1,2,4,8: CLIFFORD ALGEBRAS, BOTT PERIODICITY, AND THE K-THEORY OF THE SPHERES

JOHN PEARSON

Abstract. I will present the basics of Clifford algebras with the specific purpose of developing the connections with K-Theory that are used in Atiyah and Singer's 1968 approach to the Index Theorem.

1. Introduction

In 1963 Michael Atiyah and Isidore Singer first proved for compact manifolds the landmark theorem that would come to be known as the Atiyah-Singer Index Theorem. The original proof modeled off of Hirzebuch's proof of the Riemann-Roch theorem was not very enlightening as it was very computational in nature. In 1968 they were able to present a proof that was more general and modeled more on the approach of Grothendieck's proof of the Riemann-Roch Theorem. This proof relied heavily on K-Theory, Clifford Algebras, and the connections between the two. This paper will present the ideas surrounding the development of these tools for the proof of the Atiyah-Singer Index Theorem.

2. Clifford Algebra Basics

A Clifford algebra is basically an algebra whose multiplication is taken from a vector space with a quadratic form. Therefore, to understand Clifford algebras it is necessary to first have a quadratic form.

2.1. Quadratic Forms.

Definition 1. Let $\mathbb{K}$ be a field of characteristic$\neq 2$. A *bilinear form* is a pair (V, B) where V is a vector space over $\mathbb{K}$ and B is a bilinear map

$$B: \quad V \times V \to \mathbb{K}$$

The function $Q : V \to \mathbb{K}$ given by $Q(v) = B(v, v)$ is called the *quadratic form* corresponding to the bilinear form B.
The bilinear form B is called *nondegenerate* if $\forall \;\; v \neq 0 \in V, \exists \;\; w \in V$ such that $B(v, w) \neq 0$.

Now it should be noted that since Q is derived from a bilinear form B,

$$Q(\lambda v) = B(\lambda v, \lambda v) = \lambda^2 B(v, v) = \lambda^2 Q(v) \text{ for } \lambda \in \mathbb{K}, v \in V.$$

With this in mind, it is also possible to define a quadratic form to be a function Q that satisfies the above relation and where $B(v, w) = \frac{1}{2}[Q(v+w) - Q(v) - Q(w)]$(this is called the polarization identity) is a bilinear form. Thus, we see that the notion of a quadratic form and bilinear form are the same since each can be recovered from the other.

One of the fundamental uses of a bilinear from (V, B) is that it allows us to identify the vector space V with its dual V^* by the map,

$$v \to B(v, .)$$

Note that this is injective if and only if B is nondegenerate.

Date: August 2004.

2.2. **Clifford Algebras.** There are several ways of defining a Clifford algebra but what follows is a constructive definition which follows in the spirit of [LawMi] and [ABS].

Definition 2. Let (V, Q) be a vector space V over $\mathbb{K}$ with quadratic form Q. The *Clifford algebra* $Cl(\mathrm{V}, \mathrm{Q})$ is constructed as follows:
Let $T(V)$ be the tensor algebra of V. Let $I(V)$ be the ideal in $T(V)$ generated by the elements of the form $v \otimes v - Q(v)1$. Then

$$Cl(\mathrm{V}, \mathrm{Q}) := \mathrm{T(V)/I(V)}.$$

To begin our understanding of $Cl(\mathrm{V}, \mathrm{Q})$ we first note that V sits inside of $Cl(\mathrm{V}, \mathrm{Q})$ as the projection of $V = \bigotimes^1 V$ sitting inside of $T(V)$. The projection $\pi : T(V) \to Cl(\mathrm{V}, \mathrm{Q})$ is actually injective when restricted to V. To see this, we show the following

Proposition 2.1. $V \cap I(V) = 0$.

Proof.
Let $\phi \in V \cap I(V)$. Since $I(V)$ is an ideal of $T(V)$ generated by $v \otimes v - Q(v)1$ then any element $\phi \in I(V)$ can be written as $\phi = \sum a_i \otimes [v_i \otimes v_i - Q(v_i)] \otimes b_i$ where the sum is finite and $a_i, b_i \in T(V)$ are of homogeneous degree. Now we look at the part of the sum of highest degree. That is $\sum a_j \otimes [v_j \otimes v_j] \otimes b_j$ where deg a_j+deg $b_j = n$ is maximal.

We now proceed by induction on n. If n=0 then deg a_i+deg $b_i = 0$ for all i. Thus deg $a_i =$ deg $b_i = 0$ and we see that $a_i, b_i \in \mathbb{K}$ for all i. So either $\sum a_i \otimes [v_i \otimes v_i] \otimes b_i$ is of degree 2 or is 0. Since $\phi \in \bigotimes^1 V$, then $\sum a_i \otimes [v_i \otimes v_i] \otimes b_i = 0$. Also, $\sum a_i \otimes [Q(v_i)] \otimes b_i \in \bigotimes^0 V = \mathbb{K}$ and thus is 0. Consequently, $\phi = \sum a_i \otimes [v_i \otimes v_i - Q(v_i)] \otimes b_i = 0$. Now assume that if deg a_i+deg $b_i < n$ for all i then $\sum a_i \otimes [v_i \otimes v_i - Q(v_i)] \otimes b_i = 0$. Let $\sum a_j \otimes [v_j \otimes v_j] \otimes b_j$ be the sum over all j such that deg a_j+deg $b_j = n$. Because $\phi \in \bigotimes^1 V$ and $a_j \otimes [v_j \otimes v_j] \otimes b_j$ is of deg≥ 2, then $\sum a_j \otimes [v_j \otimes v_j] \otimes b_j = 0$. Thus $\sum a_j \otimes [Q(v_j)] \otimes b_j = \sum a_k \otimes [v_k \otimes v_k - Q(v_k)] \otimes b_k$ where deg a_k+deg $b_k < n$ for all k . But by the inductive assumption $\sum a_k \otimes [v_k \otimes v_k - Q(v_k)] \otimes b_k = 0$. Thus $\sum a_j \otimes [Q(v_j)] \otimes b_j) = 0$ and consequently $\phi = 0$.

□

Corollary 2.2. *$Cl(\mathrm{V}, \mathrm{Q})$ is generated by the vector space V (and 1) subject to $v \cdot v = -Q(v)1$.*

Proof. Since $T(V)$ is generated by V and $V \hookrightarrow Cl(\mathrm{V}, \mathrm{Q})$ then V clearly generates $Cl(\mathrm{V}, \mathrm{Q}) = \mathrm{T(V)/I(V)}$. That the relation holds is automatic from the definition. □

Note: Since we assumed the characteristic of $\mathbb{K}$ was not 2, then for $v, w \in V$

$$v \cdot w + w \cdot v = -2B(v, w) \tag{1}$$

This simply follows from the polarization identity of quadratic forms.

Clifford algebras also satisfy a nice universal property which we now show.

Proposition 2.3. *Let $f : V \to \mathcal{A}$ be a linear map into an associative $\mathbb{K}$-algebra with unit, such that*

$$f(v) \cdot f(v) = -Q(v)1 \quad \forall v \in V. \tag{2}$$

Then f extends uniquely to a $\mathbb{K}$-algebra homomorphism $\bar{f} : Cl(\mathrm{V}, \mathrm{Q}) \to \mathcal{A}$. Moreover, $Cl(\mathrm{V}, \mathrm{Q})$ is universal with respect to algebras $\mathcal{C}$ that have embeddings $j : V \hookrightarrow \mathcal{C}$ and satisfy the above property.

Proof.
Now we know that $T(V)$ is a universal object so any map $f : V \to \mathcal{A}$ extends to a unique algebra homomorphism $\tilde{f} : T(V) \to \mathcal{A}$. But equation (2) above gurantees that $\tilde{f} = 0$ when restricted to $I(V) \subset T(V)$. Thus $\tilde{f}$ descends to a map $\bar{f} : Cl(\mathrm{V}, \mathrm{Q}) \to \mathcal{A}$. To see that $Cl(\mathrm{V}, \mathrm{Q})$

is unique in this respect, suppose that $\mathcal{C}$ is an associative $\mathbb{K}$-algebra with unit such that for any $f : V \to \mathcal{A}$ for which equation (2) holds then f extends to a $\mathbb{K}$-algebra homomorphism $\hat{f} : \mathcal{C} \to \mathcal{A}$ and for which we have $j : V \hookrightarrow \mathcal{C}$. Now since $j(V)$ and $i(V)$ are isomorphic where $i : V \hookrightarrow Cl(\mathrm{V}, \mathrm{Q})$ is the embedding discussed in Prop. 2.1 above, then this isomorphism extends to an isomporphism between $\mathcal{C}$ and $Cl(\mathrm{V}, \mathrm{Q})$.

□

Now to continue our study of $Cl(\mathrm{V}, \mathrm{Q})$ we look at the special case where $Q(v) = 0$ for all $v \in V$.

Proposition 2.4. $Cl(\mathrm{V}, 0) \cong \Lambda^*\mathrm{V}$ *as algebras*

Proof. Using the identity (1), we see that $Cl(\mathrm{V}, 0)$ is generated by V subject to the relation $v \cdot w = -w \cdot v \ \ \forall v, w \in V$. This clearly implies that every element is skew-symmetric. □

With this in mind, we realize that for a general quadratic form Q on V the structure of $Cl(\mathrm{V}, \mathrm{Q})$ should not be that different from $Cl(\mathrm{V}, 0)$. This is in fact true.

To begin we note that $T(V)$ is a filtered algebra with filtration $\tilde{\mathcal{F}}^0 \subset \tilde{\mathcal{F}}^1 \subset \cdots \subset T(V)$ where $\tilde{\mathcal{F}}^i = \sum_{s=1}^{i} \bigotimes^s V$. Then if we let $\mathcal{F}^i = \pi(\tilde{\mathcal{F}}^i)$ where $\pi : T(V) \to T(V)/I(V) = Cl(\mathrm{V}, \mathrm{Q})$, we get a filtration of $Cl(\mathrm{V}, \mathrm{Q})$. More importantly, if we let $\mathcal{G}^r = \mathcal{F}^r/\mathcal{F}^{r-1}$, then $\mathcal{G}^* = \bigoplus_{r \geq 0} \mathcal{G}^r$ turns $Cl(\mathrm{V}, \mathrm{Q})$ into a graded algebra. This grading is in fact the same grading as on the exterior algebra.

Theorem 2.5. *For any quadratic form Q, the associated graded algebra $\mathcal{G}^*$ of $Cl(\mathrm{V}, \mathrm{Q})$ is naturally isomorphic to Λ^*V. Moreover, there is a vector space isomorphism from $Cl(\mathrm{V}, \mathrm{Q})$ to Λ^*V which is compatible with the filtrations.*

Proof.

We refer the reader to [LawMi] for a proof. For a similar more algebraic proof, the reader is referred to [Ch]. □

In addition to the above grading, there is another grading obtained through the above isomorphism between $Cl(\mathrm{V}, \mathrm{Q})$ and Λ^*V. Let $\alpha : \Lambda^*V \to \Lambda^*V$ be such that α is the identity on Λ^iV where i is even and is -1 on Λ^iV where i is odd. Then if we let $Cl^0(\mathrm{V}, \mathrm{Q})$ be the eigenspace for 1 and $Cl^1(\mathrm{V}, \mathrm{Q})$ be the eigenspace for -1 under the isomorphism with Λ^*V, this gives a $\mathbb{Z}_2$-grading of $Cl(\mathrm{V}, \mathrm{Q}) = Cl^0(\mathrm{V}, \mathrm{Q}) \oplus Cl^1(\mathrm{V}, \mathrm{Q})$.

Definition 3. $Cl^0(\mathrm{V}, \mathrm{Q})$ is called the *even* part of $Cl(\mathrm{V}, \mathrm{Q})$ and is a subalgebra of $Cl(\mathrm{V}, \mathrm{Q})$. $Cl^1(\mathrm{V}, \mathrm{Q})$ is called the *odd* part of $Cl(\mathrm{V}, \mathrm{Q})$.

3. Clifford Algebras over $\mathbb{R}$

To begin recall that every non-degenerate quadratic form over $\mathbb{R}^n$ has the form $Q(x) = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_n^2$ after an appropriate choice of basis. If we let $s = n - r$ then we write $Cl_{\mathrm{r,s}}$ for the Clifford algebra $Cl(\mathbb{R}^{\mathrm{n}}, \mathrm{Q})$ associated to this quadratic form. Of particular importance are the Clifford algebras $Cl_{\mathrm{n}} = Cl_{\mathrm{n,0}}$ and $Cl_{\mathrm{n}}^* = Cl_{\mathrm{0,n}}$. We shall now begin the study of these algebras.

When studying Cl_{n} and Cl_{n}^*, the $\mathbb{Z}_2$-grading plays an important role. Recall that for two $\mathbb{Z}_2$-graded algebras $\mathcal{A} = \mathcal{A}^0 \oplus \mathcal{A}^1$ and $\mathcal{B} = \mathcal{B}^0 \oplus \mathcal{B}^1$ we can take the $\mathbb{Z}_2$-graded tensor product of $\mathcal{A}$ and $\mathcal{B}$ denoted by $\mathcal{A}\hat{\otimes}\mathcal{B}$. Multiplication in $\mathcal{A}\hat{\otimes}\mathcal{B}$ is given by

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{(deg(a')deg(b))}(aa' \otimes bb'). \tag{3}$$

This $\mathbb{Z}_2$-graded tensor product is again $\mathbb{Z}_2$ graded with $(\mathcal{A}\hat{\otimes}\mathcal{B})^0 = \mathcal{A}^0 \otimes \mathcal{B}^0 + \mathcal{A}^1 \otimes \mathcal{B}^1$ and $(\mathcal{A}\hat{\otimes}\mathcal{B})^1 = \mathcal{A}^0 \otimes \mathcal{B}^1 + \mathcal{A}^1 \otimes \mathcal{B}^0$.

Proposition 3.1. *Let $V = V_1 \oplus V_2$ be a B-orthogonal decomposition of the vector space V where B is a bilinear form on V. Then we have a natural isomorphism of Clifford algebras*

$$Cl(\mathrm{V}, \mathrm{Q}) \cong Cl(\mathrm{V}_1, \mathrm{Q}_1)\hat{\otimes} Cl(\mathrm{V}_2, \mathrm{Q}_2) \tag{4}$$

where Q is the quadratic form associated to the bilinear form B and Q_i is the restriction of Q to V_i.

Proof.

Let $f : V \to Cl(\mathrm{V}_1, \mathrm{Q}_1)\hat{\otimes} Cl(\mathrm{V}_2, \mathrm{Q}_2)$ be given by $f(v) = v_1 \otimes 1 + 1 \otimes v_2$ where $v = v_1 + v_2$ is the decomposition of v with respect to the splitting $V = V_1 \oplus V_2$. Now $f(v) \cdot f(v) = (v_1 \otimes 1 + 1 \otimes v_2)^2 = v_1^2 \otimes 1 + 1 \otimes v_2^2 = -(Q_1(v_1) + Q_2(v_2))1 \otimes 1 = -Q(v)1 \otimes 1$. Thus by Prop. (2.3), f extends to an algebra homomorphism $\tilde{f} : Cl(\mathrm{V}, \mathrm{Q}) \to Cl(\mathrm{V}_1, \mathrm{Q}_1) \otimes Cl(\mathrm{V}_2, \mathrm{Q}_2)$. Now $\tilde{f}$ is surjective since its image is a subalgebra that contains $Cl(\mathrm{V}_1, \mathrm{Q}_1) \otimes 1$ and $1 \otimes Cl(\mathrm{V}_2, \mathrm{Q}_2)$. Now if we form a basis $e_1, \cdots, e_k$ of V such that $e_1, \cdots, e_r$ is a basis for V_1 and $e_{r+1}, \cdots, e_k$ is a basis for V_2. Then we clearly see that f is injective on this basis since e_i gets mapped to $e_i \otimes 1$ for $1 \leq i \leq r$ and $1 \otimes e_i$ for $r < i \leq k$. Thus $\tilde{f}$ is also injective. □

The decomposition provided in this proposition now allows a nice $\mathbb{Z}_2$-decomposition of $Cl_{\mathrm{r,s}}$.

Corollary 3.2. *There is an isomorphism*

$$Cl_{\mathrm{r,s}} = Cl_1 \hat{\otimes} \cdots \hat{\otimes} Cl_1 \hat{\otimes} Cl_1^* \cdots \hat{\otimes} Cl_1^* \tag{5}$$

where Cl_1 appears r times and Cl_1^* appears s times.

Proof. If we decompose V into 1-dimensional subspaces and then restrict our quadratic form Q to each of them, we can apply to above proposition.

□

Now to study the first few examples of the real Clifford algebras.

Example 4. $Cl_1 \cong \mathbb{C}$
To see this note that Cl_1 is generated by 1 and e_1 subject to the relation $e_1^2 = -1$. The isomorphism is clear by sending 1 to 1 and e_1 to i.

Example 5. $Cl_1^* \cong \mathbb{R} \oplus \mathbb{R}$.
Cl_1^* is generated by 1 and e_1 subject to the relation $e_1^2 = 1$. Thus we can think of e_1 as either 1 or -1. Either way, we get $\mathbb{R} \oplus \mathbb{R}$.

Corollary 3.3. *dim*$(Cl_{\mathrm{r,s}}) = 2^{\mathrm{r+s}}$

Example 6. $Cl_2 \cong \mathbb{H}$
Cl_2 is generated by e_1, e_2 subject to three relations: $e_1^2 = -1$, $e_2^2 = -1$, $e_1e_2 + e_2e_1 = 0$. Thus Cl_2 has as a basis the four elements $1, e_1, e_2$, and e_1e_2. If we map e_1 to i, e_2 to j, and e_1e_2 to k, we see that this is an isomorphism since the rules for multiplication are the same.

Example 7. $Cl_2^* \cong \mathbb{R}(2)$ (the 2×2 matrices with entries in $\mathbb{R}$)
Cl_2^* is generated by e_1, e_2 subject to three relations: $e_1^2 = 1$, $e_2^2 = 1$, $e_1e_2 + e_2e_1 = 0$. Thus Cl_2^* also has as a basis the four elements $1, e_1, e_2$, and e_1e_2. We get an isomorphism by mapping 1 to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, e_1 to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, e_2 to $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, and e_1e_2 to $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Surprisingly these few examples are enough to provide us with a complete classification of Cl_{n} and Cl_{n}^* for all $n \geq 0$. The classification basically follows from the following theorem.

Theorem 3.4. *There are isomorphisms*

$$Cl_n \otimes Cl_2^* \cong Cl_{n+2}^* \tag{6}$$

$$Cl_n^* \otimes Cl_2 \cong Cl_{n+2} \tag{7}$$

Proof.
Let $e_1, \cdots, e_{n+2}$ be a basis for $\mathbb{R}^{n+2}$ with standard inner product and with $Q(x) = ||x^2||$ for $x \in \mathbb{R}^{n+2}$. Let $e_1', ..., e_n'$ denote the generators of Cl_n and e_1'', e_2'' denote the standard generators of Cl_2^*. Let $f : \mathbb{R}^{n+2} \to Cl_n \otimes Cl_2^*$ be defined by $f(e_i) = e_i' \otimes e_1'' e_2''$ for $1 \leq i \leq n$ and $f(e_i) = 1 \otimes e_{i-n}''$ for $i = n+1, n+2$ and then extend by linearity.

For $1 \leq i, j \leq n$, $f(e_i) \cdot f(e_j) + f(e_j) \cdot f(e_i) = (e_i' \cdot e_j' + e_j' \cdot e_i' \otimes (-1)) = 2\delta_{ij} 1 \otimes 1$. For $n+1 \leq \alpha, \beta \leq n+2$, $f(e_\alpha) \cdot f(e_\beta) + f(e_\beta) \cdot f(e_\alpha) = 1 \otimes (e_{\alpha-n}'' e_{\beta-n}'' + e_{\beta-n}'' e_{\alpha-n}'') = 2\delta_{\alpha\beta} 1 \otimes 1$. Finally, $f(e_i) \cdot f(e_\alpha) + f(e_\alpha) \cdot f(e_i) = e_i \otimes e_1'' e_2'' e_{\alpha-n}'' + e_i \otimes e_{\alpha-n}'' e_1'' e_2'' = 0$. Thus we have that $f(x) \cdot f(x) = ||x^2||$ for all $x \in \mathbb{R}^{n+2}$. Thus by the universal property in Prop. (2.3) we have that f extends to an algebra homomorphism $\tilde{f} : Cl_{n+2}^* \to Cl_n \otimes Cl_2^*$. It is surjective since $\tilde{f}$ maps onto a set generators. It is injective since both spaces have the same dimension. □

To complete the classification we recall an elementry result from linear algebra.

Proposition 3.5.

$$\mathbb{R}(n) \otimes \mathbb{R}(m) \cong \mathbb{R}(nm) \quad \forall n, m \tag{8}$$

$$\mathbb{R}(n) \otimes_{\mathbb{R}} \mathbb{K} \cong \mathbb{K}(n) \qquad \mathbb{K} = \mathbb{C}, \mathbb{H} \ \ \textit{and} \ \ \forall n \tag{9}$$

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C} \tag{10}$$

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{C}(2) \tag{11}$$

$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{R}(4) \tag{12}$$

This proposition makes computing the real Clifford algebras much easier.

Example 8. $Cl_8 \cong Cl_8^* \cong \mathbb{R}(16)$
This follows since $Cl_8 \cong Cl_8^* \cong Cl_2 \otimes Cl_2 \otimes Cl_2^* \otimes Cl_2^* \cong \mathbb{H} \otimes \mathbb{H} \otimes \mathbb{R}(2) \otimes \mathbb{R}(2) \cong \mathbb{R}(4) \otimes \mathbb{R}(4) \cong \mathbb{R}(16)$

The classification becomes a lot simpler after we have the following 'periodicity' for real Clifford algebras.

Theorem 3.6. *For all $n \geq 0$,*

$$Cl_{n+8} \cong Cl_n \otimes Cl_8 \tag{13}$$

$$Cl_{n+8}^* \cong Cl_n^* \otimes Cl_8^* \tag{14}$$

Proof. Because of Theorem (3.4), $Cl_{n+8} \cong Cl_n \otimes Cl_2 \otimes Cl_2 \otimes Cl_2^* \otimes Cl_2^* \cong Cl_n \otimes Cl_8$. Similarly, $Cl_{n+8}^* \cong Cl_n^* \otimes Cl_2 \otimes Cl_2 \otimes Cl_2^* \otimes Cl_2^* \cong Cl_n^* \otimes Cl_8^*$ □

We can now write down the classification of the real Clifford algebras Cl_n and Cl_n^* in the following table. The previous theorem tells us how to obtain the results for $n > 8$.

	1	2	3	4	5	6	7	8
Cl_n	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$
Cl_n^*	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$

4. Clifford Algebras over $\mathbb{C}$

It turns out that the classification for the Clifford algebras Cl_{n} and Cl^*_{n} above actually provides the classification for Clifford algebras over $\mathbb{C}$. However, the classification for Clifford algebras over $\mathbb{C}$ is much simpler due to the fact that every non-degenerate quadratic form over $\mathbb{C}^n$ can be written in the from $x_1^2 + \cdots + x_n^2$. Thus there is really only one Clifford algebra over $\mathbb{C}$ in each dimension, the Clifford algebra $Cl(\mathbb{C}^{\mathrm{n}}, \mathrm{Q}^{\mathbb{C}}_{\mathrm{n}})$ where $Q^{\mathbb{C}}_n$ is the above quadratic form. We denote $Cl(\mathbb{C}^{\mathrm{n}}, \mathrm{Q}^{\mathbb{C}}_{\mathrm{n}})$ by $Cl^{\mathbb{C}}_{\mathrm{n}}$.

Now given any Clifford algebra $Cl_{\mathrm{r,s}}$ over $\mathbb{R}^{r+s}$ we can complexify $Cl_{\mathrm{r,s}}$ by extending the quadratic form from $\mathbb{R}$ to $\mathbb{C}$ simply by having the form take in complex numbers instead of real numbers. We then obtain $Cl_{\mathrm{r,s}} \otimes \mathbb{C} \cong Cl^{\mathbb{C}}_{\mathrm{r+s}}$. The results from the classification for Clifford algebras over $\mathbb{R}$ now gives us the classification for Clifford algebras over $\mathbb{C}$ by taking the complexification of the real algebras. Thus, Theorem (3.4) above gives the following theorem.

Theorem 4.1. *For all $n \geq 0$,*

$$Cl^{\mathbb{C}}_{\mathrm{n+2}} \cong Cl^{\mathbb{C}}_{\mathrm{n}} \otimes Cl^{\mathbb{C}}_2 \tag{15}$$

Thus the 'periodicty' of Clifford algebras over $\mathbb{C}$ has 'period' 2. Now since $Cl^{\mathbb{C}}_1 \cong Cl_1 \otimes \mathbb{C} \cong \mathbb{C} \otimes \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$ and $Cl^{\mathbb{C}}_2 \cong Cl_2 \otimes \mathbb{C} \cong \mathbb{R}(2) \otimes \mathbb{C} \cong \mathbb{C}(2)$. We thus get the classification for Clifford algebras over $\mathbb{C}$.

$$Cl^{\mathbb{C}}_1 \cong \mathbb{C} \oplus \mathbb{C}, \quad Cl^{\mathbb{C}}_2 \cong \mathbb{C}(2) \tag{16}$$

5. Representations of Clifford algebras over $\mathbb{R}$ and $\mathbb{C}$

In order to get the connections between Clifford algebras and K-Theory, we will need to use representations of the Clifford algebras over $\mathbb{R}$ and $\mathbb{C}$. We begin by recalling the basic facts concerning representations.

Definition 9. Let V be a vector space over $\mathbb{K}'$ and let Q be a quadratic form on V. Let $\mathbb{K} \supset \mathbb{K}'$ be a field containing $\mathbb{K}'$. Then a $\mathbb{K}$-*representation* of $Cl(\mathrm{V,Q})$ is a $\mathbb{K}'$-algebra homomorphism

$$\rho : Cl(\mathrm{V,Q}) \to \mathrm{Hom}_{\mathbb{K}}(\mathrm{W,W})$$

into the algebra of linear transformations of a finite dimensional vector space W over $\mathbb{K}$.
The vector space W is called a $Cl(\mathrm{V,Q})$-*module* . In order to simplify notation, we often write

$$\phi \cdot w := \rho(\phi)(w)$$

where $\phi \in Cl(\mathrm{V,Q}), \mathrm{w} \in \mathrm{W}$. The product $\phi \cdot w$ is referred to as *Clifford multiplication.*

It is important how the $Cl(\mathrm{V,Q})$-module W decomposes. To that end we have the following definition.

Definition 10. With $V, Q, \mathbb{K}', \mathbb{K}, W$ as in the above definition, the representation ρ is called *reducible* if the vector space W can be written as a non-trivial direct sum $W = W_1 \oplus W_2$ such that $\phi \cdot W_i \subset W_i$ for $i = 1, 2$ and $\forall \phi \in Cl(\mathrm{V,Q})$. In this case we write the representation ρ as $\rho = \rho_1 \oplus \rho_2$ where ρ_i is ρ restricted to W_i. A representation is called *irreducible* if it is not reducible.

Warning: Although confusion is easy and often not that misleading, irreducibility applies to representations but is defined in terms of the $Cl(\mathrm{V,Q})$-module W.

We now see that irreducible representations are the building blocks of all representations.

Proposition 5.1. *Every $\mathbb{K}$-representation ρ of a Clifford algebra $Cl(\mathrm{V,Q})$ can be decomposed into a direct sum $\rho = \rho_1 \oplus \cdots \oplus \rho_k$ of irreducible representations.*

Proof.

Let ρ be a representation of $Cl(\mathrm{V},\mathrm{Q})$ on W. Now if ρ is reducible then ρ decomposes as $\rho = \rho_1 \oplus \rho_2$. We then see if ρ_i is an irreducible representation on the $Cl(\mathrm{V},\mathrm{Q})$-module W_i. If not, then it decomposes into two representations. Eventually this process must stop since W is finite dimensional. □

Of course, there is a notion of when two representations are the same.

Definition 11. Two representation $\rho_i : Cl(\mathrm{V},\mathrm{Q}) \to \mathrm{Hom}_{\mathbb{K}}(\mathrm{W},\mathrm{W})$ for i=1,2 are said to be *equivalent* if there exists a $\mathbb{K}$-linear isomorphism $F : W_1 \to W_2$ such that $F \circ \rho_1(\phi) \circ F^{-1} = \rho_2(\phi)$ for all $\phi \in Cl(\mathrm{V},\mathrm{Q})$.

Much of representation theory is concerned with determining the equivalence classes of irreducible representations of certain algebraic objects. The following theorem completely determines the equivalence classes of irreducible representations of matrix algebras over $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$.

Theorem 5.2. *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$ and consider $\mathbb{K}(n)$ as an algebra over $\mathbb{R}$. Then the natural representation ρ_n of $\mathbb{K}(n)$ on $\mathbb{K}^n$ is the only irreducible real representation of $K(n)$ up to equivalence.*

The algebra $\mathbb{K}(n)\oplus\mathbb{K}(n)$ has exactly two equivalence classes of irreducible real representations. They are given by

$$\rho_1(\phi_1, \phi_2) := \rho(\phi_1) \text{ and } \rho_2(\phi_1, \phi_2) := \rho(\phi_2)$$

Proof. It turns out that the algebras $\mathbb{K}(n)$ are simple and therefore their representation theory is easy. For proofs, the reader is referred to [Lang] □

Because of the classification in the previous two sections we can easily find all the irreducible representations of Clifford algebras over $\mathbb{R}, \mathbb{C}$. Once we know all the irreducible representations, we realize that the irreducible representations form a semigroup under direct product of vector spaces. We can complete this semigroup to form the Grothendieck group $\mathfrak{M}_n(\mathfrak{M}_n^{\mathbb{C}})$ of irreducible representations of $Cl_{\mathrm{n}}(Cl_{\mathrm{n}}^{\mathbb{C}})$.

We will now define a group that will give the connection between Clifford algebras and the Bott Periodicty. To begin, realize that $\mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1} \subset Cl_{\mathrm{n+1}}$ induces a map $i : Cl_{\mathrm{n}} \hookrightarrow Cl_{\mathrm{n+1}}$. We can now think of Cl_{n} sitting inside of $Cl_{\mathrm{n+1}}$ and moreover if we are careful and use the algebraic structure of $Cl_{\mathrm{n+1}}$ we can make it the even part of $Cl_{\mathrm{n+1}}$.

Proposition 5.3. *For all $n \geq 0$,*

$$Cl_{\mathrm{n}} \cong Cl^0_{\mathrm{n+1}} \tag{17}$$

Proof.

First, choose an orthonormal basis $e_1, \cdots, e_n$ of $\mathbb{R}^n$ such that $Q(e_i) = 1$ for all $i = 1, \ldots, n$. Let $\mathbb{R}^n = \mathrm{span}\{e_i | i \neq n\}$. Now let $f : \mathbb{R}^n \to Cl^0_{\mathrm{n+1}}$ by $f(e_i) = e_n e_i$ and then extending by linearity. For $x = \sum_{i=1}^{n-1} x_i e_i$, we see that:

$$\begin{aligned} f(x)^2 &= \sum_{i,j} x_i x_j e_n e_i e_n e_j \\ &= \sum_{i,j} x_i x_j e_i e_j \\ &= x \cdot x \\ &= -Q(x) \cdot 1 \end{aligned}$$

Thus by the universal property of Clifford algebras, f extends to an algebra homomorphism

$$\tilde{f} : Cl_{\mathrm{n}} \to Cl^0_{\mathrm{n+1}}$$

which is clearly injective and is surjective based on dimension.

□

Now because we have a map $i : Cl_{\mathrm{n}} \hookrightarrow Cl_{\mathrm{n+1}}$, this induces a map $i^* : \mathfrak{M}_{n+1} \to \mathfrak{M}_n$ by restriction of a representation in $\mathfrak{M}_{n+1}$ to Cl_{n}. Let $\mathfrak{Q}_n := \mathfrak{M}_{n-1}/\mathfrak{M}_n$. Since $\mathfrak{M}_{n+8} \cong \mathfrak{M}_n$, it is clear that $\mathfrak{Q}_{n+8} \cong \mathfrak{Q}_n$. We now begin calculating $\mathfrak{Q}_n$ for $n = 1, \cdots, 8$.

Example 12. $\mathfrak{Q}_1 \cong \mathfrak{Q}_2 \cong \mathbb{Z}_2$
Any representation in $\mathfrak{M}_0$ is a real vector space whereas a representation in $\mathfrak{M}_1$ is a complex vector space. The map $i^* : \mathfrak{M}_1 \to \mathfrak{M}_0$ is defined by treating a representation in $\mathfrak{M}_1$ with complex dimension r as a $2r$-dimensional real vector space. Similarly every representation in $\mathfrak{M}_2$ is a quaternionic vector space.

Example 13. $\mathfrak{Q}_4 \cong \mathfrak{Q}_8 \cong \mathbb{Z}$

Example 14. $\mathfrak{Q}_3 \cong \mathfrak{Q}_5 \cong \mathfrak{Q}_6 \cong \mathfrak{Q}_7 \cong 0$

We now see the Bott Periodicity in terms of representations of Clifford algebras over $\mathbb{R}$. Similar results occur for the complex Clifford algebras and give:

$$\mathfrak{Q}_1^{\mathbb{C}} \cong 0 \text{ and } \mathfrak{Q}_2^{\mathbb{C}} \cong \mathbb{Z}$$

Although these results do give the Bott Periodicity, they do not give the exact ring structure that we want for the connection with K-Theory since $Cl_{\mathrm{n}} \otimes Cl_{\mathrm{m}}$ is not in general a Clifford algebra. In order to get the proper ring structure, we must modify $\mathfrak{M}_n$ just a little. Instead of considering irreducible representations, we consider irreducible $\mathbb{Z}_2$-graded representations and the associated Grothendieck group $\hat{\mathfrak{M}}_n$.

A module W for Cl_{n} is a $\mathbb{Z}_2$*-graded module* if it has a decomposition $W = W^0 \oplus W^1$ that satisfies:

$$Cl^{\mathrm{i}}_{\mathrm{n}} \cdot \mathrm{W}^{\mathrm{j}} \subset \mathrm{W}^{\mathrm{(i+j)mod2}}$$

for $0 \leq i, j \leq 1$. Now we form the $\mathbb{Z}_2$-graded tensor product of two $\mathbb{Z}_2$-graded modules $W = W^0 \oplus W^1$ over Cl_{n} and $V = V^0 \oplus V^1$ over Cl_{m} by:

$$(W \hat{\otimes} V)^0 = W^0 \otimes V^0 + W^1 \otimes V^1$$
$$(W \hat{\otimes} V)^1 = W^0 \otimes V^1 + W^1 \otimes V^0$$

The representation of $Cl_{\mathrm{n}} \hat{\otimes} Cl_{\mathrm{m}}$ on $V \hat{\otimes} W$ is given by:

$$(\phi \otimes \psi) \cdot (w \otimes v) := (-1)^{pq} (\phi \cdot w) \otimes (\psi \cdot v)$$

where $\deg(\phi) = p$ and $\deg(\psi) = q$ with $\phi \in Cl_{\mathrm{n}}, \psi \in Cl_{\mathrm{m}}, w \in \mathrm{W}, v \in \mathrm{V}$. Now since $Cl_{\mathrm{n+m}} \cong Cl_{\mathrm{n}} \hat{\otimes} Cl_{\mathrm{m}}$ (think of the decompositions into Cl_1), then $V \hat{\otimes} W$ is a $\mathbb{Z}_2$-graded module for $Cl_{\mathrm{n+m}}$. We can then form $\hat{\mathfrak{M}}^* = \bigoplus_{n \geq 0} \hat{\mathfrak{M}}^n$ and realize that this is a graded ring with addition induced by $\oplus$ on the irreducible representations and multiplication induced by $\hat{\otimes}$. Fortunately, $\hat{\mathfrak{M}}^n$ is closely related to $\mathfrak{M}^{n-1}$.

Proposition 5.4. *For all* $n \geq 1$,

$$\hat{\mathfrak{M}}^n \cong \mathfrak{M}^{n-1}$$

Proof.
First recall that by Prop. (5.3), $Cl^0_{\mathrm{n}} \cong Cl_{\mathrm{n-1}}$. Thus we get a homomorphism from $\hat{\mathfrak{M}}^n$ to $\mathfrak{M}^{n-1}$ by mapping a $\mathbb{Z}_2$-graded module $W = W^0 \oplus W^1$ to the ungraded Cl^0_{n}-module W^0.

The inverse homomorphism is defined in a slightly more complicated manner. First realize that Cl_{n} is in fact a $\mathbb{Z}_2$-graded Cl_{n}-module under multiplication on itself. Any even element in Cl_{n} will preserve both the even and odd parts of Cl_{n} while any odd element will permute the two. We now form the $\mathbb{Z}_2$-graded module W by:

$$W := Cl_{\rm n} \otimes_{Cl^0_{\rm n}} \mathrm{W}^0$$

where the action of $Cl_{\rm n}$ is on itself. □

We thus see that $\mathfrak{Q}_n \cong \hat{\mathfrak{M}}_n/\hat{\mathfrak{M}}_{n+1}$ so that we get the same Bott Periodicity when considering the ring $\hat{\mathfrak{M}}^*$. It is this ring that exhibits the connection with K-Theory and in fact we have the following theorem.

Theorem 5.5. *Let $\hat{\mathfrak{M}}_*/\hat{\mathfrak{M}}_{*+1} := \bigoplus_{n\geq 0} \hat{\mathfrak{M}}_n/\hat{\mathfrak{M}}_{n+1}$. Then we have:*

$$\hat{\mathfrak{M}}_*/\hat{\mathfrak{M}}_{*+1} \cong KO^*(pt.)$$

Of course, everything discussed so far naturally carries over to the complex case so we also have the following theorem.

Theorem 5.6. *Let $\hat{\mathfrak{M}}^{\mathbb{C}}_*/\hat{\mathfrak{M}}^{\mathbb{C}}_{*+1} := \bigoplus_{n\geq 0} \hat{\mathfrak{M}}^{\mathbb{C}}_n/\hat{\mathfrak{M}}^{\mathbb{C}}_{n+1}$. Then we have:*

$$\hat{\mathfrak{M}}^{\mathbb{C}}_*/\hat{\mathfrak{M}}^{\mathbb{C}}_{*+1} \cong K^*(pt.)$$

These isomorphisms are far from trivial and require several new tools. First, it turns out that there is an important group $Spin(n) \subset Cl_{\rm n}$. After developing this group, we use its representations to form an associated $Spin(n)$- principal bundle on S^{n-1}. Then after a careful reworking of K-Theory to make it more compatible with Clifford algebras we use the above vector bundle on S^{n-1} to get our map into the K-Theory of the point which is completely determined by the K-Theory of the spheres.

References

[ABS] M.F. Atiyah, R. Bott, A. Shapiro "Clifford Modules". *Topology 3, Suppl. 1* (1964) pg. 3-38.

[AS0] M.F. Atiyah, I. Singer "The index of elliptic operators on compact manifolds". *Bull. A.M.S. 69* (1963) pg. 422-433.

[AS1] M.F. Atiyah, I. Singer "The index of elliptic operators I". *Ann. of Math. 87* (1968) pg. 484-530.

[Ch] Chevalley, Claude *The Algebraic Theory of Spinors*. Columbia University Press (1954).

[RSS] R. Delanghe, F. Sommen, V. Soucek *Clifford Algebra and Spinor-Valued Functions*. Kluwer Academic Publishers (1992).

[Fr] Friedrich, Thomas *Dirac Operators in Riemannian Geometry*. American Mathematical Society (2000).

[Ko] Kostant, Bertram "Clifford Algebra Analogue of the Hopf-Koszul-Samelson Theorem, the ρ- Decomposition $C(\mathfrak{g})$ =End $V_\rho \otimes C(P)$ and the $\mathfrak{g}$-Module structure of $\bigwedge \mathfrak{g}$" *Adv. in Math. 125* (1997) pg.275-350.

[Lang] Lang, Serge *Algebra*. Springer-Verlag (2002).

[LawMi] H.B. Lawson, M.L. Michelson *Spin Geometry*. Princeton University Press (1989).