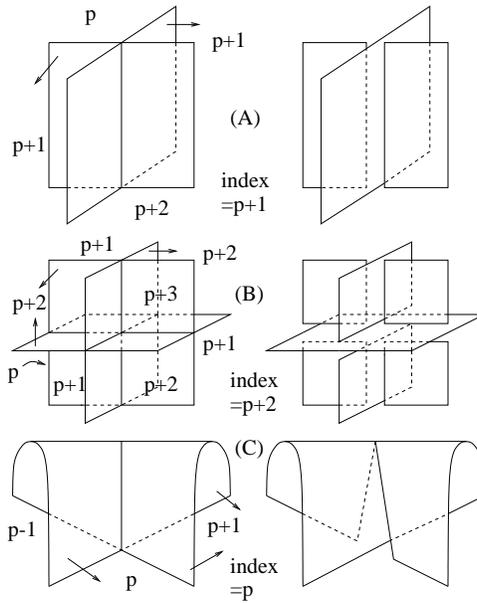
FIGURE 5. Projections, broken surface diagrams and indices

The closure of the above set is called *the double point set.*      The image of the boundary points of the 1-manifold are the branch points of the map $p \circ f$. At a triple point three curves meet transversely, so a triple point is a 6-valent vertex.

In the *diagram of a knotted surface*, one of the two sheets along the double point arcs to indicate that it is higher than the other sheet in the $w$-direction. Broken surfaces are indicated in Fig. 5.

The double point curves are oriented in such a way that the orientation $\vec{v}$ together with normal vectors of top and bottom sheets $(\vec{n_1}, \vec{n_2}, \vec{v})$ matches the orientation of the space. The double point curve depicted in Fig. 5 (A) is oriented downward.

2.2. SIGNS OF BRANCH AND TRIPLE POINTS. The *sign of a triple point* is defined as follows [8]. For the normal vectors $v_1, v_2, v_3$ of top, middle, bottom sheet, respectively, if the triple $(v_1, v_2, v_3)$ matches the orientation of $\mathbf{R}^3$, then the sign is positive, and negative if otherwise.

The *sign of a branch point* is defined as follows [4, 5]. If the the double curve ending at the given branch point is oriented towards (resp. away from) the branch point, then the sign is negative (resp. positive). The triple point depicted in Fig. 5 is negative, and the branch point is positive.

2.3. CHARTS OF SURFACE BRAIDS. The notion of surface braids, a generalization of Artin's braid theory, was originally proposed by Viro, and a similar notion had been used by Rudolph [25]. We review the chart description of surface braids as developed in [6, 7, 14, 15].

Figure 6 shows how to express certain generic surfaces in 3-space by means of planar graphs. We consider a surface $S$ (this $S$ corresponds to $f(F)$ in the preceding section) in a box $B = I_1 \times I_2 \times I_3 \subset \mathbf{R}^3$ a schematic of which is depicted on the
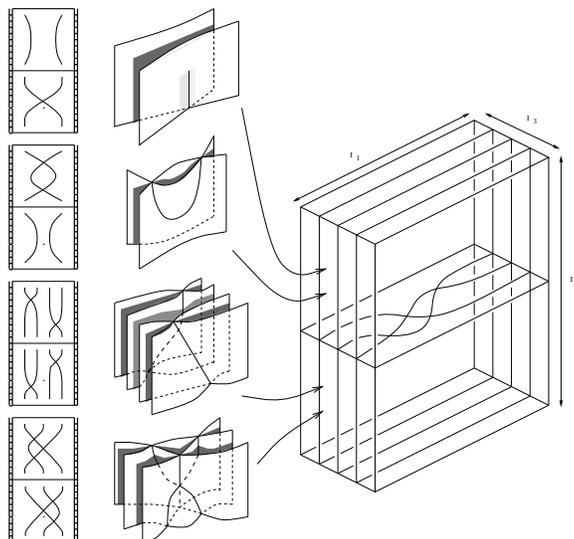
FIGURE 6.   Generic surfaces, movies, and surface braids

right of Figure 6, where $I_j$ denotes a copy of the unit interval, for $j = 1, 2, 3$. We require that the surface $S$ in $B$ satisfies the following conditions.

- $S$ is generic.
- The boundary $\partial S$ of $S$ is a closed trivial braid contained in $\partial(I_1 \times I_2) \times I_3$.
- The projection $p : I_1 \times I_2 \times I_3 \to I_1 \times I_2$ restricted to $S$ is a branched covering such that each branch point is *simple*. The preimage of a branch point, then, consists of a unique branch point of degree 2 and a collection of points around which the projection is a local homeomorphism.

Let $D$ be the double point set of $S$. Then $p(D) \subset I_1 \times I_2$ is a planar graph. Let us call this an *un-oriented chart*. An un-oriented chart has univalent vertices corresponding to branch points, 4-valent vertices that correspond to the crossings of the projections of double arcs, and 6-valent vertices corresponding to triple points of $S$. A generic intersection $S \cap I_1 \times \{t\} \times I_3$, consists of intersecting strings in $I_1 \times \{t\} \times I_3$. Such intersections are shown in the figure by movie strips.

When the surface $S$ is the projection of a surface braid in the 4-disk $I_1 \times I_2 \times I_3 \times I_4$, the fourth coordinate can be indicated in the movies by broken arcs. Then the arcs in a braid chart are oriented to agree with the orientation of the corresponding double curves. A downward pointing arrow in the chart (with respect to the height direction in the page) that is labeled with an integer $i$ corresponds to the $i$th braid generator in which the $i$th string passes over the $(i + 1)$st string. In this way, an oriented braid chart for surface braids is defined [**15**].

The univalent vertices of a chart correspond to branch points and are called *black vertices*. The 6-valent vertices correspond to triple points and are called *white vertices*. The signs of triple points on a braid chart can be determined as follows: if the indices of arcs at a triple point is $p$ and $p + 1$ and the middle incoming arc is labeled by $p + 1$, then the triple point (white vertex) is positive, and otherwise negative. By capping off the nested boundary circles on the boundary of the box $B$ by nested disjoint disks, we get a closed surface in 3-space. If it is a projection
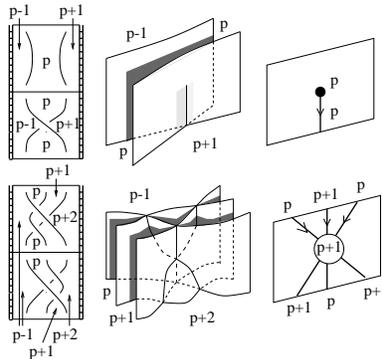
FIGURE 7.   Indices at braid generators

of a knotted surface, taking such a closure gives rise to a closed surface braid in 4-space.

The figures 2 and 3 represent the braid charts of 3-twist-spun trefoil (see [**24**] for example) and Fox's Example 12 in [**10**].

## 3. Alexander numberings

Let $F$ denote an oriented surface that is embedded in 4-space via a map $f :$ $F \to \mathbf{R}^4$. Consider a general position projection of the knotting $f$ into 3-space and the associated knotted surface diagram. Choose a normal orientation in 3-space for the projection such that the tangent orientation followed the normal orientation agrees with the right-handed orientation of 3-space. Number with integers the 3-dimensional regions in the complement of the projection according to the convention that (a) the unbounded region is numbered 0, (b) regions that are separated by a 2-dimensional face are numbered consecutively, and (c) the normal vector to the surface points towards the region with largest number. Such an indexing is called an *Alexander numbering*. This was defined in [**11, 23**] and was called *index* or *degree*.

If the regions adjacent to a double arc are numbered $p$, $p + 1$, $p + 1$, and $p + 2$, then *the Alexander number of the double arc* is $p+1$ (Fig. 5 (A)). In a neighborhood of a triple point, one of the adjacent regions is numbered $p$, three are numbered $p + 1$, three are numbered $p + 2$, and one is numbered $p + 3$. The *Alexander number of the triple point* is $p + 2$ (Fig. 5 (B)). The *Alexander number of a branch point* is the number of the region that is not interior to the figure 8. Thus if the regions interior to the figure 8 are labeled $p - 1$ and $p + 1$, then the Alexander number of the branch point is $p$ (Fig. 5 (C)).

When a knotted surface is given as a surface braid, then the cross sectional braids are closed with arcs that pass to the right as in Fig. 1. We may assume that the surface is oriented so that the normal direction points consistently to the right on the "braid portion of the diagram" (On the closure part, the normal points left: Fig. 1). The *braid index of a branch point* is the index of the braid generator that is created or distroyed at the black branch point in the chart. The *braid index of a triple point* is the largest braid index among the indices that appear adjacent to the triple point. Thus at a white vertex in a braid chart 3 edges each are incident

with indices $p$ and $p + 1$. The braid index of the vertex is $p + 1$. A white vertex in the chart corresponds to the braid relation $\sigma_p \sigma_{p+1} \sigma_p = \sigma_{p+1} \sigma_p \sigma_{p+1}$; the braid index is $(p + 1)$. The situation is depicted in Fig. 7.

3.1. OBSERVATION. *The Alexander numberings of the double, branch, and triple points agree with the braid index of the corresponding points when the surface is in braid form.*

*Proof.* The region containing $\infty$ is immediately to the left of the first arc in a braid word.

The Alexander numberings of double, branch, triple points are indicated in Figs. 2 and 3 in their charts.

## 4. Statement of Main Result

4.1. DEFINITION. Let $T(p, \delta)$ denote the number of triple points of Alexander index $p$ with sign $\delta = \pm$. Let $B(p, \sigma)$ denote the number of branch points of Alexander index $p$ with sign $\sigma = \pm$.

4.2. THEOREM. *For a diagram of a knotted oriented closed surface,*

$$\sum_{p,\sigma} \sigma x_p B(p, \sigma) + \sum_{q,\delta} \delta y_q T(q, \delta) = 0$$

*provided $y_p = x_p - x_{p-1}$.*

*Proof.* Consider an arc of double points of index $p$. Such an arc can start at a branch point of index $p$ or start at a triple point of index $p$ or index $(p + 1)$. There are two double arcs of index $p$ that start at each positive triple point of index $p$ and each negative triple of index $(p + 1)$. One such arc starts at a negative triple point of index $p$, positive triple of index $(p + 1)$, or branch point of index $p$. Thus the number of edges of index $p$ is

$$E(p) = B(p, +) + 2T(p, +) + T(p, -) + T(p + 1, +) + 2T(p + 1, -)$$

The ending points of such arcs can be determined similiarly. So,

$$E(p) = B(p, -) + 2T(p, -) + T(p, +) + T(p + 1, -) + 2T(p + 1, +).$$

We obtain,

$$B(p, +) - B(p, -) = T(p + 1, +) - T(p + 1, -) - (T(p, +) - T(p, -)).$$

We call this relation *equation* $(\star)$. Multiply through by $x_p$, and sum over all $p$ to get:

$$\begin{aligned}
\sum_{p,\sigma} \sigma x_p B(p, \sigma) &= \sum_{p,\delta} \delta x_p T(p + 1, \delta) - \sum_{p,\delta} \delta x_p T(p, \delta) \\
&= \sum_{p,\delta} \delta(x_{p-1} - x_p) T(p, \delta)
\end{aligned}$$

This completes the proof.

4.3. COROLLARY. *If $y_p = 0$ for all $p$, then $x_p = x_{p+1}$. Therefore,*

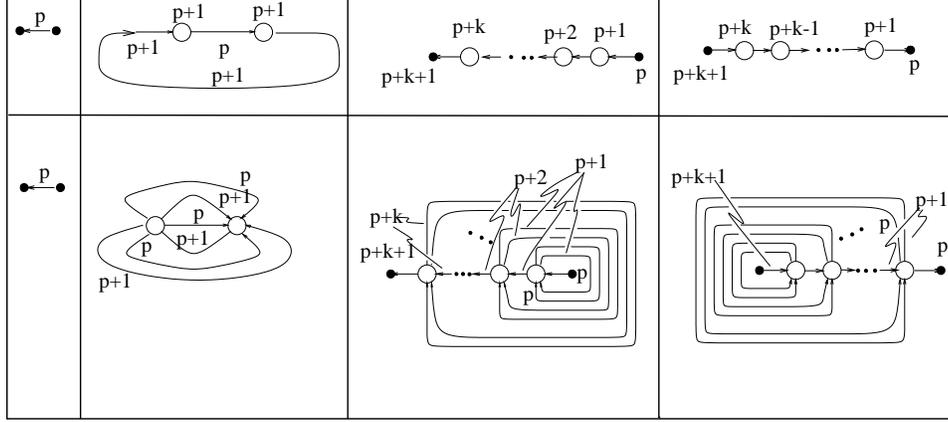$$\sum_{p,\sigma} \sigma B(p, \sigma) = 0.$$

FIGURE 8.   Some charts of unknotted surfaces

So we recover the fact that oriented surfaces have trivial normal Euler classes because the number of signed branch points add up to the normal Euler number [**5**].

4.4. COROLLARY. *If $y_p = p$, and $x_1 = 1$, then $x_p = p(p+1)/2$, and*

$$\sum_{p,\sigma} \sigma \frac{p(p+1)}{2} B(p,\sigma) + \sum_{q,\delta} \delta q T(q,\delta) = 0.$$

4.5. COROLLARY. *If $y_p = 1$, and $x_1 = 1$, then $x_p = p$, and*

$$\sum_{p,\sigma} \sigma p B(p,\sigma) + \sum_{q,\delta} \delta T(q,\delta) = 0.$$

4.6. COROLLARY. *If $B(p,\sigma) = 0$ for all $p$, then $T(p,+) = T(p,-)$. In particular, this is true if the projection of the surface is immersed.*

4.7. COROLLARY.
$T(p,+) - T(p,-) = \sum_{i<p} B(i,+) - B(i,-) = -\sum_{i\geq p} B(i,+) - B(i,-).$

*Proof.* This is a consequence of equation $\star$. Observe that these sums are finite.

4.8. THEOREM. *There exists a diagram of an unknot or unlink that realizes prescibed values for $B(p,\pm)$ and $T(p,\pm)$ that satisfy the formula in Theorem 4.2.*

*Proof.* By adding nested spheres near $\infty$, we may assume that the labels are all positive. We construct a labeled planar graph with the correct number of black and white vertices, and then we use this to construct a chart of an unlink. Between lines $x = p$ and $x = p+1$ put B(p,+) (resp. B(p,-)) black vertices with horizontal arrows that point away from (resp. toward) the vertex. Put $T(p+1,+)$ (resp. $T(p+1,-)$) white vertices in the same region with horizontal right-pointing (resp. left-pointing) incoming (resp. outgoing) arrows labeled $p$ and right-pointing (resp. left pointing) outgoing (resp. incoming) arrows labeled $(p+1)$. Along the line $x = p$ stack $|T(p,+) - T(p,-)|$ arrows that point to the right if $T(p,+) - T(p,-) > 0$ arrows and to the left otherwise. Connect the arrows according to the rules that
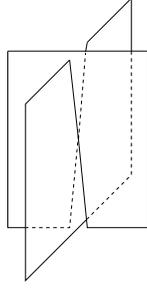
FIGURE 9.   A broken surface diagram at a singular point

(1) lines only cross the line $x = p$ along the stacked arrows, (2) each possible pair of white vertices are connected in a loop, (3) a pair of black vertices are linearly chained by a (possibly empty) set of white vertices. The arcs in the graph may intersect. The only loops in the graph connect a pair of white vertices.

Embed the graph in the plane. Figure 8 indicates how to replace the graph with bits of charts. The resulting surface can be seen to be unknotted by means of chart moves. See [**9, 14**] for a description of the moves. This completes the proof.

4.9. ORIENTED SURFACES WITH SINGULAR POINTS. In this section we generalize Theorem 4.2 to generically immersed oriented surfaces in 4-space. Such surfaces have isolated transverse double points that we call *singular points* to distinguish them from arcs of double points in the projection. These can be indicated by broken surface diagrams as indicated in Fig. 9. The *sign of such a singular point* is negative if the oriented double arcs both point toward the singular point. The sign is positive if the double arcs point away from the singular point. Say that two of the regions adjacent to the singular point have indices $(p + 1)$, the other two have indices $p$ and $p + 2$. Then the *index of singular point* is $(p + 1)$.

Let $D(p, \epsilon)$ denote the number of singular points of index $p$ and sign $\epsilon$.

4.10. THEOREM. *For a diagram of a knotted oriented closed surface with singular points,*

$$\sum_{p,\sigma} \sigma x_p B(p,\sigma) + \sum_{q,\delta} \delta y_q T(q,\delta) + \sum_{r,\epsilon} \epsilon z_r D(r,\epsilon) \quad = \quad 0$$

*provided that $y_p = x_p - x_{p-1}$ and $z_p = 2x_p$.*

*Proof.* By counting edges of label $p$, as before, we obtain

$$E(p) = B(p,+) + 2T(p,+) + T(p,-) + T(p+1,+) + 2T(p+1,-) + 2D(p,+)$$

$$= B(p,-) + 2T(p,-) + T(p,+) + T(p+1,-) + 2T(p+1,+) + 2D(p,-).$$

Then

$$B(p,+) - B(p,-) + 2(D(p,+) - D(p,-))$$

$$= T(p+1,+) - T(p+1,-) - (T(p,+) - T(p,-)).$$

Multiply through by $x_p$, and sum over all $p$ to obtain the result.

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