Basic Measure and Integration Theory

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Sep 22, 2002

Measure Theory: Introduction

- What is measure theory?
- Why bother to learn measure theory?

What is measure theory?

- Measure theory is concerned with one of the most basic of all scientific activities: measuring

- Familiar kinds of measures: lengths, volumes, time intervals, mass

What is measure theory?

- Fundamental, intuitive property of measures: Measures are generally non-negative real numbers and additive
  - whole is equal to the sum of its parts
  - not always non-negative (signed measures are possible)
  - not always real (complex measures are possible)

Why bother to learn measure theory?

- Stochastic phenomena require a notion of probability

- Probability is a way of measuring the likelihood of events

- Hence, primitive notions of measure are fundamental to probability theory

Why bother to learn measure theory?

- We need to integrate stochastic differential equations that involve random processes
  - white noise, random walk
- Random processes are time-dependent random variables
- Random variables are functions on the underlying probability space
- Such functions are not integrable in the ordinary Riemannian sense

Why bother to learn measure theory?

- To integrate random processes over time, we need a more general kind of integration
- To define a more general integral, we need to have a more general notion of measurement, since integration is a kind of measurement

Basic Measure Theory: σ-algebras

In this section the letter I will always denote a countable index set.

Definition. *A σ-algebra on a set Ω is a non-empty collection $\mathcal{M}$ of subsets of Ω satisfying the following conditions:*

$$\emptyset \in \mathcal{M} \tag{1}$$

$$A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M} \tag{2}$$

$$A_i \in \mathcal{M}, \forall i \in I \Rightarrow \bigcup_{i \in I} A_i \in \mathcal{M} \tag{3}$$

Thus a σ-algebra is closed under complementation and countable unions, and it follows easily that it is also closed under countable intersections and contains the carrier set Ω as well.

Note that the countable intersection of σ-algebras is again a σ-algebra.

This is important, because it says that given any set $A \subseteq \Omega$, there is a unique, smallest σ-algebra containing A, namely, the intersection of all σ-algebras containing A.

We shall call this the σ-**algebra generated by** A and denote it by $< A >_\sigma$.

Examples. *σ-algebras: Let Ω be any non-empty set. Then:*

1. $\mathcal{M} = \{\emptyset, \Omega\}$

2. $\mathcal{M} = \mathcal{P}(\Omega)$

3. $\Omega = \{a, b, c\}$, $\mathcal{M} = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$

4. $\Omega =$ *any uncountable set, i.e.,* $|\Omega| > \aleph_0$, *and* $\mathcal{M} = \{A \subseteq \Omega : |A| \leq \aleph_0 \vee |A^c| < \aleph_0\}$

5. *For any* $a, b \in \mathbf{R}_\infty$ *such that* $a < b$, *let* $[a, b) = \{x \in \mathbf{R} : a \leq x < b\}$, *and let* $\mathcal{M} =< \{A \subseteq \mathbf{R} : A = \bigcup_{i=1}^{n} [a_i, b_i), a_1, b_1, \ldots a_n, b_n \in \mathbf{R}_\infty\} >_\sigma$ *This σ-algebra occurs so frequently that we give it a special name: the* **Borel** *algebra, and its sets are called Borel sets. We also use the special notation $\mathcal{B}$ instead of $\mathcal{M}$.*

Note that in probability theory, the set of all events associated with an experiment is a σ-algebra on the sample space Ω. In the case of finite sample spaces such as those occurring in die-throwing experiments, the σ-algebra is merely the set of all subsets of the sample space, i.e., $\mathcal{P}(\Omega)$.

The elements of a σ-algebra $\mathcal{M}$ are called **measurable** sets.

Definition. *Let* $\mathcal{M}_1 \subseteq \mathcal{P}(\Omega_1)$ *and* $\mathcal{M}_2 \subseteq \mathcal{P}(\Omega_2)$ *be two* σ*-algebras, and let* f *be a function from* Ω_1 *into* Ω_2*, i.e.,* $f : \Omega_1 \to \Omega_2$*. Then* f *is called a* **measurable** *function if* $f^{-1}(A_2) \in \mathcal{M}_1$ *whenever* $A_2 \in \mathcal{M}_2$*.*

Thus, under a measurable function, the inverse image of a measurable set is always measurable.

Examples.

1. *Let* $\Omega = \{a, b, c\}$, $\mathcal{M}_1 = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$ *and* $\mathcal{M}_2 = \{\emptyset, \{c\}, \{a, b\}, \{a, b, c\}\}$. *Let* f *map* a *into* c *and vice versa, and leave* b *fixed.*

2. *Every function mapping* Ω *into itself can be considered a measurable function with respect to the* σ*-algebras* $\mathcal{M}_1 = \mathcal{P}(\Omega)$ *and* $\mathcal{M}_2 = \{\emptyset, \Omega\}$. *Note that this is no longer true if we reverse the roles of these two* σ*-algebras.*

3. **Characteristic Function**: *For any* $E \in \mathcal{M}$ *let* χ_E *be defined as follows:*

$$\chi_E(\omega) = \begin{cases} 1 & \text{if } \omega \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Definition. *A* **measure** *on a σ-algebra $\mathcal{M}$ is a function μ that assigns to each measurable set a non-negative, extended real number in such a way that it is countably additive on disjoint sets, i.e.,*
if A_i is a countable sequence of pairwise disjoint measurable sets in $\mathcal{M}$, i.e., $A_i \cap A_j = \emptyset$ whenever $i \neq j$, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i). \tag{4}$$

Examples. *Measures:*

1. *Let* $\mathcal{B} =$ $< \{A \subseteq \mathbf{R} : A = \bigcup_{i=1}^{n} [a_i, b_i), a_1, b_1, \ldots a_n, b_n \in \mathbf{R}_\infty\} >_\sigma$ *be the Borel algebra and define* μ *on* $\mathcal{B}$ *such that the restriction of* μ *to unions of disjoint semiclosed intervals is given by* $\mu(\bigcup_{i=1}^{n} [a_i, b_i)) = \sum_{i=1}^{n} (b_i - a_i)$. *It can be shown that there is only one such measure that satisfies this property. We call this measure the* **Lebesgue** *measure on* $\mathcal{B}$ *and instead of* μ *we use* λ.

2. *Let* Ω *be a finite set and consider the power set* σ*-algebra* $\mathcal{M} = \mathcal{P}(\Omega)$. *The* **counting measure** *is defined by* $\mu(A) = |A|$.

3. **Dirac Measure**. *Let* $\mathcal{M}$ *be any* σ*-algebra and let* $\omega \in \Omega$ *be an arbitrary but fixed element of* Ω. *For each* $A \in \mathcal{M}$ *define*

$$\mu_\omega(A) = \begin{cases} 1 & \textit{if } \omega \in A, \\ 0 & \textit{otherwise.} \end{cases}$$

4. **Probability Measure**. *If* $\mu(\Omega) = 1$, *then* μ *is called a* **probability measure** *on* Ω. *In this course, we shall be concerned almost exclusively with probability and Lebesgue measures. Also, we shall see that measurable mappings from a probability space into the Borel algebra on the real line are of considerable importance – they are called* **random variables**. *More on that later ...*

Basic Integration Theory

- A measure gives us information about the relative sizes of measurable sets, i.e., it measures them.
- Integration theory is concerned with measuring the sizes of sets defined by measurable functions on measurable sets, e.g., volumes and areas.

Simple Functions

- **Definition.** *A* **simple function** *is a measurable function that takes on only finitely many distinct real values. It can be written as:*

$$f = \sum_{i=1}^{n} a_i \chi_i \tag{5}$$

where $\chi_i = \chi_{f^{-1}\{a_i\}}$.