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Zeros of Polynomials and Fractional Order Differences of Their Coefficients

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I. INTRODUCTION

In 1893 Eneström [1] proved that, if $c_0, c_1, \cdots, c_n$ $(n \geq 1)$ are real numbers (not all zero) satisfying

$$c_0 \geq c_1 \geq c_2 \cdots \geq c_n \geq 0, \tag{1}$$

then no zero of the polynomial $E(z) \equiv \Sigma_{k=0}^{n} c_k z^k$ lies in the disk $|z| < 1$. The interested reader may consult Professor Marden's treatise [2, § 30] on this and related results. A generalization of Eneström's theorem for power series with complex coefficients was given by Krishnaiah [3].

Let ∇ denote the backward-difference operator defined by $\nabla a_k \equiv a_k - a_{k-1}$ (see, e.g., [4, pp. 207-208]). Then (1) is equivalent to the condition

$$\nabla c_k \leq 0 \qquad (k = 1, 2, \cdots, n+1) \tag{2}$$

where c_{n+1} is taken to be zero. Eneström's conclusion follows from the observation that

$$(1 - z)\, E(z) \equiv \sum_{k=1}^{n+1} \{\nabla c_k\}\, (z^k - 1)$$

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which implies that

$$\operatorname{Re}\{(1-z)E(z)\} \equiv \sum_{k=1}^{n+1} \{\nabla c_k\} \operatorname{Re}(z^k - 1) > 0$$

whenever $|z| < 1$.

In this paper we extend Eneström's theorem by replacing (2) by similar conditions involving fractional order differences.

Let $\mathbf{E}$ denote the displacement operator defined by $\mathbf{E}a_k \equiv a_{k+1}$, and let $\mathbf{I}$ be the identity operator: $\mathbf{I}a_k \equiv a_k$. Then, symbolically,

$$\nabla^\alpha = (\mathbf{I} - \mathbf{E}^{-1})^\alpha = \sum_{m=0}^{\infty} (-1)^m \binom{\alpha}{m} \mathbf{E}^{-m}$$

for every complex α.

Accordingly, we define ∇^α by means of the identity (see, e.g., [5, § 5.5])

$$\nabla^\alpha a_k \equiv \sum_{m=0}^{\infty} (-1)^m \binom{\alpha}{m} a_{k-m}. \tag{3}$$

If $a_k = 0$ for $k = -1, -2, \cdots$, then (3) yields, for $k = 0, 1, 2, \cdots$,

$$\nabla^\alpha a_k = \sum_{m=0}^{k} (-1)^m \binom{\alpha}{m} a_{k-m}. \tag{4}$$

Given complex numbers α, $a_0, a_1, \cdots, a_k$, we shall always mean by $\nabla^\alpha a_k$ the right-hand side of (4).

II. Polynomials with Positive Coefficients

THEOREM 1. *Let* $E(z) \equiv \sum_{k=0}^{n} c_k z^k$ ($\not\equiv 0$, $n \geq 1$) *be a polynomial, and let* $0 < \alpha \leq 1$. *Assume that* $c_k \geq 0$ ($k = 0, 1, \cdots, n$) *and that* $\nabla^\alpha c_k \leq 0$ ($k = 1, 2, \cdots, n$). *Then no zero of* $E(z)$ *lies in* $|z| < 1$.

PROOF. Consider the function

$$(1-z)^\alpha \equiv \sum_{m=0}^{\infty} (-1)^m \binom{\alpha}{m} z^m \qquad |z| \leq 1. \tag{5}$$

Setting $c_k = 0$ for $k = n+1, n+2, \cdots$, we have throughout $|z| \leq 1$

$$(1-z)^\alpha E(z) = \sum_{k=0}^{\infty} \left\{ \sum_{m=0}^{k} (-1)^m \binom{\alpha}{m} c_{k-m} \right\} z^k$$

$$= \sum_{k=0}^{\infty} z^k \nabla^\alpha c_k = \sum_{k=1}^{\infty} (z^k - 1) \nabla^\alpha c_k, \tag{6}$$

since

$$\sum_{k=0}^{\infty} \nabla^{\alpha} c_k = 0.$$

For $k = n + 1, n + 2, \cdots$, we have

$$\nabla^{\alpha} c_k = \sum_{m=k-n}^{k} (-1)^m \binom{\alpha}{m} c_{k-m} . \tag{7}$$

Since

$$(-1)^m \binom{\alpha}{m} \leq 0, \qquad m = 1, 2, \cdots, \tag{8}$$

$\nabla^{\alpha} c_k \leq 0$ for $k = n + 1, n + 2, \cdots$. Thus $\nabla^{\alpha} c_k \leq 0$ $(k = 1, 2, \cdots)$, and by (6), not all of these numbers $\nabla^{\alpha} c_k$ are zero. Hence, for $|z| < 1$,

$$\operatorname{Re} \{(1 - z)^{\alpha} E(z)\} = \sum_{k=1}^{\infty} \{\nabla^{\alpha} c_k\} \operatorname{Re} (z^k - 1) > 0,$$

and so $E(z) \neq 0$.

Remarks. (a) Theorem 1 with $\alpha = 1$ is just Eneström's theorem. If $n = 1$ and α is a given number satisfying $0 < \alpha < 1$, then the test of Theorem 1 is weaker than Eneström's test.

(b) Let $0 < \alpha < 1, c_0 = 16$, $c_1 = 2\alpha$, and $c_2 = \alpha(8 - 6\alpha)$. One can apply Theorem 1 to $E(z) \equiv c_0 + c_1 z + c_2 z^2$ and conclude that it has no zero in $|z| < 1$. Eneström's theorem, however, is not applicable to this $E(z)$.

(c) Let $0 < \alpha_1 < \alpha_2 \leq 1$, let n be an integer larger than 1, and set $E_1(z) \equiv (-1)^{n+1} \binom{\alpha_1}{n}^{-1} + z^n$, $E_2(z) \equiv \sum_{k=0}^{n} \alpha_2^k z^k$. We can conclude that no zero of $E_1(z)$ lies in $|z| < 1$ by means of Theorem 1 with $\alpha = \alpha_1$, but not by means of Eneström's theorem. On the other hand, one can apply to $E_2(z)$ Eneström's theorem, but not Theorem 1 with $\alpha = \alpha_1$, for the condition $\nabla^{\alpha_1} c_1 = \alpha_2 - \alpha_1 \leq 0$ is not fulfilled.

(d) Let $0 < \alpha_1 < \alpha_2 \leq 1$, and set $E(z) \equiv 4 + 2(\alpha_1 + \alpha_2) z + \alpha_2(1 + \alpha_1) z^2$. Then one can apply to this $E(z)$ Theorem 1 with $\alpha = \alpha_2$. One cannot, however, apply Theorem 1 with $\alpha = \alpha_1$.

(e) Let $\frac{1}{2} \leq \alpha_1 < \alpha_2 < 1$, and set

$$E(z) \equiv [\alpha_1(1 - \alpha_1)]^{-1} + [\alpha_2(1 - \alpha_2)]^{-1} + z^2.$$

Then one can apply to this $E(z)$ Theorem 1 with $\alpha = \alpha_1$, but one cannot apply Theorem 1 with $\alpha = \alpha_2$.

III. Complex Coefficients

We shall now consider polynomials with complex coefficients.

By a sector with vertex at the origin we mean a set of the form

$$\{\rho e^{i\varphi} : \rho \geq 0, \varphi_1 \leq \varphi \leq \varphi_2\}.$$

Theorem 2. *Let* $E(z) \equiv \sum_{k=0}^{n} c_k z^k$ $(\not\equiv 0, n \geq 1)$ *be a polynomial with complex coefficients, and let* $0 < \alpha \leq 1$. *Set* $c_k = 0$ $(k = n + 1, n + 2, \cdots)$, *and let S be a sector with vertex at the origin whose angular measure* 2θ *satisfies* $0 \leq 2\theta < \pi$. *Then each of the following three hypotheses implies that* $E(z)$ *has no zero in* $|z| < \cos\theta$: (I) $-c_k$ $(k = 0, 1, \cdots, n)$ *and* $\nabla^\alpha c_k$ $(k = 1, 2, \cdots, n)$ *belong to S;* (II) $\nabla^\alpha c_k \in S$ $(k = 1, 2, \cdots, n, n + 1, n + 2, \cdots)$; (III) $\nabla c_k \in S$ $(k = 1, 2, \cdots, n + 1)$.

Theorem 2 with hypothesis I and with S taken as the negative real axis (including the origin) is our previous Theorem 1.

To establish Theorem 2, we prove first the following

Lemma. *Let* $E(z) \equiv \sum_{k=0}^{\infty} c_k z^k$ $(\not\equiv 0)$ *be a power series with complex coefficients converging at* $z = 1$, *and let* $0 < \alpha \leq 1$. *Let* γ *and* r $(0 < r \leq 1)$ *be real numbers such that, for* $k = 1, 2, \cdots$, $\nabla^\alpha c_k = |\nabla^\alpha c_k| e^{i(\varphi_k + \gamma)}$ *where* φ_k *is real and* $|\varphi_k| \leq \arccos r^k$. *Then* $E(z)$ *has no zero in* $|z| < r$.

(Whenever arccos, arcsin, or arg appears, its principal value is being used.)

Proof of the Lemma. Due to the absolute convergence of the right-hand member of (5) at $z = 1$, (6) holds in the present case throughout $|z| < 1$. Let κ be a positive integer for which $\nabla^\alpha c_\kappa \neq 0$. (Such a κ exists, for otherwise we would get from (6) that $E(z) \equiv 0$.) Then throughout $|z| < r$ we have

$$|\arg[e^{-i\gamma}(1 - z^\kappa)\nabla^\alpha c_\kappa]| \leq |\arg(e^{-i\gamma}\nabla^\alpha c_\kappa)| + |\arg(1 - z^\kappa)|$$

$$< \arccos r^\kappa + \arcsin r^\kappa = \pi/2,$$

and therefore

$$\operatorname{Re}\{e^{-i\gamma}(1 - z^\kappa)\nabla^\alpha c_\kappa\} > 0.$$

If $|z| < r$, then by (6),

$$\operatorname{Re}\{e^{i(\pi-\gamma)}(1 - z)^\alpha E(z)\} = \sum_{k=1}^{\infty} \operatorname{Re}\{e^{-i\gamma}(1 - z^k)\nabla^\alpha c_k\} > 0;$$

and, consequently, $E(z) \neq 0$. This proves the Lemma.

PROOF OF THEOREM 2. For a suitable real constant γ, every $z \in S$ can be written in the form $|z| e^{i(\varphi(z)+\gamma)}$ where $-\theta \le \varphi(z) \le \theta$. To prove Theorem 2 (with hypothesis II), observe that for every $k \ge 1$ we have

$$\nabla^\alpha c_k = |\nabla^\alpha c_k| e^{i(\varphi_k+\gamma)}$$

where ϕ_k is real and

$$|\varphi_k| \le \theta = \arccos \cos \theta \le \arccos (\cos \theta)^k.$$

By the lemma, $E(z)$ has no zero in $|z| < \cos \theta$. Next, we prove Theorem 2 with hypothesis I. It is sufficient to show that $\nabla^\alpha c_k \in S$ for every $k > n$. Now, for such a k, we see from (7) and (8) that $\nabla^\alpha c_k$ is a weighted sum of $-c_0$, $-c_1, \cdots, -c_n$ with real, nonnegative weights. Since $-c_m \in S$ for $m = 0, 1, \cdots, n$, it follows that $\nabla^\alpha c_k$ also belongs to S. Finally, hypothesis III obviously implies II (with $\alpha = 1$).

Let c_0 be a positive number, and let $c_1, c_2, \cdots, c_n$ be nonnegative real numbers. Let $0 < \alpha < 1$, and for every $r \ge 0$, let $\mu(r) = \max_{1 \le k \le n} \nabla^\alpha(r^k c_k)$. Since $\mu(0) = \max_{1 \le k \le n} [(-1)^k \binom{\alpha}{k} c_0] < 0$, there exists a positive r for which $\mu(r) \le 0$. *Every such r has the property that all zeros of $E(z) \equiv \sum_{k=0}^n c_k z^k$ lie in $|z| \ge r$.* Indeed, by Theorem 1, no zero of $E(rz) \equiv \sum_{k=0}^n c_k r^k z^k$ lies in $|z| < 1$.

THEOREM 3. *Let $E(z) \equiv \sum_{k=0}^\infty c_k z^k$ $(\not\equiv 0)$ converge at $z = 1$. Let $0 < \alpha \le 1$, and assume that all numbers $\nabla^\alpha c_k$ $(k = 1, 2, \cdots)$ lie in some sector with vertex at the origin whose angular measure 2θ satisfies $0 \le 2\theta < \pi$. Then no zero of $E(z)$ lies in $|z| < \cos \theta$.*

The proof is the same as that of Theorem 2 (with hypothesis II).

IV. FURTHER RESULTS

Let S be a sector as in Theorem 2, let $0 < \alpha \le 1$, and let $c_0, c_1, \cdots$ be complex numbers (not all zero) such that $-c_k$ $(k = 0, 1, \cdots)$ and $\nabla^\alpha c_k$ $(k = 1, 2, \cdots)$ lie in S. Then, as one easily concludes, $c_0 \ne 0$. For $n = 1, 2, \cdots$, let $E_n(z) \equiv \sum_{k=0}^n c_k z^k$. By Theorem 2 (with hypothesis I), no $E_n(z)$ can have a zero in $|z| < \cos \theta$. If $\overline{\lim} |c_n|^{1/n}$ were larger than $(\cos \theta)^{-1}$, then we could find an n such that $|c_n/c_0|^{1/n} > (\cos \theta)^{-1}$. For such an n, the geometric mean of the moduli of the zeros of $E_n(z)$ would be smaller than $\cos \theta$; and therefore $E_n(z)$ would have at least one zero in $|z| < \cos \theta$, contradicting our above observation. Thus, *the radius of convergence of $\sum_{k=0}^\infty c_k z^k$ is $\ge \cos \theta$*; and, by Hurwitz's theorem relating the zeros of a power series to those of its partial sums, $\sum_{k=0}^\infty c_k z^k \ne 0$ throughout $|z| < \cos \theta$.

Taking a more general consideration, let $0 < \alpha \leq 1$, let γ and r be real numbers ($0 < r \leq 1$), and let $C_0, C_1, \cdots$ be complex numbers (not all zero) such that $-C_k = |C_k| e^{i(\psi_k+\gamma)}$, ψ_k real, $|\psi_k| \leq \arccos r^{k+1}$ ($k = 0, 1, \cdots$), and such that $\nabla^\alpha C_k = |\nabla^\alpha C_k| e^{i(\Phi_k+\gamma)}$, Φ_k real, $|\Phi_k| \leq \arccos r^k$ ($k = 1, 2, \cdots$). Again it follows that $C_0 \neq 0$. For $n = 1, 2, \cdots$, set $E_n(z) \equiv \Sigma_{k=0}^n C_k z^k$, and consider some arbitrary $E_n(z)$. Let $c_k = C_k$ and $\phi_k = \Phi_k$ for $k = 0, 1, \cdots, n$, and let $c_k = 0$ for $k = n+1, n+2, \cdots$. If k is larger than n, then all the numbers $|\psi_0|, |\psi_1|, \cdots, |\psi_n|$ are equal to or less than $\arccos r^k$, and therefore $-c_0, -c_1, \cdots, -c_n$ lie in the sector $\{\rho e^{i\varphi} : \rho \geq 0, \gamma - \arccos r^k \leq \varphi \leq \gamma + \arccos r^k\}$. Thus, in view of (7) and (8), $\nabla^\alpha c_k$ lies in that sector; and, consequently, we may write $\nabla^\alpha c_k = |\nabla^\alpha c_k| e^{i(\varphi_k+\gamma)}$, where φ_k is real and $|\varphi_k| \leq \arccos r^k$. Applying the lemma to $E_n(z)$ we conclude that $E_n(z)$ has no zero in $|z| < r$. Now, exactly as before we can show that the radius of convergence of $\Sigma_{\nu=0}^\infty C_\nu z^\nu$ is at least r and that this power series is different from zero throughout $|z| < r$.

Assume again the hypotheses of Eneström's theorem. Since $\Sigma_{k=0}^n c_k > 0$, $E(1) \neq 0$. Therefore, by considerations as presented in the Introduction, if $\nabla c_1 \neq 0$, $E(z)$ has no zero on $|z| = 1$. If $E(z) = 0$, $|z| = 1$, then $\nabla c_1 = 0$, and $z^k = 1$ for each k satisfying $2 \leq k \leq n+1$, $\nabla c_k \neq 0$; in particular, $z^{n+1} = 1$ if $c_n \neq 0$. See [6]. Furthermore, if for some k with $1 \leq k \leq n+1$, $\nabla c_k \neq 0$, a zero z of $E(z)$ satisfies $\operatorname{Re}(z^k) < 1$, then there exists a k' ($1 \leq k' \leq n+1$, $\nabla c_{k'} \neq 0$) such that $\operatorname{Re}(z^{k'}) > 1$.

Similarly, assume the hypothesis of Theorem 1 with $\alpha < 1$. Again $E(1) \neq 0$. However, now we have the result that $E(z)$ *has no zero on the unit circumference*. Indeed, by (7) and (8) it follows that $\nabla^\alpha c_k < 0$ for $k = n+1$, $n+2, \cdots$. Also, by hypothesis, $\nabla^\alpha c_k \leq 0$, $k = 1, 2, \cdots, n$. If $E(z) = 0$, $|z| = 1$, then since $\Sigma_{k=1}^\infty \{\nabla^\alpha c_k\} \operatorname{Re}(z^k - 1) = 0$, we have $z^k = 1$, $k = n+1, n+2, \cdots$. Therefore $z = 1$, contradicting our above remark that $E(1) \neq 0$.

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