Appl. Math. Mech. -Engl. Ed. **30**(1), 13–18 (2009)
DOI: 10.1007/s10483-009-0102-9

Applied Mathematics and Mechanics
(English Edition)

3D analysis of functionally graded material plates with complex shapes and various holes *

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(Communicated by Hao-jiang DING)

Abstract In this paper, the basic formulae for the semi-analytical graded FEM on FGM members are derived. Since FGM parameters vary along three space coordinates, the parameters can be integrated in mechanical equations. Therefore with the parameters of a given FGM plate, problems of FGM plate under various conditions can be solved. The approach uses 1D discretization to obtain 3D solutions, which is proven to be an effective numerical method for the mechanical analyses of FGM structures. Examples of FGM plates with complex shapes and various holes are presented.

Key words functionally graded material, semi-analytical method, complex shape, 3D analysis, hole

Chinese Library Classification O343.8
2000 Mathematics Subject Classification 74E10

Introduction

FGM was born for the need of aerospace engineering[1]. The idea is to control each component of FGM throughout the producing process to obtain a gradient distribution of macroscopic property, and ultimately, reduce or eliminate stress concentration. With flexible designs, FGM's volume distribution of each component can adjust to various requirements for structural optimization[2]. With structures tailored to its specific function, FGM is widely used in aerospace, energy resources, electronic and chemical engineering, etc[3].

Until now, most studies of FGM structures have been on beam, plate and shell. Mainstream methods are laminated method, asymptotic method, precise method, simplified model technique, etc. These methods, however, are applicable only to certain structures under specific boundary conditions[4]. Among them, the simplified model technique is most applicable to more complicated structures, yet it neither satisfactorily reflects the gradient variation in FGM nor produces 3D solutions of FGM members[5].

Since FGM parameters vary along three space coordinates, the equations are of variable-coefficient. As a result, currently published results are mainly on beam, plate and shell of square or rectangular shape without holes[6]. In this paper, the semi-analytical graded FEM is used to analyze FGM plates with complicated shapes and holes. It differs from the general semi-analytical method[7] in that it uses 1D discretization to obtain 3D analytical results.

* Received Jul. 23, 2008 / Revised Nov. 5, 2008
Project supported by the National Natural Science Foundation of China (No. 10432030)
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1 Basic formula of semi-analytical graded FEM

1.1 Semi-analytical element

The structure analyzed is a plate, and 1D discretization element (in x direction) is adopted (Fig. 1). The two ends of the element are $L_1(x)$ and $L_2(x)$, where the boundary conditions are simply supported or fixed or free. The formulae of the element are based on 3D elastic theory.

1.2 Displacement mode of semi-analytical element

The displacement mode of semi-analytical element uses analytical function in the y direction, and numerical method in the x and z directions. It can be expressed as

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{Bmatrix} \sum_{n=1}^{q} Y_n(y) \sum_{k=1}^{s} N_k(x,z) u_{kn}(t) \\ \sum_{n=1}^{q} \overline{Y}_n(y) \sum_{k=1}^{s} N_k(x,z) v_{kn}(t) \\ \sum_{n=1}^{q} Y_n(y) \sum_{k=1}^{s} N_k(x,z) w_{kn}(t) \end{Bmatrix}, \tag{1}$$

where $Y_n(y)$ is the analytical function in the y direction, which can be the beam functions corresponding to the boundary conditions of both ends; $\overline{Y}_n(y)$ is the derivative of $Y_n(y)$. For a simply supported element, for example, $Y_n(y) = \sin \frac{n\pi}{L} y$, $\overline{Y}_n(y) = \cos \frac{n\pi}{L} y$, where L is the length of the element in y direction. $N_k(x,z)$ is the plane shape function with 12 nodes (Fig. 2), which could be derived using 2D shape functions of general FEM[8]. u_{kn}, v_{kn}, w_{kn}, are the general degrees of freedom of the node lines, which will be calculated later.

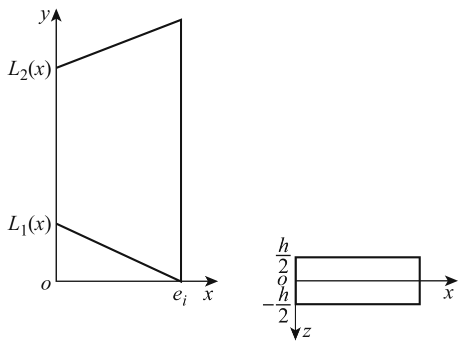

Fig. 1 Semi-analytical element

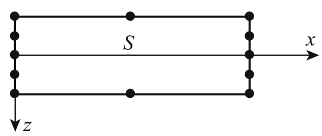

Fig. 2 Node distribution on element section

1.3 Derivation of basic formulae

(1) Displacement vector

Formula (1) can be expressed in terms of matrix form as

$$[u, v, w]^{\mathrm{T}} = \sum_{n=1}^{q} [N]_n \{\delta\}_n = [N]\{\delta\}, \tag{2a}$$

where

$$[N]_n = \begin{bmatrix} Y_n(y)N_1 & & & \cdots & & & Y_n(y)N_{12} & & \\ & \overline{Y}_n(y)N_1 & & & \cdots & & & \overline{Y}_n(y)N_{12} & \\ & & Y_n(y)N_1 & & & \cdots & & & Y_n(y)N_{12} \end{bmatrix}, \tag{2b}$$

$$\{\delta\}_n = \begin{bmatrix} u_{1n} & v_{1n} & w_{1n} & \cdots & \cdots & \cdots & u_{12n} & v_{12n} & w_{12n} \end{bmatrix}^{\mathrm{T}}, \tag{2c}$$

$$[N] = [[N]_1[N]_2 \cdots [N]_q], \tag{2d}$$

$$\{\delta\} = [\{\delta\}_1\{\delta\}_2 \cdots \{\delta\}_q]. \tag{2e}$$

(2) Strain vector

$$\begin{aligned} \{\delta\} &= \begin{bmatrix} \varepsilon_x & \varepsilon_y & \varepsilon_z & \gamma_{xy} & \gamma_{yz} & \gamma_{zx} \end{bmatrix}^{\mathrm{T}} \\ &= \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial z} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} & \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \end{bmatrix}^{\mathrm{T}} \\ &= \sum_{n=1}^{q} [B]_n \{\delta\}_n = [B]\{\delta\}, \end{aligned} \tag{3}$$

where $[B]_n$ and $[B]$ can be obtained by substituting the displacement expressions (2a) into (3).

(3) Stress vector

$$\{\sigma\} = \begin{Bmatrix} \sigma_x & \sigma_y & \sigma_z & \sigma_{xy} & \sigma_{yz} & \sigma_{zx} \end{Bmatrix}^{\mathrm{T}} = \begin{bmatrix} [D_1] & [0] \\ [0] & [D_2] \end{bmatrix} \{\varepsilon\} = [D]\{\varepsilon\} = [D][B]\{\delta\}, \tag{4a}$$

where

$$[D_1] = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1 & \mu/(1-\mu) & \mu/(1-\mu) \\ \mu/(1-\mu) & 1 & \mu/(1-\mu) \\ \mu/(1-\mu) & \mu/(1-\mu) & 1 \end{bmatrix}, \tag{4b}$$

$$[D_2] = \begin{bmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & G \end{bmatrix}, \tag{4c}$$

and the FGM properties, as a function of z, are

$$E = E(z) = E_0 \mathrm{e}^{\alpha z/b}, \quad \mu = \mu(z) = \mu_0 \mathrm{e}^{\alpha z/b}, \quad G = G(z) = E/[2(1+\mu)]. \tag{5}$$

Here, α is a coefficient of change in material; a and b are the length and width of the plate, respectively.

1.4 Basic equations in matrix form

Substituting formulae (2), (3) and (4) into the functional, which is composed of deformation energy, kinetic energy, and external-force work; and taking the variation of $\{\delta\}$; we get the basic equation of an element:

$$[M_i]\{\ddot{\delta}\} + [K_i]\{\delta\} = \{F_i\}, \tag{6}$$

where the sub-matrix of the element stiffness matrix $[K_i]$ is

$$[K_i]_{mn} = \iint_S \int_{L_1(x)}^{L_2(x)} [B]_m^{\mathrm{T}} [D] [B]_n dyds; \tag{7a}$$

the sub-matrix of the element mass matrix $[M_i]$ is

$$[M_i]_{mn} = \iint_S \int_{L_1(x)}^{L_2(x)} [N]_m^{\mathrm{T}} \rho [N]_n dyds; \tag{7b}$$

the sub-array of the element loading vector $\{F_i\}$ is

$$\{F_i\}_n = \iint_S \int_{L_1(x)}^{L_2(x)} [N]_n^{\mathrm{T}} \{q\} dyds. \tag{7c}$$

Here, the external load vector $\{q\} = \begin{bmatrix} q_x & q_y & q_z \end{bmatrix}^{\mathrm{T}}$, mass density $\rho = \rho(z)$, and s is the element section field in x and z.

The process of assembling element stiffness, element mass, and element loading matrixes into general stiffness, mass, and loading matrixes is the same as that of general semi-analytical numerical method[7].

1.5 Method to realize complex shape and holes

When the plate is of complex shape, it is divided into several elements parallel to the y-axis, and the length of each element is a function of x. Then the upper and lower limits of integration of $[K_i]$, $[M_i]$, $\{F_i\}$ in the y direction become functions of x. These functions are usually integrated by the numerical integral method.

Plate with holes is divided into parts with holes and parts without holes, each part divided into several elements respectively. The stiffness, mass, and loading matrixes of the elements with holes are obtained by the segmented integral in y direction. Elements of parts without holes are handled with the usual method. Assembly of these element matrixes produces the total matrix.

2 3D analyses of FGM plates with complex shape

The semi-analytical graded FEM is suitable for 3D analyses of FGM plates of various shapes. Below are the solutions of six simply supported plates with the different shapes.

The common properties of six plates are: thickness h=0.02 m, modulus of elasticity $E(z) = E_0 \mathrm{e}^{\alpha z/b}$, uniform loading $q = 10$ N/m^2, the number of elements is 13, and the Poisson's ratio $\mu(z) = \mu_0 \mathrm{e}^{\alpha z/b}$, where $E_0 = 100$ GPa, $\mu_0 = 0.3$, and $\alpha = 5$. Their shapes and sizes are as follows:

(a) parallelogram plate: bottom side is 0.5 m, height is 1 m, base angle is 80°;

(b) right triangular plate: bottom side is 0.5 m, height is 1 m;

(c) octagonal plate: side length is 0.5 m;

(d) circular plate: radius is 0.5 m;

(e) hemicycle plate: radius is 0.5 m;

(f) elliptical plate: the semiminor axis is 0.25 m, the semimajor axis is 0.5 m.

The distribution values along the thickness direction of the dimensionless displacements $w(a/2,\, b/2)\ E_0/qb$, $v(a/2, 0)E_0/qb$, and the dimensionless stresses $\sigma_x(a/2, b/2)/q$ of the above six FGM plates are given in Table 1.

The following conclusions can be drawn from Table 1:

(1) The distribution features of the basic mechanical quantities: w is constant in thickness direction, and v and σ_x are approximately anti-symmetrical about the middle plane. The results generally coincide with the 2D plate theory, proving the above method reliable, since it is based on 3D theory and makes no assumption in the thickness direction.

(2) The mechanical quantities deviate slightly from those of 2D plate theory: w moderately exceeds that of 2D plate theory on the middle plane, and v and σ_x deviate slightly from linear distribution.

Table 1 Mechanics quantity distributions in thickness direction of FGM plates of different shapes

Dimensionless quantity	Types	$-h/2$	$-h/4$	0	$h/4$	$h/2$
$w\left(\frac{a}{2}, \frac{b}{2}\right)E_0/qb$	a	265.9	266.1	266.2	266.1	265.9
	b	151.0	151.2	151.3	151.2	150.9
	c	612.9	613.2	613.3	613.2	612.9
	d	504.4	504.3	504.2	504.3	504.4
	e	208.8	208.8	208.8	208.8	208.7
	f	498.0	498.4	498.5	498.4	498.9
$v\left(\frac{a}{2}, 0\right)E_0/qb$	a	9.515	5.352	1.197	−2.959	−7.122
	b	9.309	4.579	−0.1381	−4.858	−9.591
	c	14.60	6.485	−1.569	−9.583	−17.53
	d	18.16	10.23	2.265	−5.737	−13.68
	e	12.73	6.141	−0.4373	−7.007	−13.68
	f	16.42	8.562	0.7078	−7.149	−14.99
$\sigma_x\left(\frac{a}{2}, \frac{b}{2}\right)/q$	a	−138.7	−72.35	−2.084	72.75	152.9
	b	−76.37	−41.43	−32.95	39.39	83.74
	c	−60.82	−7.624	29.78	61.32	102.5
	d	115.2	65.91	16.29	−36.32	−95.24
	e	16.45	12.66	4.346	−4.839	−10.22
	f	−224.5	−116.7	−2.621	118.4	249.5

These deviations from classical plate theory reflect the 3D properties[9] of the mechanical quantities of FGM members, of which the material parameters vary along each coordinate.

3 3D analyses for FGM plates with various holes

The semi-analytical graded finite element method is suitable for 3D analyses of FGM plates with various holes. The solutions of six simply supported rectangular plates with the different holes are given as follows.

Table 2 Mechanics quantity distributions in thickness direction of FGM plates with different holes

Dimensionless quantity	Types	$-h/2$	$-h/4$	0	$h/4$	$h/2$
$w\left(\frac{a}{4}, \frac{b}{2}\right)E_0/qb$	a	411.3	411.6	411.7	411.6	411.3
	b	495.4	495.8	495.9	495.8	495.4
	c	246.2	246.2	246.1	245.9	245.7
	d	252.6	252.5	252.3	252.1	251.9
	e	399.3	399.5	399.6	399.5	399.2
	f	468.4	468.7	468.8	468.7	468.4
$v\left(\frac{a}{2}, 0\right)E_0/qb$	a	18.59	9.460	0.3478	−4.858	−17.89
	b	23.03	11.91	0.8255	−10.25	−21.36
	c	4.856	−0.7148	−6.172	−11.63	−17.16
	d	5.498	−0.1510	−5.759	−11.35	−17.00
	e	18.01	9.082	0.1524	−8.781	−17.72
	f	22.40	12.03	1.666	−8.706	−19.09
$\sigma_x\left(\frac{a}{4}, \frac{b}{2}\right)/q$	a	−169.4	−86.17	3.111	98.24	198.4
	b	−249.1	−131.9	−8.679	120.6	256.3
	c	8.218	31.29	62.74	92.76	120.4
	d	44.64	49.22	65.18	83.94	92.51
	e	−124.3	−64.72	0.6270	71.06	145.1
	f	−153.4	−81.75	−5.510	76.31	167.9

All the plates have: width a=0.5 m, length b=1.0 m, thickness h=0.02 m, uniform loading $q = 10$ N/m^2, and the number of elements is 10. The module of elasticity and the Poisson's ratio are the same as those given in the previous section. The positions and sizes of the six holes are as follows:

(a) central square hole 0.1 m×0.1 m;

(b) central rectangular hole 0.1 m×0.2 m;

(c) central circle hole $r = 0.125$ m;

(d) central elliptical hole: semimajor axis=0.125 m, semiminor axis=0.0625 m;

(e) double square holes (x direction): 0.05 m×0.05 m, in the 4$^{\text{th}}$ and 7$^{\text{th}}$ elements;

(f) triple square holes (x direction): 0.05 m×0.05 m, in the 4$^{\text{th}}$, 6$^{\text{th}}$, and 8$^{\text{th}}$ elements.

The distribution values in the thickness direction of dimensionless displacement $w(a/4,\ b/2)E_0/qb$, $v(a/2,0)E_0/qb$, and dimensionless stress $\sigma_x(a/4,b/2)/q$ of the above six FGM plates are given in Table 2.

It can be found from Table 2 that the distribution characteristic of mechanics quantities of FGM plate is almost the same as that of thin plate theory, which means that the proposed method is correct and reliable. The slight deviation resembles the deviation in the solutions of analytical solution of FGM plate[9] from classical plate theory.

4 Conclusions

A semi-analytical graded finite element method for the mechanical analyses of FGM members has been presented in the paper. The FGM plates with different shapes and holes are analyzed and 3D distribution shape of mechanical quantities is given. As an efficient numerical method for the analysis of FGM members, it has the following properties and advantages:

(1) It is applicable to mechanical analyses of the FGM plates with various shapes and different holes.

(2) It utilizes 1D numerical calculation to produce 3D distribution of mechanic quantities, with less elements and lower degrees of freedom than those of general FEM.

(3) It reflects the distribution of mechanical quantities of FGM plates and differentiates itself from classic plate theory, and could serve as an important reference for further research in special plate and shell theory of FGM.

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